Integrable Nonlocal PT-Symmetric Modified Korteweg-de Vries Equations Associated with so(3, R)

Wen-Xiu Ma

Abstract: We construct integrable PT-symmetric nonlocal reductions for an integrable hierarchy associated with the special orthogonal Lie algebra so(3, R). The resulting typical nonlocal integrable equations are integrable PT-symmetric nonlocal complex reverse-spacetime and real reverse-spacetime modified Korteweg-de Vries equations associated with so(3, R).

Keywords: zero curvature equation; Liouville integrability; Nonlocal integrable reduction

PACS: 02.30.Ik

MSC: 37K06; 37K10; 35Q53

1. Introduction

The modified Korteweg-de Vries (mKdV) equation appears in the study of acoustic waves in an anharmonic lattice [1] and the Alfvén wave in a cold collision-free plasma [2]. Its relation with the Korteweg-de Vries (KdV) equation is discussed [3], and its N-soliton solutions is presented by the inverse scattering transform [4]. The Lax pair in the associated matrix spectral problems comes from sl(2, R) [5]. There is also a system of mKdV equations associated with so(3, R) [6], which is not well studied.

Recently, two kinds of nonlocal integrable mKdV equations associated with sl(2, R) have been explored, which are Liouville integrable, i.e., possess infinitely many symmetries and conservation laws [7,8]. The resulting model equations can relate function values at point (x, t) to its function values at a mirror-reflection point (−x, −t). This is one of the complex types, since it involves the Hermitian transpose, and the other is of real type, since it only uses the matrix transpose. Such studies have opened new avenues for studying mKdV type integrable equations [9,10].

It is known that, based on matrix loop Lie algebras, integrable equations can be generated from matrix spectral problems [11] and their reduced problems [12]. Lax pairs [13] play a crucial role in the formulation of integrable equations and their solutions to Cauchy problems [11]. The trace identity [14] and the variational identity [15] can be used to establish Hamiltonian structures, which exhibit the Liouville integrability of the underlying model equations. Among the well-known integrable equations associated with simple Lie algebras are the KdV equation, the Ablowitz–Kaup–Newell–Segur system of nonlinear Schrödinger (NLS) equations, and the derivative NLS equation [11,16]. More generally, there are integrable couplings associated with non-semisimple Lie algebras, which bring various types of hereditary recursion operators in block matrix form.

In our construction, we will use the special orthogonal Lie algebra g = so(3, R), presented by all 3 × 3 trace-free, skew-symmetric real matrices, with a basis:
whose structure equations read
\[ [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \]  
(2)

Obviously, the derived algebra \([g, g] = [so(3, \mathbb{R}), so(3, \mathbb{R})]\) is \(so(3, \mathbb{R})\) itself. The algebra is one of the only two three-dimensional real Lie algebras with this property. The other one is the special linear algebra \(sl(2, \mathbb{R})\), which is used as a starting point for the study of integrable equations [5]. It is interesting to note that the two complex Lie algebras, \(sl(2, \mathbb{C})\) and \(so(3, \mathbb{C})\), are isomorphic to each other. The corresponding matrix loop algebra that we will use is
\[ \tilde{g} = \tilde{so}(3, \mathbb{R}) = \{ M \in so(3, \mathbb{R}) \mid \text{entries of } M - \text{Laurent series in } \lambda \}, \]  
(3)

where \(\lambda\) is a spectral parameter. This matrix loop algebra has also been recently used to construct integrable equations [6,17].

In this paper, we would like to revisit a hierarchy of integrable equations associated with \(so(3, \mathbb{R})\) [6]. We will then construct two pairs of nonlocal PT-symmetric integrable reductions for the adopted spectral matrix, to generate two reduced hierarchies of scalar integrable equations associated with \(so(3, \mathbb{R})\). Two typical such reduced nonlocal PT-symmetric integrable equations are the complex reverse-spacetime mKdV equation
\[ p_t = -p_{xxx} + \frac{3}{2} p^2 p_x + \frac{3}{2} p_x (p^* (-x_e - t))^2, \]
where \(p^*\) denotes the complex conjugate of \(p\), and the real reverse-spacetime mKdV equation
\[ p_t = -p_{xxx} + \frac{3}{2} p^2 p_x + \frac{3}{2} p_x (p (-x_e - t))^2, \]
which are all associated with \(so(3, \mathbb{R})\).

