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Integrable Nonlocal PT-Symmetric Modified Korteweg-de Vries Equations Associated with $\mathrm{so}(3,\mathbb{R})$

Wen-Xiu Ma [1,2,3,4]

[1] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China; mawx@cas.usf.edu
[2] Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[3] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
[4] School of Mathematical and Statistical Sciences, Mafikeng Campus, North-West University, Private Bag X2046, Mmabatho 2735, South Africa

Abstract: We construct integrable PT-symmetric nonlocal reductions for an integrable hierarchy associated with the special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$. The resulting typical nonlocal integrable equations are integrable PT-symmetric nonlocal complex reverse-spacetime and real reverse-spacetime modified Korteweg-de Vries equations associated with $\mathrm{so}(3,\mathbb{R})$.

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1. Introduction

The modifed Korteweg-de Vries (mKdV) equation appears in the study of acoustic waves in an anharmonic lattice [1] and the Alfven wave in a cold collision-free plasma [2]. Its relation with the Korteweg-de Vries (KdV) equation is discussed [3], and its N-soliton solutions is presented by the inverse scattering transform [4]. The Lax pair in the associated matrix spectral problems comes from sl(2,$\mathbb{R}$) [5]. There is also a system of mKdV equations associated with so(3,$\mathbb{R}$) [6], which is not well studied.

Recently, two kinds of nonlocal integrable mKdV equations associated with sl(2,$\mathbb{R}$) have been explored, which are Liouville integrable, i.e., possess infinitely many symmetries and conservation laws [7,8]. The resulting model equations can relate function values at point (x,t) to its function values at a mirror-reflection point $(-x,-t)$. This is one of the complex types, since it involves the Hermitian transpose, and the other is of real type, since it only uses the matrix transpose. Such studies have opened new avenues for studying mKdV type integrable equations [9,10].

It is known that, based on matrix loop Lie algebras, integrable equations can be generated from matrix spectral problems [11] and their reduced problems [12]. Lax pairs [13] pay a crucial role in the formulation of integrable equations and their solutions to Cauchy problems [11]. The trace identity [14] and the variational identity [15] can be used to establish Hamiltonian structures, which exhibit the Liouville integrability of the underlying model equations. Among the well-known integrable equations associated with simple Lie algebras are the KdV equation, the Ablowitz–Kaup–Newell–Segur system of nonlinear Schrödinger (NLS) equations, and the derivative NLS equation [11,16]. More generally, there are integrable couplings associated with non-semisimple Lie algebras, which bring various types of hereditary recursion operators in block matrix form.

In our construction, we will use the special orthogonal Lie algebra $\mathfrak{g} = \mathrm{so}(3,\mathbb{R})$, presented by all 3×3 trace-free, skew-symmetric real matrices, with a basis:

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

whose structure equations read

$$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2. \tag{2}$$

Obviously, the derived algebra $[\mathfrak{g}, \mathfrak{g}] = [\mathrm{so}(3,\mathbb{R}), \mathrm{so}(3,\mathbb{R})]$ is $\mathfrak{g} = \mathrm{so}(3,\mathbb{R})$ itself. The algebra is one of the only two three-dimensional real Lie algebras with this property. The other one is the special linear algebra $\mathrm{sl}(2,\mathbb{R})$, which is used as a starting point for the study of integrable equations [5]. It is interesting to note that the two complex Lie algebras, $\mathrm{sl}(2,\mathbb{C})$ and $\mathrm{so}(3,\mathbb{C})$, are isomorphic to each other. The corresponding matrix loop algebra that we will use is

$$\tilde{\mathfrak{g}} = \widetilde{\mathrm{so}}(3,\mathbb{R}) = \{M \in \mathrm{so}(3,\mathbb{R}) \,|\, \text{entries of } M \text{ - Laurent series in } \lambda\}, \tag{3}$$

where λ is a spectral parameter. This matrix loop algebra has also been recently used to construct integrable equations [6,17].

