Studies in Nonlinear Sciences 2 (4): 140-144, 2011
ISSN 2221-3910

Generalized Bilinear Differential Equations

Wen-Xiu Ma

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
mawx@cas.usf.edu

Abstract: We introduce a kind of bilinear differential equations by generalizing Hirota bilinear operators, and explore when the linear superposition principle can apply to the resulting generalized bilinear differential equations. Together with an algorithm using weights, two examples of generalized bilinear differential equations are computed to shed light on the presented general scheme for constructing bilinear differential equations which possess linear subspaces of solutions.

Key words: Bilinear differential equation, Linear superposition principle, Subspace of solutions

INTRODUCTION

It is known that the Hirota bilinear form provides an efficient tool to solve nonlinear differential equations of mathematical physics [1]. Particularly, based on the Hirota bilinear form, soliton solutions can be generated by the Hirota perturbation technique [1] or the multiple exp-function algorithm [2]. It is interesting to note that the linear superposition principle can also apply to Hirota bilinear equations and present their resonant soliton solutions [3]. Such solutions would be dense in the solution set of a function space, appropriately equipped with an inner product, and play an important role in formulating basic approximate solutions to initial value problems.

Many important equations of mathematical physics are rewritten in the Hirota bilinear form through dependent variable transformations [1,4]. For instance, the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

the Boussinesq equation

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0,$$

and the KP equation

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0,$$

which can be expressed as

$$(D_x D_t + D_x^4) f \cdot f = 0$$

under the transformation $u = 2(\ln f)_{xx}$,

$$(D_t^2 + D_x^4) f \cdot f = 0$$

under the transformation $u = 6(\ln f)_{xx}$, and

$$(D_t D_x + D_x^4 + D_y^2) f \cdot f = 0$$

under the transformation $u = 2(\ln f)_{xx}$, respectively. In the above equations, D_x, D_y and D_t are Hirota bilinear operators, and generally, we have

$$D_x^m D_y^n D_t^k f \cdot g = \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big)^m \Big(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\Big)^n \times \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\Big)^k f(x,y,t) g(x',y',t')\big|_{x'=x, y'=y, t'=t},$$

for nonnegative integers m, n and k. Wronskian solutions, including solitons, positons and complexitons [5]-[8], can be presented precisely, on the basis of the Hirota bilinear form.

In this letter, we would like to introduce a kind of generalized bilinear differential operators and explore when the linear superposition principle applies to the corresponding bilinear differential equations. The resulting theory paves a way to construct a new kind of bilinear differential equations which possess linear subspaces of solutions. The considered solutions are linear combinations of exponential traveling wave solutions, and the involved exponential wave solutions may and may not satisfy the corresponding dispersion relations. All the obtained results will exhibit that there are bilinear differential equations different from Hirota bilinear equations, which share some common features with linear differential equations.

The letter is structured as follows. In the 2nd section, we will generalize Hirota bilinear operators and introduce a kind of generalized bilinear differential equations. In the 3rd section, we will analyze the linear superposition principle for exponential traveling waves and establish a criterion for guaranteeing the existence of linear subspaces of exponential traveling wave solutions to generalized bilinear differential equations. In the 4th section, we will present two examples of newly introduced bilinear differential equations, together with an algorithm using weights to compute. In the final section, we will make a few concluding remarks.

BILINEAR DIFFERENTIAL OPERATORS AND EQUATIONS

Let $M, p \in \mathbb{N}$ be given. We introduce a kind of bilinear differential operators:

$$\prod_{i=1}^{M} D_{p,x_i}^{n_i} f \cdot g = \prod_{i=1}^{M} \left(\frac{\partial}{\partial x_i} + \alpha \frac{\partial}{\partial x_i'}\right)^{n_i} f(x) g(x')\big|_{x'=x}, \quad (1)$$

where $x = (x_1, \cdots, x_M)$, $x' = (x_1', \cdots, x_M')$, $n_1, \cdots, n_M$ are arbitrary nonnegative integers, and for an integer m, the mth power of α is defined by

$$\alpha^m = (-1)^{r(m)}, \text{ if } m \equiv r(m) \bmod p \quad (2)$$

with $0 \le r(m) < p$.