2. The Integrable Hierarchy Revisited
2.1. Integrable Hierarchy

We would like to revisit an integrable hierarchy associated with the matrix loop algebra \(\tilde{so}(3, \mathbb{R})\) [6]. Let \(i\) be the unit imaginary number. We start from a slightly different spatial matrix spectral problem
\[ -i \phi_x = U \phi \quad \text{or} \quad \phi_x = i U \phi, \]
(4)

with
\[ U = U(u, \lambda) = \lambda e_1 + p e_2 + q e_3 = \begin{bmatrix} 0 & -q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}, \]
(5)

where \(\lambda\) is a spectral parameter, \(u = (p, q)^T\) is a potential and \(\phi = (\phi_1, \phi_2, \phi_3)^T\) is a column eigenfunction. We have adopted the spectral matrix \(i U\), involving a constant factor \(i\). This will bring us convenience in finding integrable nonlocal reductions.

As usual, we solve the stationary zero curvature equation
\[ W_x = i[U, W] \]  
(6)
for $W = W(u, \lambda) \in \mathfrak{s}\mathfrak{o}(3, \mathbb{R})$. This equivalently requires

$$a_x = i(pc - qb), \quad b_x = i(-\lambda c + qa), \quad c_x = i(\lambda b - pa), \quad (7)$$

as long as $W$ is determined as follows:

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix} = \sum_{m \geq 0} W_{0,m} \lambda^{-m}, \quad (8)$$

with

$$W_{0,m} = a_m e_1 + b_m e_2 + c_m e_3 = \begin{bmatrix} 0 & -c_m & -a_m \\ c_m & 0 & -b_m \\ a_m & b_m & 0 \end{bmatrix}, \quad m \geq 0. \quad (9)$$

Upon taking the initial values

$$a_0 = -1, \quad b_0 = c_0 = 0, \quad (10)$$

the system (7) gives

$$b_{m+1} = -ic_{m,x} + pa_m, \quad c_{m+1} = ib_{m,x} + qa_m, \quad a_{m+1} = i(pc_{m+1} - qb_{m+1}), \quad m \geq 0, \quad (11)$$

and defines the sequence of $\{a_m, b_m, c_m \mid m \geq 1\}$ uniquely, when taking all constants of integration as zero:

$$a_m \mid_{u=0} = 0, \quad m \geq 1. \quad (12)$$

which implies

$$b_m \mid_{u=0} = c_m \mid_{u=0} = 0, \quad m \geq 1. \quad (13)$$

The first few sets are as follows:

$$b_1 = -p, \quad c_1 = -q, \quad a_1 = 0;$$

$$b_2 = iq_x, \quad c_2 = -ip_x, \quad a_2 = \frac{1}{2}(p^2 + q^2);$$

$$b_3 = -px + \frac{1}{2}p^3 + \frac{1}{2}pq^2,$$

$$c_3 = -qx + \frac{1}{2}p^2q + \frac{1}{2}q^3,$$

$$a_3 = i(p_3q - pq_3);$$

$$b_4 = i(q_{xxx} - \frac{3}{2}p^2q_x - \frac{3}{2}q^2q_x),$$

$$c_4 = i(p_{xxx} + \frac{3}{2}p^2p_x + \frac{3}{2}p_xq^2),$$

$$a_4 = pp_{xx} + qq_{xx} - \frac{1}{2}p_x^2 - \frac{1}{2}q_x^2 - \frac{3}{2}(p^2 + q^2)^2.$$
generate a hierarchy of integrable equations:

\[ u_t = K_m = i \begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} iq \\ -ip \end{bmatrix}, \quad m \geq 0, \tag{17} \]

where the operator \( \Phi \) can be determined by the recursion relation (11):

\[ \Phi = i \begin{bmatrix} q\partial^{-1} p & -\partial + q\partial^{-1} q \\ \partial - p\partial^{-1} p & -p\partial^{-1} q \end{bmatrix}, \quad \partial = \frac{\partial}{\partial x}. \tag{18} \]