In this paper, we would first like to revisit a hierarchy of integrable equations associated with $\mathrm{so}(3,\mathbb{R})$ [6]. We will then construct two pairs of nonlocal PT-symmetric integrable reductions for the adopted spectral matrix, to generate two reduced hierarchies of scalar integrable equations asscoated with $\mathrm{so}(3,\mathbb{R})$. Two typical such reduced nonlocal PT-symmetric integrable equations are the complex reverse-spacetime mKdV equation

$$p_t = -p_{xxx} + \frac{3}{2}p^2 p_x + \frac{3}{2}p_x(p^*(-x,-t))^2,$$

where p^* denotes the complex conjugate of p, and the real reverse-spacetime mKdV equation

$$p_t = -p_{xxx} + \frac{3}{2}p^2 p_x + \frac{3}{2}p_x(p(-x,-t))^2,$$

which are all associated with so(3,$\mathbb{R}$).

2. The Integrable Hierarchy Revisited

2.1. Integrable Hierarchy

We would like to revisit an integrable hierarchy associated with the matrix loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ [6]. Let i be the unit imaginary number. We start from a slightly different spatial matrix spectral problem

$$-i\phi_x = U\phi \ \text{ or } \ \phi_x = iU\phi, \tag{4}$$

with

$$U = U(u,\lambda) = \lambda e_1 + p e_2 + q e_3 = \begin{bmatrix} 0 & -q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}, \tag{5}$$

where λ is a spectral parameter, $u = (p,q)^T$ is a potential and $\phi = (\phi_1, \phi_2, \phi_3)^T$ is a column eigenfunction. We have adopted the spectral matrix iU, involving a constant factor i. This will bring us convenience in finding integrable nonlocal reductions.

As usual, we solve the stationary zero curvature equation

$$W_x = i[U, W] \tag{6}$$

for $W = W(u,\lambda) \in \widetilde{\mathrm{so}}(3,\mathbb{R})$. This equivalently requires

$$a_x = i(pc - qb),\ b_x = i(-\lambda c + qa),\ c_x = i(\lambda b - pa), \tag{7}$$

as long as W is determined as follows:

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix} = \sum_{m\geq 0} W_{0,m}\lambda^{-m}, \tag{8}$$

with

$$W_{0,m} = a_m e_1 + b_m e_2 + c_m e_3 = \begin{bmatrix} 0 & -c_m & -a_m \\ c_m & 0 & -b_m \\ a_m & b_m & 0 \end{bmatrix},\ m \geq 0. \tag{9}$$

Upon taking the initial values

$$a_0 = -1,\ b_0 = c_0 = 0, \tag{10}$$

the system (7) gives

$$b_{m+1} = -ic_{m,x} + pa_m,\ c_{m+1} = ib_{m,x} + qa_m,\ a_{m+1,x} = i(pc_{m+1} - qb_{m+1}),\ m \geq 0, \tag{11}$$

and defines the sequence of $\{a_m, b_m, c_m |\, m \geq 1\}$ uniquely, when taking all constants of integration as zero:

$$a_m|_{u=0} = 0,\ m \geq 1. \tag{12}$$

which implies

$$b_m|_{u=0} = c_m|_{u=0} = 0,\ m \geq 1. \tag{13}$$

The first few sets are as follows:

$$\begin{aligned}
&b_1 = -p,\ c_1 = -q,\ a_1 = 0; \\
&b_2 = iq_x,\ c_2 = -ip_x,\ a_2 = \tfrac{1}{2}(p^2 + q^2); \\
&b_3 = -p_{xx} + \tfrac{1}{2}p^3 + \tfrac{1}{2}pq^2, \\
&c_3 = -q_{xx} + \tfrac{1}{2}p^2 q + \tfrac{1}{2}q^3, \\
&a_3 = i(p_x q - pq_x); \\
&b_4 = i(q_{xxx} - \tfrac{3}{2}p^2 q_x - \tfrac{3}{2}q^2 q_x), \\
&c_4 = i(-p_{xxx} + \tfrac{3}{2}p^2 p_x + \tfrac{3}{2}p_x q^2), \\
&a_4 = pp_{xx} + qq_{xx} - \tfrac{1}{2}p_x^2 - \tfrac{1}{2}q_x^2 - \tfrac{3}{8}(p^2 + q^2)^2.
\end{aligned}$$