For example, if $p = 2k$ ($k \in \mathbb{N}$), all above bilinear differential operators are Hirota bilinear operators, since $D_{2k,x} = D_x$ [1,9]. If $p = 3$, we have

$$\alpha = -1,\ \alpha^2 = \alpha^3 = 1,\ \alpha^4 = -1,\ \alpha^5 = \alpha^6 = 1,\ \cdots,$$

which tells the pattern of symbols

$$-,\ +,\ +,\ -,\ +,\ +,\ \cdots\ (p = 3);$$

and if $p = 5$, we have

$$\alpha = -1,\ \alpha^2 = 1,\ \alpha^3 = -1,\ \alpha^4 = \alpha^5 = 1,$$
$$\alpha^6 = -1,\ \alpha^7 = 1,\ \alpha^8 = -1,\ \alpha^9 = \alpha^{10} = 1,\ \cdots,$$

which tells the pattern of symbols

$$-,\ +,\ -,\ +,\ +,\ -,\ +,\ -,\ +,\ +,\ \cdots\ (p = 5).$$

Note that the pattern of symbols for defining Hirota bilinear operators is

$$-,\ +,\ -,\ +,\ -,\ +,\ \cdots\ (p = 2),$$

and similarly when $p = 7$, the pattern of symbols is

$$-,\ +,\ -,\ +,\ -,\ +,\ +,$$
$$-,\ +,\ -,\ +,\ -,\ +,\ +,\ \cdots\ (p = 7).$$

Following those patterns of symbols, some new bilinear differential operators can be computed:

$$D_{3,x} f \cdot g = f_x g - f g_x,$$
$$D_{3,x} D_{3,t} f \cdot g = f_{xt} g - f_x g_t - f_t g_x + f g_{xt},$$
$$D_{3,x}^3 f \cdot g = f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} + f g_{xxx},$$
$$D_{3,x}^2 D_{3,t} f \cdot g = f_{xxt} g - f_{xx} g_t - 2 f_{xt} g_x + 2 f_x g_{xt} + f_t g_{xx} + f g_{xxt},$$
$$D_{3,x}^3 D_{3,t} f \cdot g = f_{xxxt} g - f_{xxx} g_t - 3 f_{xxt} g_x + 3 f_{xt} g_{xx} + 3 f_{xx} g_{xt} + 3 f_x g_{xxt} + f_t g_{xxx} - f g_{xxxt},$$
$$D_{3,x}^4 f \cdot g = f_{xxxx} g - 4 f_{xxx} g_x + 6 f_{xx} g_{xx} + 4 f_x g_{xxx} - f g_{xxxx},$$
$$D_{3,x}^5 f \cdot g = f_{xxxxx} g - 5 f_{xxxx} g_x + 10 f_{xxx} g_{xx} + 10 f_{xx} g_{xxx} - 5 f_x g_{xxxx} + f g_{xxxxx};$$

and

$$D_{5,x} f \cdot g = f_x g - f g_x,$$
$$D_{5,x} D_{5,t} f \cdot g = f_{xt} g - f_x g_t - f_t g_x + f g_{xt},$$
$$D_{5,x}^3 f \cdot g = f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx},$$
$$D_{5,x}^2 D_{5,t} f \cdot g = f_{xxt} g - f_{xx} g_t - 2 f_{xt} g_x + 2 f_x g_{xt} + f_t g_{xx} - f g_{xxt},$$
$$D_{5,x}^3 D_{5,t} f \cdot g = f_{xxxt} g - f_{xxx} g_t - 3 f_{xxt} g_x + 3 f_{xt} g_{xx} + 3 f_{xx} g_{xt} - 3 f_x g_{xxt} - f_t g_{xxx} + f g_{xxxt},$$
$$D_{5,x}^4 f \cdot g = f_{xxxx} g - 4 f_{xxx} g_x + 6 f_{xx} g_{xx} - 4 f_x g_{xxx} + f g_{xxxx},$$
$$D_{5,x}^5 f \cdot g = f_{xxxxx} g - 5 f_{xxxx} g_x + 10 f_{xxx} g_{xx} - 10 f_{xx} g_{xxx} + 5 f_x g_{xxxx} + f g_{xxxxx}.$$

Immediately from those formulas, we see that except $D_{5,x}^3 f \cdot f = 0$, $D_{3,x}^3 f \cdot f$, $D_{3,x}^5 f \cdot f$ and $D_{5,x}^5 f \cdot f$ do not equal to zero, which is different from the Hirota case: $D_x^3 f \cdot f = D_x^5 f \cdot f = 0$.