2.2. Hamiltonian Structure and the Liouville Integrability

We use the trace identity [14] for our spectral matrix \( iU \):

\[ \delta \int \text{tr}(W \frac{\partial U}{\partial \lambda}) dx = \lambda^{-\gamma} \frac{\lambda}{\lambda} \partial \lambda \gamma \text{tr}(W \frac{\partial U}{\partial u}), \tag{19} \]

where the constant \( \gamma \) is given by

\[ \gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\langle W, W \rangle|, \tag{20} \]

to construct Hamiltonian structures, which will exhibit the Liouville integrability of the hierarchy (17). We readily see that the corresponding trace identity (19) reads

\[ \delta \int a dx = \lambda^{-\gamma} \frac{b}{c}. \]

Simply applying this, we obtain the following Hamiltonian structures for the hierarchy (17):

\[ u_t = K_m = i \begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \tag{21} \]

with the Hamiltonian operator and the Hamiltonian functionals

\[ J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{H}_m = \int \left( -\frac{iA_{m+1}}{m+1} \right) dx, \quad m \geq 0. \tag{22} \]

These tell that there exist infinitely many conservation laws for each system in the hierarchy (17), which can often be generated through symbolic computation by computer algebra systems (see, e.g., [18]).

To exhibit its Liouville integrability, let us show that the operator \( \Phi \) given by (18) is a common hereditary recursion operator for the hierarchy (17).

First, a straightforward calculation can verify that the operator \( \Phi \) is hereditary (see [19] for definition), that is to say, it satisfies

\[ \Phi'(u)[\Phi K]S - \Phi'(u)K[\Phi S] = \Phi'(u)[\Phi S]K - \Phi K[\Phi S], \tag{23} \]

and \( \Phi \) is a recursion operator for \( u_{t_0} = K_0 \):

\[ L_{K_0} \Phi = 0, \quad K_0 = i \begin{bmatrix} q \\ -p \end{bmatrix}. \tag{24} \]
where \((L_K \Phi)S = \Phi[K, S] - [K, \Phi S]\), in which \(K\) and \(S\) are arbitrary vector fields and \([\cdot, \cdot]\) is the Lie bracket of vector fields. Another direct result is that the pair of operators, \(J\) and 
\[ M = \Phi J = \begin{bmatrix} -\partial + q \partial^{-1} q & -q \partial^{-1} p \\ -p \partial^{-1} q & -\partial + p \partial^{-1} p \end{bmatrix}, \]
constitutes a Hamiltonian pair (see [20] for details). The hereditary property (23) equivalently requires
\[ L_{\Phi K} \Phi = \Phi L_K \Phi, \]
where \(K\) is an arbitrary vector field. Thus, we have
\[ L_{K_m} \Phi = L_{\Phi K_{m-1}} \Phi = \Phi L_{K_{m-1}} \Phi = 0, \quad m \geq 1, \]
where the \(K_m\)’s are given by (17). This implies that the operator \(\Phi\) defined by (18) is a common hereditary recursion operator for the hierarchy (17) (see also [21] for symbolic computation).

Now, the hierarchy (17) is bi-Hamiltonian (see, e.g., [20, 22, 23] for details):
\[ u_t = K_m = \int \frac{\delta H_m}{\delta u} = M \frac{\delta H_{m-1}}{\delta u}, \quad m \geq 1, \]
where \(J, M\) and \(H_m\) are defined by (22) and (25). In this way, every member in the hierarchy is Liouville integrable, i.e., it possesses infinitely many commuting symmetries and conservation laws. Particularly, we have the Abelian symmetry algebra:
\[ [K_k, K_l] = K_k[u][K_l] - K_l[u][K_k] = 0, \quad k, l \geq 0, \]
and the Abelian algebras of conserved functionals:
\[ \{H_k, H_l\}_J = \int \left( \frac{\delta H_k}{\delta u} \right)^T \frac{\delta H_l}{\delta u} \, dx = 0, \quad k, l \geq 0, \]
and
\[ \{H_k, H_l\}_M = \int \left( \frac{\delta H_k}{\delta u} \right)^T M \frac{\delta H_l}{\delta u} \, dx = 0, \quad k, l \geq 0. \]

The third-order nonlinear integrable system in the hierarchy (17) is a system of mKdV equations associated with so(3, \(\mathbb{R}\)):
\[ p_t = -p_{xxx} + \frac{3}{2} p^2 p_x + \frac{3}{2} q_x p_x^2, \quad q_t = -q_{xxx} + \frac{3}{4} p^2 q_x + \frac{3}{2} q_x^2 q_x. \]