Let us then take

$$V^{[m]} = V^{[m]}(u,\lambda) = (\lambda^m W)_+ = \sum_{l=0}^{m} W_{0,l}\lambda^{m-l},\ m \geq 0, \tag{14}$$

to introduce the temporal matrix spectral problems:

$$-i\phi_t = V^{[m]}\phi \ \text{ or } \ \phi_t = iV^{[m]}\phi,\ m \geq 0. \tag{15}$$

It is now direct to see that the compatibility conditions of (4) and (15), i.e., the zero curvature equations

$$U_t - V_x^{[m]} + i[U, V^{[m]}] = 0,\ m \geq 0, \tag{16}$$

generate a hierarchy of integrable equations:

$$u_t = K_m = i\begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} iq \\ -ip \end{bmatrix},\ m \geq 0, \tag{17}$$

where the operator Φ can be determined by the recursion relation (11):

$$\Phi = i\begin{bmatrix} q\partial^{-1}p & -\partial + q\partial^{-1}q \\ \partial - p\partial^{-1}p & -p\partial^{-1}q \end{bmatrix},\ \partial = \frac{\partial}{\partial x}. \tag{18}$$

2.2. Hamiltonian Structure and the Liouville Intergrability

We use the trace identity [14] for our spectral matrix iU:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(W\frac{\partial U}{\partial \lambda})\,dx = \lambda^{-\gamma}\frac{\lambda}{\partial \lambda}\lambda^{\gamma}\,\mathrm{tr}(W\frac{\partial U}{\partial u}), \tag{19}$$

where the constant γ is given by

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|, \tag{20}$$

to construct Hamiltonian structures, which will exhibit the Liouville integrability of the hierarchy (17). We readily see that the corresponding trace identity (19) reads

$$\frac{\delta}{\delta u}\int a\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} b \\ c \end{bmatrix}.$$

Simply applying this, we obtain the following Hamiltonian structures for the hierarchy (17):

$$u_t = K_m = i\begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u},\ m \geq 0, \tag{21}$$

with the Hamiltonian operator and the Hamiltonian functionals

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \mathcal{H}_m = \int (-\frac{ia_{m+2}}{m+1})\,dx,\ m \geq 0. \tag{22}$$

These tell that there exist infinitely many conservation laws for each system in the hierarchy (17), which can often be generated through symbolic computation by computer algebra systems (see, e.g., [18]).

To exhibit its Liouville integrability, let us show that the operator Φ given by (18) is a common hereditary recursion operator for the hierarchy (17).

First, a straightforward calculation can verify that the operator Φ is hereditary (see [19] for definition), that is to say, it satisfies

$$\Phi'(u)[\Phi K]S - \Phi\Phi'(u)[K]S = \Phi'(u)[\Phi S]K - \Phi\Phi'(u)[S]K, \tag{23}$$

and Φ is a recursion operator for $u_{t_0} = K_0$:

$$L_{K_0}\Phi = 0,\ K_0 = i\begin{bmatrix} q \\ -p \end{bmatrix}, \tag{24}$$

where $(L_K\Phi)S = \Phi[K,S] - [K,\Phi S]$, in which K and S are arbitrary vector fields and $[\cdot,\cdot]$ is the Lie bracket of vector fields. Another direct result is that the pair of operators, J and

$$M = \Phi J = i\begin{bmatrix} -\partial + q\partial^{-1}q & -q\partial^{-1}p \\ -p\partial^{-1}q & -\partial + p\partial^{-1}p \end{bmatrix}, \tag{25}$$

constitutes a Hamiltonian pair (see [20] for details). The hereditary property (23) equivalently requires

$$L_{\Phi K}\Phi = \Phi L_K\Phi, \tag{26}$$

where K is an arbitrary vector field. Thus, we have

$$L_{K_m}\Phi = L_{\Phi K_{m-1}}\Phi = \Phi L_{K_{m-1}}\Phi = 0,\ m \geq 1, \tag{27}$$

where the K_m's are given by (17). This implies that the operator Φ defined by (18) is a common hereditary recursion operator for the hierarchy (17) (see also [21] for symbolic computation).