Now let P be a polynomial in M variables and introduce a generalized bilinear differential equation:

$$P(D_{p,x_1}, \cdots, D_{p,x_M}) f \cdot f = 0. \quad (3)$$

Particularly, when $p = 3$, we have the generalized bilinear KdV equation

$$(D_{3,x} D_{3,t} + D_{3,x}^4) f \cdot f = 2 f_{xt} f - 2 f_x f_t + 6 f_{xx}{}^2 = 0, \quad (4)$$

the generalized bilinear Boussinesq equation

$$(D_{3,t}^2 + D_{3,x}^4) f \cdot f = 2 f_{tt} f - 2 f_t{}^2 + 6 f_{xx}{}^2 = 0, \quad (5)$$

and the generalized bilinear KP equation

$$(D_{3,t} D_{3,x} + D_{3,x}^4 + D_{3,y}^2) f \cdot f = 2 f_{xt} f - 2 f_x f_t + 6 f_{xx}{}^2 + 2 f_{yy} f - 2 f_y{}^2 = 0. \quad (6)$$

We would like to discuss linear subspaces of solutions to the generalized bilinear differential equations defined by (3). More exactly, as in the Hirota case [3], we want to explore when the linear superposition principle will apply to the generalized bilinear differential equations (3).

LINEAR SUPERPOSITION PRINCIPLE

Let us now fix $N \in \mathbb{N}$ and introduce N wave variables

$$\eta_i = k_{1,i} x_1 + \cdots + k_{M,i} x_M,\ 1 \le i \le N, \quad (7)$$

and N exponential wave functions

$$f_i = e^{\eta_i} = e^{k_{1,i} x_1 + \cdots + k_{M,i} x_M},\ 1 \le i \le N, \quad (8)$$

where the $k_{j,i}$'s are constants. Note that we have a bilinear identity:

$$P(D_{p,x_1},\cdots,D_{p,x_M})\,e^{\eta_i}\cdot e^{\eta_j} = P(k_{1,i}+\alpha k_{1,j},\cdots,k_{M,i}+\alpha k_{M,j})\,e^{\eta_i+\eta_j}, \quad (9)$$

where the powers of α obey the rule (2). Then we consider a linear combination solution to the bilinear differential equation (3):

$$f=\varepsilon_1 f_1+\cdots+\varepsilon_N f_N=\varepsilon_1\,e^{\eta_1}+\cdots+\varepsilon_N\,e^{\eta_N}, \quad (10)$$

where $\varepsilon_i,\ 1\le i\le N$, are arbitrary constants. Thanks to (9), we can compute that

$$\begin{aligned} &P(D_{p,x_1},\cdots,D_{p,x_M})f\cdot f\\ &=\sum_{i,j=1}^{N}\varepsilon_i\varepsilon_j P(D_{p,x_1},\cdots,D_{p,x_M})\,e^{\eta_i}\cdot e^{\eta_j}\\ &=\sum_{i,j=1}^{N}\varepsilon_i\varepsilon_j P(k_{1,i}+\alpha k_{1,j},\cdots,k_{M,i}+\alpha k_{M,j})\,e^{\eta_i+\eta_j}\\ &=\sum_{i=1}^{N}\varepsilon_i^{\,2}\left[P(k_{1,i}+\alpha k_{1,i},\cdots,k_{M,i}+\alpha k_{M,i})\,e^{2\,\eta_i}\right.\\ &\ +\sum_{1\le i<j\le N}\varepsilon_i\varepsilon_j\,[P(k_{1,i}+\alpha k_{1,j},\cdots,k_{M,i}+\alpha k_{M,j})\\ &\ +P(k_{1,j}+\alpha k_{1,i},\cdots,k_{M,j}+\alpha k_{M,i})]\;e^{\eta_i+\eta_j}. \quad (11)\end{aligned}$$