Based on (28), it possesses the following bi-Hamiltonian structure
\[ u_t = K_3 = \int \frac{\delta H_3}{\delta u} = M \frac{\delta H_2}{\delta u}, \]
where the Hamiltonian pair \(\{J, M\}\) is defined by (22) and (25), and the Hamiltonian functionals, \(H_2\) and \(H_3\), are given by
\[ H_2 = -\frac{i}{3} \int \left( p_{xxx} q - \frac{3}{2} q_x p_{xx} - \frac{3}{8} p^2 \left( q_x^2 + q_x^2 \right) \right) \, dx, \]
\[ H_3 = \frac{1}{4} \int \left( p_x q_{xxx} - p_{xxx} q_x + p_{xx} q_{xx} \right) - \frac{3}{2} p^2 p_x q + \frac{3}{2} p q_x^2 q_x + \frac{3}{2} p^2 q_x - \frac{3}{2} p q^3 \, dx. \]

The Hamiltonian formulation is extremely important in carrying out the Whitham modulation [24].
We point out that there is the transformation
\[
\hat{p}(x,t) = \frac{1}{2}(-p + iq)(ix,it), \quad \hat{q}(x,t) = \frac{1}{2}(-p - iq)(ix,it),
\]
between the system (32) of mKdV equations associated with so(3) and the system of mKdV equations
\[
\hat{p}_t = \hat{p}_{xxx} + 6\hat{p}\hat{q}\hat{p}_x, \quad \hat{q}_t = \hat{q}_{xxx} + 6\hat{p}\hat{q}\hat{q}_x,
\]
which is associated with sl(2), but there is no nontrivial transformation if we consider only real potentials \(p,q,\hat{p},\hat{q}\). This reflects the fact that the two complex Lie algebras, \(\text{sl}(2,\mathbb{C})\) and \(\text{so}(3,\mathbb{C})\), are isomorphic, but not so are the two real Lie algebras, \(\text{sl}(2,\mathbb{R})\) and \(\text{so}(3,\mathbb{R})\).

3. Integrable Nonlocal Reductions

3.1. Complex Reverse-Spacetime Reductions

Let us first consider a pair of specific complex reverse-spacetime reductions for the spectral matrix:
\[
U^\dagger(-x,-t,−\lambda^*) = -CU(x,t,\lambda)C^{-1}, \quad C = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ \delta & 0 & 0 \end{bmatrix}, \quad \delta = \pm 1,
\]
where \(\dagger\) and * stand for the Hermitian transpose and the complex conjugate, respectively. They generate the potential reductions
\[
p^\ast(-x,-t) = -\delta \hat{q}(x,t), \quad \delta = \pm 1.
\]
Once assigning these potential reductions, we have the reduction property for \(W\):
\[
W^\dagger(-x,-t,−\lambda^*) = CW(x,t,\lambda)C^{-1},
\]
since two matrices on both sides of the above equation solve the stationary zero curvature Equation (6) with the same initial values. This engenders that
\[
a^\ast(-x,-t,−\lambda^*) = a(x,t,\lambda), \quad b^\ast(-x,-t,−\lambda^*) = \delta c(x,t,\lambda),
\]
namely,
\[
a_m^\ast(-x,-t) = (-1)^m a_m(x,t), \quad b_m^\ast(-x,-t) = (-1)^m \delta c_m(x,t), \quad m \geq 1.
\]
Therefore, we obtain
\[
(V^{[m]})^\dagger(-x,-t,−\lambda^*) = (-1)^m CV^{[m]}(x,t,\lambda)C^{-1}, \quad m \geq 1,
\]
and then
\[
((U_t - V_x^{[2l+1]} + i[U, V^{[2l+1]}])(-x,-t,−\lambda^*)^\dagger = C(U_t - V_x^{[2l+1]} + i[U, V^{[2l+1]}])(x,t,\lambda)C^{-1}, \quad l \geq 1.
\]
This implies that the potential reductions given by (39) are compatible with the \((2l + 1)\)-th zero curvature equation of the integrable hierarchy (17). In this way, one obtains two reduced scalar integrable hierarchies associated with \(\text{so}(3,\mathbb{R})\):
\[
p_t = K_{2l+1,1}|_{q(x,t)} = -\delta p^\ast(-x,-t), \quad l \geq 1,
\]
where $K_m = (K_{m,1}, K_{m,2})^T$, $m \geq 1$, are defined by (17). The infinitely many symmetries and conservation laws for the integrable hierarchy (17) will be reduced to infinitely many ones for the above integrable hierarchies in (45), under (39).