Now, the hierarchy (17) is bi-Hamiltonian (see, e.g., [20,22,23] for details):

$$u_t = K_m = J\frac{\delta\mathcal{H}_m}{\delta u} = M\frac{\delta\mathcal{H}_{m-1}}{\delta u},\ m \geq 1, \tag{28}$$

where J, M and $\mathcal{H}_m$ are defined by (22) and (25). In this way, every member in the hierarchy is Liouville integrable, i.e., it possesses infinitely many commuting symmetries and conservation laws. Particularly, we have the Abelian symmetry algebra:

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0,\ k,l \geq 0, \tag{29}$$

and the Abelian algebras of conserved functionals:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int (\frac{\delta\mathcal{H}_k}{\delta u})^T J\frac{\delta\mathcal{H}_l}{\delta u}\,dx = 0,\ k,l \geq 0, \tag{30}$$

and

$$\{\mathcal{H}_k, \mathcal{H}_l\}_M = \int (\frac{\delta\mathcal{H}_k}{\delta u})^T M\frac{\delta\mathcal{H}_l}{\delta u}\,dx = 0,\ k,l \geq 0. \tag{31}$$

The third-order nonlinear integrable system in the hierarchy (17) is a system of mKdV equations associated with so(3,$\mathbb{R}$):

$$p_t = -p_{xxx} + \frac{3}{2}p^2p_x + \frac{3}{2}p_xq^2,\ q_t = -q_{xxx} + \frac{3}{2}p^2q_x + \frac{3}{2}q^2q_x. \tag{32}$$

Based on (28), it possesses the following bi-Hamiltonian structure

$$u_t = K_3 = J\frac{\delta\mathcal{H}_3}{\delta u} = M\frac{\delta\mathcal{H}_2}{\delta u}, \tag{33}$$

where the Hamiltonian pair $\{J, M\}$ is defined by (22) and (25), and the Hamiltonian functionals, $\mathcal{H}_2$ and $\mathcal{H}_3$, are given by

$$\mathcal{H}_2 = -\frac{i}{3}\int [pp_{xx} + qq_{xx} - \frac{1}{2}p_x^2 - \frac{1}{2}q_x^2 - \frac{3}{8}(p^2+q^2)^2]\,dx, \tag{34}$$

$$\begin{aligned}\mathcal{H}_3 = \frac{1}{4}\int (&p_{xxx}q - pq_{xxx} - p_{xx}q_x + p_xq_{xx} \\ &- \frac{3}{2}p^2p_xq + \frac{3}{2}pq^2q_x + \frac{3}{2}p^3q_x - \frac{3}{2}p_xq^3)\,dx.\end{aligned} \tag{35}$$

The Hamiltonian formulation is extremely important in carrying out the Whitham modulation [24].

We point out that there is the transformation

$$\tilde{p}(x,t) = \frac{1}{2}(-p+iq)(ix,it),\ \tilde{q}(x,t) = \frac{1}{2}(-p-iq)(ix,it), \tag{36}$$

between the system (32) of mKdV equations associated with so(3) and the system of mKdV equations

$$\tilde{p}_t = \tilde{p}_{xxx} + 6\tilde{p}\tilde{q}\tilde{p}_x,\ \tilde{q}_t = \tilde{q}_{xxx} + 6\tilde{p}\tilde{q}\tilde{q}_x, \tag{37}$$

which is associated with sl(2), but there is no nontrivial transformation if we consider only real potentials $p, q, \tilde{p}, \tilde{q}$. This reflects the fact that the two complex Lie algebras, sl(2,$\mathbb{C}$) and so(3,$\mathbb{C}$), are isomorphic, but not so are the two real Lie algebras, sl(2,$\mathbb{R}$) and so(3,$\mathbb{R}$).