It thus follows that a linear combination function f defined by (10) solves the generalized bilinear differential equation (3) if and only if for $1\le i\le j\le N$, all

$$\begin{aligned}&P(k_{1,i}+\alpha k_{1,j},\cdots,k_{M,i}+\alpha k_{M,j})\\ &+P(k_{1,j}+\alpha k_{1,i},\cdots,k_{M,j}+\alpha k_{M,i})=0 \quad (12)\end{aligned}$$

are satisfied, where the powers of α obey the rule (2). The conditions in (12) present a system of nonlinear algebraic equations on the wave related numbers $k_{i,j}$'s and the coefficients of the polynomial P. Generally, it is not easy to solve (12). But in many cases, such systems have various sets of solutions. In the next section, we will present an idea to solve.

We now summarize our result as follows.

Thm 1. (**Criterion for linear superposition principle**) *Let $P(x_1,\cdots,x_M)$ be a polynomial in the indicated variables and the N wave variables η_i, $1\le i\le N$, be defined by $\eta_i=k_{1,i}x_1+\cdots+k_{M,i}x_M$, $1\le i\le N$, where the $k_{i,j}$'s are all constants. Then any linear combination of the exponential waves e^{η_i}, $1\le i\le N$, solves the generalized bilinear differential equation $P(D_{p,x_1},\cdots,D_{p,x_M})f\cdot f=0$ if and only if the conditions in* (12) *are satisfied.*

This theorem tells us that the linear superposition principle can apply to generalized bilinear differential equations defined by (3). It also paves a way of constructing N-wave solutions to generalized bilinear differential equations. The system (12) is a resonance condition we need to deal with (see [10] for resonance of 2-solitons for Hirota bilinear equations). As soon as we find a solution of the wave related numbers $k_{i,j}$'s to the system (12), we can present an N-wave solution, formed by (10), to the considered generalized bilinear differential equation (3).

APPLICATIONS

Let us now consider how to compute examples of generalized bilinear differential equations, defined by (3), with linear subspaces of solutions, by applying Theorem 1. The problem is how to construct a multivariate polynomial $P(x_1,\cdots x_M)$ such that

$$\begin{aligned}&P(k_{1,1}+\alpha k_{1,2},\cdots,k_{M,1}+\alpha k_{M,2})\\ &+P(k_{1,2}+\alpha k_{1,1},\cdots,k_{M,2}+\alpha k_{M,1})=0, \quad (13)\end{aligned}$$

holds for two sets of constants $k_{1,i},\cdots,k_{M,i}$, $i=1,2$, where the powers of α obey the rule (2). Our basic idea to solve is to introduce weights for the independent variables and then use parameterizations of wave numbers and frequencies (see [11] for the case of Hirota bilinear equations).

Let us first introduce the weights for the independent variables:

$$(w(x_1),\cdots,w(x_M))=(n_1,\cdots,n_M), \quad (14)$$

where each weight $w(x_i)=n_i$ is an integer, and then form a polynomial $P(x_1,\cdots,x_M)$ being homogeneous in some weight. Second, for $i=1,2$, we parameterize the constants $k_{1,i},\cdots,k_{M,i}$, consisting of wave numbers and frequencies, using a free parameter k_i as follows:

$$k_{j,i}=b_j k_i^{\,n_i},\ 1\le j\le M, \quad (15)$$

where the b_j's are constants to be determined. This parameterization balances the degrees of the free parameters in the system (13). Then, plugging the parameterized constants (15) into (13), we collect terms by powers of the parameters k_1 and k_2, and set the coefficient of each power to be zero, to obtain algebraic equations on the constants b_j's and the coefficients of the polynomial P. Finally, solve the resulting algebraic equations to determine the polynomial P and the parameterization.

Now, the resulting solution obviously yields a generalized bilinear differential equation defined by (3) and a linear subspace of its solutions given by

$$f=\sum_{i=1}^{N}\varepsilon_i f_i=\sum_{i=1}^{N}\varepsilon_i\,e^{b_1 k_i^{\,n_1}x_1+\cdots+b_M k_i^{\,n_M}x_M},\ N\ge 1, \quad (16)$$

where the ε_i's and k_i's are arbitrary constants.