With $\delta = 1$, the third-order nonlinear reduced scalar integrable equation is a nonlocal complex reverse-spacetime PT-symmetric mKdV equation associated with $\mathfrak{so}(3, \mathbb{R})$:

$$p_t = -p_{xxx} + \frac{3}{2} p_x p_x^2 + \frac{3}{2} p_x (p^* (-x, -t))^2$$  \hspace{1cm} (46)

where $p^*$ denotes the complex conjugate of $p$. Observe that the two components of $K_m$, $m \geq 1$, have even and odd properties with respect to $x$ and $q$. In fact, $K_{2l,1}$, $l \geq 1$, are odd with respect to $q$ and even with respect to $p$, and $K_{2l+1,1}$, $l \geq 1$, are even with respect to $q$ and odd with respect to $p$. Similarly, we see that $K_{2l,1}$, $l \geq 1$, are odd with respect to $p$ and even with respect to $q$, and $K_{2l+1,2}$, $l \geq 1$, are even with respect to $p$ and odd with respect to $q$. Therefore, the third-order reduced scalar integrable equation with $\delta = -1$ in (45) is exactly the same as the complex nonlocal reverse-space mKdV Equation (46).

### 3.2. Real Reverse-Spacetime Reductions

Secondly, let us consider a pair of specific real reverse-spacetime reductions for the spectral matrix:

$$U^T (-x, -t, \lambda) = C U(x, t, \lambda) C^{-1}, \quad C = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ \delta & 0 & 0 \end{bmatrix}, \quad \delta = \pm 1,$$  \hspace{1cm} (47)

where $T$ means taking the matrix transpose. They engender the potential reductions

$$p(-x, -t) = \delta q(x, t), \quad \delta = \pm 1.$$  \hspace{1cm} (48)

As before, under these potential reductions, $W$ satisfies the following reduction property:

$$W^T (-x, -t, \lambda) = CW(x, t, \lambda) C^{-1},$$  \hspace{1cm} (49)

because two matrices on both sides of the equation solve the stationary zero curvature Equation (6) with the same initial values. Thus, one has

$$a(-x, -t, \lambda) = a(x, t, \lambda), \quad b(-x, -t, \lambda) = \delta c(x, t, \lambda),$$  \hspace{1cm} (50)

namely,

$$a_m(-x, -t) = a_m(x, t), \quad b_m(-x, -t) = \delta c_m(x, t), \quad m \geq 1.$$  \hspace{1cm} (51)

Then, we arrive at

$$(V^{[m]})^T (-x, -t, \lambda) = C V^{[m]}(x, t, \lambda) C^{-1}, \quad m \geq 1,$$  \hspace{1cm} (52)

and therefore, we obtain

$$((U_t - V^{[m]}_x + i[U, V^{[m]}])(-x, -t, \lambda))^T$$

$$= -C (U_t - V^{[m]}_x + i[U, V^{[m]}])(x, t, \lambda) C^{-1}, \quad m \geq 1.$$  \hspace{1cm} (53)

This guarantees that the potential reductions in (48) are compatible with the zero curvature equations of the integrable hierarchy (17).

In this way, one obtains two reduced scalar integrable hierarchies associated with $\mathfrak{so}(3, \mathbb{R})$:

$$p_t = K_{m,1} q(x, t) = \delta p(-x, -t), \quad m \geq 1,$$  \hspace{1cm} (54)
where $K_m = (K_{m,1}, K_{m,2})^T$, $m \geq 1$, are defined by (17). Moreover, the infinitely many symmetries and conservation laws for the integrable hierarchy (17) are reduced to infinitely many ones for the above integrable hierarchies in (54), under (48).

With $\delta = 1$, the third-order nonlinear reduced scalar integrable equation is a nonlocal real reverse-spacetime PT-symmetric mKdV equation associated with $\text{so}(3, \mathbb{R})$:

$$p_t = -p_{xxx} + \frac{3}{2} p^2 p_x + \frac{3}{2} p_x (p(-x, -t))^2. \quad (55)$$

Even and odd properties with respect to $p$ and $q$ in the two components of $K_m$, $m \geq 1$, tell that the third-order nonlinear reduced scalar integrable equation with $\delta = -1$ in (54) is exactly the same as the nonlocal real reverse-spacetime mKdV Equation (55).