3. Integrable Nonlocal Reductions

3.1. Complex Reverse-Spacetime Reductions

Let us first consider a pair of specific complex reverse-spacetime reductions for the spectral matrix:

$$U^\dagger(-x,-t,-\lambda^*) = -CU(x,t,\lambda)C^{-1},\ C = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ \delta & 0 & 0 \end{bmatrix},\ \delta = \pm 1, \tag{38}$$

where $\dagger$ and $*$ stand for the Hermitian transpose and the complex conjugate, respectively. They generate the potential reductions

$$p^*(-x,-t) = -\delta q(x,t),\ \delta = \pm 1. \tag{39}$$

Once assigning these potential reductions, we have the reduction property for W:

$$W^\dagger(-x,-t,-\lambda^*) = CW(x,t,\lambda)C^{-1}, \tag{40}$$

since two matrices on both sides of the above equation solve the stationary zero curvature Equation (6) with the same initial values. This engenders that

$$a^*(-x,-t,-\lambda^*) = a(x,t,\lambda),\ b^*(-x,-t,-\lambda^*) = \delta c(x,t,\lambda), \tag{41}$$

namely,

$$a_m^*(-x,-t) = (-1)^m a_m(x,t),\ b_m^*(-x,-t) = (-1)^m \delta c_m(x,t),\ m \geq 1. \tag{42}$$

Therefore, we obtain

$$(V^{[m]})^\dagger(-x,-t,-\lambda^*) = (-1)^m CV^{[m]}(x,t,\lambda)C^{-1},\ m \geq 1, \tag{43}$$

and then

$$\begin{aligned} &((U_t - V_x^{[2l+1]} + i[U, V^{[2l+1]}])(-x,-t,-\lambda^*))^\dagger \\ =&C(U_t - V_x^{[2l+1]} + i[U, V^{[2l+1]}])(x,t,\lambda)C^{-1},\ l \geq 1. \end{aligned} \tag{44}$$

This implies that the potential reductions given by (39) are compatible with the $(2l+1)$-th zero curvature equation of the integrable hierarchy (17). In this way, one obtains two reduced scalar integrable hierarchies associated with so(3,$\mathbb{R}$):

$$p_t = K_{2l+1,1}|_{q(x,t)=-\delta p^*(-x,-t)},\ l \geq 1, \tag{45}$$

where $K_m = (K_{m,1}, K_{m,2})^T$, $m \geq 1$, are defined by (17). The infinitely many symmetries and conservation laws for the integrable hierarchy (17) will be reduced to infinitely many ones for the above integrable hierarchies in (45), under (39).

With $\delta = 1$, the third-order nonlinear reduced scalar integrable equation is a nonlocal complex reverse-spacetime PT-symmetric mKdV equation associated with so(3,$\mathbb{R}$):

$$p_t = -p_{xxx} + \frac{3}{2}p^2 p_x + \frac{3}{2}p_x(p^*(-x,-t))^2 \tag{46}$$

where p^* denotes the complex conjugate of p. Observe that the two components of K_m, $m \geq 1$, have even and odd properties with respect to p and q. In fact, $K_{2l,1}$, $l \geq 1$, are odd with respect to q and even with respect to p, and $K_{2l+1,1}$, $l \geq 1$, are even with respect to q and odd with respect to p. Similarly, we see that $K_{2l,2}$, $l \geq 1$, are odd with respect to p and even with respect to q, and $K_{2l+1,2}$, $l \geq 1$, are even with respect to p and odd with respect to q. Therefore, the third-order reduced scalar integrable equation with $\delta = -1$ in (45) is exactly the same as the complex nonlocal reverse-space mKdV Equation (46).