In what follows, we present two illustrative examples in 3+1 dimensions, which apply the above parameterization achieved by using one free parameter.

Example 1. Example with positive weights:

Let us introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 2, 3, 4). \quad (17)$$

Then, a general polynomial being homogeneous in weight 5 reads

$$P = c_1x^5 + c_2x^3y + c_3x^2z + c_4xt + c_5yz. \quad (18)$$

Following the parameterization of wave numbers and frequency in (15), the wave variables read

$$\eta_i = k_ix + b_1{k_i}^2y + b_2{k_i}^3z + b_3{k_i}^4t,\ 1 \le i \le N,$$

where k_i, $1 \le i \le N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined. In this example, the corresponding generalized bilinear differential equation reads

$$\begin{aligned} &P(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t})f \cdot f \\ &= 2\,c_1f_{xxxxx}f - 10\,c_1f_{xxxx}f_x + 20\,c_1f_{xxx}f_{xx} \\ &+6\,c_2f_{xx}f_{xy} + 2\,c_3f_{xxz}f + 2\,c_4f_{xt}f \\ &-2\,c_4f_xf_t + 2\,c_5f_{yz}f - 2\,c_5f_yf_z = 0. \end{aligned} \quad (19)$$

The corresponding linear subspace of N-wave solutions is given by

$$f = \sum_{i=1}^{N} \varepsilon_if_i = \sum_{i=1}^{N} \varepsilon_i\, e^{k_ix+b_1{k_i}^2y+b_2{k_i}^3z+b_3{k_i}^4t}, \quad (20)$$

where ε_i, $1 \le i \le N$, are arbitrary constants but b_1, b_2, b_3 need to satisfy

$$\begin{cases} c_5b_1b_2 + c_4b_3 + c_1 + c_3b_2 = 0, \\ -5\,c_1 - c_4b_3 = 0, \\ 10\,c_1 - c_5b_1b_2 + 3\,c_2b_1 = 0. \end{cases} \quad (21)$$

A special solution to (21) is

$$b_1 = -\frac{5\,c_3}{c_5},\ b_2 = -\frac{3\,c_2}{c_5},\ b_3 = -\frac{15\,c_2c_3}{c_4c_5}, \quad (22)$$

when the coefficients of the polynomial P satisfy

$$c_1 = \frac{3\,c_2c_3}{c_5}. \quad (23)$$

If $c_4 = 0$, then $c_1 = 0$, and further a non-trivial (e.g., $b_1b_2 \ne 0$) solution of b_1 and b_2 is given by

$$b_1 = -\frac{c_3}{c_5},\ b_2 = \frac{3\,c_2}{c_5}, \quad (24)$$

but b_3 is arbitrary.

If $c_4 \ne 0$, then

$$b_3 = -\frac{5\,c_1}{c_4}, \quad (25)$$

and further two non-trivial (e.g., $b_1b_2 \ne 0$) solutions of b_1 and b_2 are determined by

$$\begin{cases} c_3b_2 - 4\,c_1 = 0, \\ 3\,c_2b_1 + 10\,c_1 = 0, \end{cases} \quad (26)$$

when $c_5 = 0$, and

$$\begin{cases} c_3c_5{b_2}^2 + (6\,c_1c_5 - 3\,c_2c_3)b_2 + 12\,c_1c_2 = 0, \\ c_5b_1b_2 + c_3b_2 - 4\,c_1 = 0, \end{cases} \quad (27)$$

when $c_5 \ne 0$. The formula (27) provides a large class of generalized bilinear equations which possess the discussed N-wave solutions.

Example 2. Example with positive and negative weights:

Let us introduce the weights of independent variables:

$$(w(x), w(y), w(t)) = (1, -1, 3). \quad (28)$$

Then, a polynomial being homogeneous in weight 2 reads

$$P = c_1x^2 + c_2x^3y + c_3x^4y^2 + c_4yt. \quad (29)$$

Following the parameterization of wave numbers and frequency in (15), the wave variables read

$$\eta_i = k_ix + b_1{k_i}^{-1}y + b_2{k_i}^3t,\ 1 \le i \le N,$$

where k_i, $1 \le i \le N$, are arbitrary constants, but b_1 and b_2 are constants to be determined.