4. Concluding Remarks

We have revisited a hierarchy of integrable equations based on zero curvature equations associated with $\text{so}(3, \mathbb{R})$ and presented two pairs of integrable nonlocal PT-symmetric reductions for the hierarchy. Two typical examples among the reduced scalar integrable equations are a nonlocal complex reverse-spacetime modified Korteweg-de Vries (mKdV) equation and a nonlocal real reverse-spacetime mKdV equation, associated with the special orthogonal Lie algebra $\text{so}(3, \mathbb{R})$. Every pair of nonlocal reductions leads to the same nonlocal integrable equations. This is a new phenomenon for integrable equations associated with $\text{so}(3, \mathbb{R})$, different from the one for integrable equations associated with $\text{sl}(2, \mathbb{R})$.

Associated with the special orthogonal Lie algebras, there are many interesting questions for integrable equations, both local and nonlocal. Firstly, what kind of general integrable equations could exist? Some interesting structures of local integrable equations associated with $\text{so}(4, \mathbb{R})$ and nonlocal integrable equations associated with $\text{so}(3, \mathbb{R})$ have been explored (see, e.g., [25,26]). Secondly, do there exist $N$-soliton solutions in such cases (see, e.g., [27] for $(1 + 1)$-dimensional models and [28–31] for $(2 + 1)$-dimensional models)? Thirdly, how can we formulate Riemann-Hilbert problems, based on associated matrix spectral problems? The above spectral matrix $iU$ in our formulation with zero potential has three eigenvalues, and this feature brings difficulty in solving related problems. We do not know how to establish Riemann-Hilbert problems for integrable systems associated with $\text{so}(3, \mathbb{R})$, different from the one for integrable equations associated with $\text{sl}(2, \mathbb{R})$. The existing examples of Riemann-Hilbert problems in the literature belong to the class of spectral matrices which possess two eigenvalues when the potential is taken as zero.

It is known that integrable couplings are constructed from zero curvature equations associated with non-semisimple Lie algebras, and their Hamiltonian structures could be established through the variational identity [32]. Bi-integrable couplings and tri-integrable couplings are such examples, and they exhibit insightful thoughts about generic mathematical structures for multi-component integrable equations. Multi-integrable couplings produce abundant examples of recursion operators in block matrix form, indeed. There are rich algebraic and geometric structures related to integrable couplings. However, non-semisimple matrix Lie algebras may not possess any non-degenerate and ad-invariant bilinear forms required in the variational identities, and this causes serious problems in furnishing integrable couplings with Hamiltonian structures. For instance, we do not even know whether there exists any Hamiltonian structure for an interesting perturbation type coupling:

$$u_t = K(u), \ v_t = K'(u)[v], \ w_t = K'(u)[w].$$

Specifically in the KdV case, we have the question whether there is any Hamiltonian structure for the following integrable coupling:

$$u_t = 6uu_x + u_{xxx}, \ v_t = 6(uv)_x + v_{xxx}, \ w_t = 6(uw)_x + w_{xxx}.$$  

We may need to develop a generalized variational identity to explore Hamiltonian structures for integrable couplings in this kind of case.
Funding: This work was supported in part by the National Natural Science Foundation of China (11975145, 11972291 and 51771083), the Ministry of Science and Technology of China (BG20190216001), and the Natural Science Foundation for Colleges and Universities in Jiangsu Province (17 KJB110020).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: The author would like to thank Alle Adjiri, Ahmed Ahmed, Mohamed Reda Ali, Nadia Cheemaa, Li Cheng, Morgan McAnally, Solomon Manukure, Rahma Sadat Moussa, Yaning Tang, Fudong Wang, Melike Kaplan Yalçın and Yi Zhang for their valuable discussions.

Conflicts of Interest: The author declares that there is no known competing financial in-terests or personal relationships that could have appeared to influence the work reported in this paper.

References


30. Ma, W.X.; Yong, X.L.; Lü, X. Soliton Solutions to the B-type Kadomtsev-Petviashvili Equation under General Dispersion Relations. *Wave Motion* 2021, 103, 102719. [CrossRef]