3.2. Real Reverse-Spacetime Reductions

Secondly, let us consider a pair of specific real reverse-spacetime reductions for the spectral matrix:

$$U^T(-x,-t,\lambda) = CU(x,t,\lambda)C^{-1},\ C = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ \delta & 0 & 0 \end{bmatrix},\ \delta = \pm 1, \tag{47}$$

where T means taking the matrix transpose. They engender the potential reductions

$$p(-x,-t) = \delta q(x,t),\ \delta = \pm 1. \tag{48}$$

As before, under these potential reductions, W satisfies the following reduction property:

$$W^T(-x,-t,\lambda) = CW(x,t,\lambda)C^{-1}, \tag{49}$$

because two matrices on both sides of the equation solve the stationary zero curvature Equation (6) with the same initial values. Thus, one has

$$a(-x,-t,\lambda) = a(x,t,\lambda),\ b(-x,-t,\lambda) = \delta c(x,t,\lambda), \tag{50}$$

namely,

$$a_m(-x,-t) = a_m(x,t),\ b_m(-x,-t) = \delta c_m(x,t),\ m \geq 1. \tag{51}$$

Then, we arrive at

$$(V^{[m]})^T(-x,-t,\lambda) = CV^{[m]}(x,t,\lambda)C^{-1},\ m \geq 1, \tag{52}$$

and therefore, we obtain

$$\begin{aligned} &((U_t - V_x^{[m]} + i[U,V^{[m]}])(-x,-t,\lambda))^T \\ =& -C(U_t - V_x^{[m]} + i[U,V^{[m]}])(x,t,\lambda)C^{-1},\ m \geq 1. \end{aligned} \tag{53}$$

This guarantees that the potential reductions in (48) are compatible with the zero curvature equations of the integrable hierarchy (17).

In this way, one obtains two reduced scalar integrable hierarchies associated with so(3,$\mathbb{R}$):

$$p_t = K_{m,1}|_{q(x,t)=\delta p(-x,-t)},\ m \geq 1, \tag{54}$$

where $K_m = (K_{m,1}, K_{m,2})^T,\ m \geq 1$, are defined by (17). Moreover, the infinitely many symmetries and conservation laws for the integrable hierarchy (17) are reduced to infinitely many ones for the above integrable hierarchies in (54), under (48).

With $\delta = 1$, the third-order nonlinear reduced scalar integrable equation is a nonlocal real reverse-spacetime PT-symmetric mKdV equation associated with so(3,$\mathbb{R}$):

$$p_t = -p_{xxx} + \frac{3}{2}p^2 p_x + \frac{3}{2}p_x(p(-x,-t))^2. \tag{55}$$

Even and odd properties with respect to p and q in the two components of $K_m,\ m \geq 1$, tell that the third-order nonlinear reduced scalar integrable equation with $\delta = -1$ in (54) is exactly the same as the nonlocal real reverse-spacetime mKdV Equation (55).

4. Concluding Remarks

We have revisited a hierarchy of integrable equations based on zero curvature equations associated with $\mathrm{so}(3,\mathbb{R})$ and presented two pairs of integrable nonlocal PT-symmetric reductions for the hierarchy. Two typical examples among the reduced scalar integrable equations are a nonlocal complex reverse-spacetime modified Korteweg-de Vries (mKdV) equation and a nonlocal real reverse-spacetime mKdV equation, associated with the special orthogonal Lie algebra so(3,$\mathbb{R}$). Every pair of nonlocal reductions leads to the same nonlocal integrable equations. This is a new phenomenon for integrable equations associated with $\mathrm{so}(3,\mathbb{R})$, different from the one for integrable equations associated with $\mathrm{sl}(2,\mathbb{R})$.

Associated with the special orthogonal Lie algebras, there are many interesting questions for integrable equations, both local and nonlocal. Firstly, what kind of general integrable equations could exist? Some interesting structures of local integrable equations associated with $\mathrm{so}(4,\mathbb{R})$ and nonlocal integrable equations associated with $\mathrm{so}(3,\mathbb{R})$ have been explored (see, e.g., [25,26]). Secondly, do there exist N-soliton solutions in such cases (see, e.g., [27] for (1 + 1)-dimensional models and [28–31] for (2 + 1)-dimensional models)? Thirdly, how can we formulate Riemann-Hilbert problems, based on associated matrix spectral problems? The above spectral matrix iU in our formulation with zero potential has three eigenvalues, and this feature brings difficulty in solving related problems. We do not know how to establish Riemann-Hilbert problems for integrable systems associated wth so(3,$\mathbb{R}$). The existing examples of Riemann-Hilbert problems in the literature belong to the class of spectral matrices which possess two eigenvalues when the potential is taken as zero.

It is known that integrable couplings are constructed from zero curvature equations associated with non-semisimple Lie algebras, and their Hamiltonian structures could be established through the variational identity [32]. Bi-integrable couplings and tri-integrable couplings are such examples, and they exhibit insightful thoughts about generic mathematical structures for multi-component integrable equations. Multi-integrable couplings produce abundant examples of recursion operators in block matrix form, indeed. There are rich algebraic and geometric structures related to integrable couplings. However, non-semisimple matrix Lie algebras may not possess any non-degenerate and ad-invariant bilinear forms required in the variational identities, and this causes serious problems in furnishing integrable couplings with Hamiltonian structures. For instance, we do not even know whether there exists any Hamiltonian structure for an interesting perturbation type coupling:

$$u_t = K(u),\ v_t = K'(u)[v],\ w_t = K'(u)[w].$$

Specifically in the KdV case, we have the question whether there is any Hamiltonian structure for the following integrable coupling:

$$u_t = 6uu_x + u_{xxx},\ v_t = 6(uv)_x + v_{xxx},\ w_t = 6(uw)_x + w_{xxx}.$$

We may need to develop a generalized variational identity to explore Hamiltonian structures for integrable couplings in this kind of case.

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References

1. Zabusky, N.J. A Synergetic Approach to Problems of Nonlinear Dispersive Wave Propagation and Interaction. In *Proceedings of the Symposium on Nonlinear Partial Differential Equations*; Ames, A., Ed.; Academic Press: New York, NY, USA, 1967; pp. 223–258.
2. Kakutani, T.; Ono, H. Weak Non-Linear Hydromagnetic Waves in a Cold Collision-Free Plasma. *J. Phys. Soc. Jpn.* **1969**, *26*, 1305–1318. [CrossRef]
3. Miura, R.M. Korteweg-de Vries Equation and Generalizations I: A Remarkable Explicit Nonlinear Transformation. *J. Math. Phys.* **1968**, *9*, 1202–1204. [CrossRef]
4. Wadati, M. The Modified Korteweg-de Vries Equation. *J. Phys. Soc. Jpn.* **1973**, *34*, 1289–1296. [CrossRef]
5. Ablowitz, M.J.; Kaup, D.J.; Newell, A.C.; Segur, H. The Inverse Scattering Transform-Fourier Analysis for Nonlinear Problems. *Stud. Appl. Math.* **1974**, *53*, 249–315. [CrossRef]
6. Ma, W.X. A Soliton Hierarchy Associated with so(3,ℝ). *Appl. Math. Comput.* **2013**, *220*, 117–122.
7. Ablowitz, M.J.; Musslimani, Z.H. Integrable Nonlocal Nonlinear Equations. *Stud. Appl. Math.* **2017**, *139*, 7–59. [CrossRef]
8. Ji, J.L.; Zhu, Z.N. On a Nonlocal Modified Korteweg-de Vries Equation: Integrability, Darboux Transformation and Soliton Solutions. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *42*, 699–708. [CrossRef]
9. Ji, J.L.; Zhu, Z.N. Soliton Solutions of an Integrable Nonlocal Modified Korteweg–de Vries Equation through Inverse Scattering Transform. *J. Math. Anal. Appl.* **2017**, *453*, 973–984. [CrossRef]
10. Ma, W.X. Inverse Scattering and Soliton Solutions of Nonlocal Complex Reverse-spacetime mKdV Equations. *J. Geom. Phys.* **2020**, *157*, 103845. [CrossRef]
11. Ablowitz, M.A.; Clarkson, P.A. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*; Cambridge University Press: Cambridge, UK, 1991.
12. Mikhailov, A.V. The Reduction Problem and the Inverse Scattering Method. *Physica D* **1981**, *3*, 73–117. [CrossRef]
13. Lax, P.D. Integrals of Nonlinear Equations of Evolution and Solitary Waves. *Comm. Pure Appl. Math.* **1968**, *21*, 467–490. [CrossRef]
14. Tu, G.Z. On Liouville Integrability of Zero-curvature Equations and the Yang Hierarchy. *J. Phys. A Math. Gen.* **1989**, *22*, 2375–2392.
15. Ma, W.X.; Chen, M. Hamiltonian and Quasi-Hamiltonian Structures Associated with Semi-direct Sums of Lie Algebras. *J. Phys. A Math. Gen.* **2006**, *39*, 10787–10801. [CrossRef]
16. Kaup, D.J.; Newell, A.C. An Exact Solution for a Derivative Nonlinear Schrödinger Equation. *J. Math. Phys.* **1978**, *19*, 798–801. [CrossRef]
17. Shen, S.F.; Jiang, L.Y.; Jin, Y.Y.; Ma, W.X. New Soliton Hierarchies Associated with the Lie Algebra so(3,ℝ) and Their Bi-Hamiltonian Structures. *Rep. Math. Phys.* **2015**, *75*, 113–133. [CrossRef]
18. Hereman, W.; Adams, P.J.; Eklund, H.; Hickman, M.S.; Herbst, B.M. Direct Methods and Symbolic Software for Conservation Laws of Nonlinear Equations. In *Advances in Nonlinear Waves and Symbolic Computation*; Nova Science Publishers: New York, NY, USA, 2009; pp. 19–78.
19. Fuchssteiner, B. Application of Hereditary Symmetries to Nonlinear Evolution Equations. *Nonlinear Anal.* **1979**, *3*, 849–862. [CrossRef]
20. Magri, F. A Simple Model of the Integrable Hamiltonian Equation. *J. Math. Phys.* **1978**, *19*, 1156–1162. [CrossRef]
21. Baldwin, D.E.; Hereman, W. A Symbolic Algorithm for Computing Recursion Operators of Nonlinear Partial Differential Equations. *Int. J. Comput. Math.* **2010**, *87*, 1094–1119. [CrossRef]
22. Fuchssteiner, B.; Fokas, A.S. Symplectic Structures, Their Bäcklund Transformations and Hereditary Symmetries. *Physica D* **1981**, *4*, 47–66. [CrossRef]
23. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1986.
24. Whitham, G.B. *Linear and Nonlinear Waves*; Pure and Applied Mathematics; Wiley-Interscience: New York, NY, USA; London, UK; Sydney, Australia, 1974.
25. Zhu, S.D.; Shen, S.F.; Jin, Y.Y.; Li, C.X.; Ma, W.X. New Soliton Hierarchies Associated with the Real Lie Algebra so(4,ℝ). *Math. Methods Appl. Sci.* **2017**, *40*, 680–698. [CrossRef]
26. Zhang, L.Q.; Ma, W.X. Nonlocal PT-Symmetric Integrable Equations of Fourth-Order Associated with so(3,ℝ). *Mathematics* **2021**, *9*, 2130. [CrossRef]

27. Ma, W.X. *N*-Soliton Solutions and the Hirota Conditions in (1 + 1)-Dimensions. *Int. J. Nonlinear Sci. Numer. Simul.* **2021**, *22*. [CrossRef]
28. Ma, W.X. *N*-Soliton Solutions and the Hirota Conditions in (2 + 1)-Dimensions. *Opt. Quantum Electron.* **2020**, *52*, 511. [CrossRef]
29. Ma, W.X. *N*-Soliton Solution of a Combined pKP-BKP Equation. *J. Geom. Phys.* **2021**, *165*, 104191. [CrossRef]
30. Ma, W.X.; Yong, X.L.; Lü, X. Soliton Solutions to the B-type Kadomtsev-Petviashvili Equation under General Dispersion Relations. *Wave Motion* **2021**, *103*, 102719. [CrossRef]
31. Ma, W.X. *N*-Soliton Solution and the Hirota Condition of a (2 + 1)-Dimensional Combined Equation. *Math. Comput. Simul.* **2021**, *190*, 270–279. [CrossRef]
32. Sakovich, S.Y. On Integrability of a (2 + 1)-dimensional Perturbed KdV Equation. *J. Nonlinear Math. Phys.* **1998**, *5*, 230–233. [CrossRef]