Now, a similar direct computation tells that the corresponding generalized bilinear differential equation

$$\begin{aligned} &(c_1D_{3,x}^2 + c_2D_{3,x}^3D_{3,y} + c_3D_{3,x}^4D_{3,y}^2 \\ &+c_4D_{3,t}D_{3,y})f \cdot f = 0 \end{aligned} \quad (30)$$

possesses the linear subspace of N-wave solutions determined by

$$f = \sum_{i=1}^{N} \varepsilon_if_i = \sum_{i=1}^{N} \varepsilon_i\, e^{k_ix+b_1{k_i}^{-1}y+b_2{k_i}^3t}, \quad (31)$$

where the ε_i's and k_i's are arbitrary, but b_1 and b_2 satisfy

$$\begin{cases} -c_1 + 6\,c_3{b_1}^2 = 0, \\ -c_4b_1b_2 + 4\,c_3{b_1}^2 = 0, \\ c_4b_1b_2 + c_3{b_1}^2 + c_1 + 3\,c_2b_1 = 0. \end{cases} \quad (32)$$

Therefore, the coefficients of the polynomial P need satisfy

$$121\,c_1c_3 = 54\,{c_2}^2, \quad (33)$$

and then the corresponding generalized bilinear differential equation

$$\begin{aligned} &P(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t})f \cdot f \\ &= 2\,c_1f_{xx}f - 2\,c_1{f_x}^2 + 6\,c_2f_{xx}f_{xy} \\ &+2\,c_3f_{xxxxyy}f + 8\,c_3f_{xxx}f_{xyy} \\ &+12\,c_3{f_{xxy}}^2 + 2\,c_4f_{yt}f - 2\,c_4f_yf_t = 0 \end{aligned} \quad (34)$$

has the N-wave solution defined by (31) with

$$b_1 = -\frac{11\,c_1}{18\,c_2},\ b_2 = -\frac{12\,c_2}{11\,c_4}. \tag{35}$$

CONCLUDING REMARKS

We introduced a new kind of bilinear differential operators and analyzed when the corresponding generalized bilinear equations possess the linear superposition principle. Particularly, we computed two examples by an algorithm using weights and their linear subspaces of exponential traveling wave solutions. The balance requirement of weights allows us to present a class of parameterizations of wave numbers and frequencies,

Theorem 1 presents a sufficient and necessary criterion for guaranteeing the applicability of the linear superposition principle to a kind of generalized bilinear differential equations. It also follows from Theorem 1 that if we begin with an arbitrary multivariate polynomial $P(x_1, \cdots, x_M)$, then a sufficient and necessary condition to guarantee the applicability of the discussed linear superposition principle is

$$P(k_{1,i} + \alpha k_{1,j}, \cdots,\ k_{M,i} + \alpha k_{M,j}) + P(k_{1,j} + \alpha k_{1,i}, \cdots,\ k_{M,j} + \alpha k_{M,i}) = 0,$$

where $1 \le i \le j \le N$. This also serves as a starting point for us to search for generalized bilinear differential equations which possess linear subspaces of solutions.

Our results generalize Hirota bilinear operators and establish a bridge between bilinear differential equations and linear differential equations, which exhibits a kind of integrability of nonlinear differential equations [12]. The existence of linear subspaces of solutions amends diversity of exact solutions generated by various analytical methods (see, for example, [13]-[19]). The generalized bilinear operators (1) definitely bring more opportunities to generate non-trivial trilinear differential equations. It is also extremely interesting to explore other mathematical properties that the generalized bilinear differential equations (3) possess and how to construct their non-resonant soliton solutions.

Acknowledgements: The work was supported in part by the Natural Science Foundation of Shanghai (No. 09ZR1410800) and the Shanghai Leading Academic Discipline Project (No. J50101), Zhejiang Innovation Project (Grant No. T200905), the State Administration of Foreign Experts Affairs of China, the National Natural Science Foundation of China (Nos. 10831003, 61072147 and 11071159), and Chunhui Plan of the Ministry of Education of China. I am also grateful to Prof. Huiqun Zhang for her fruitful discussions and valuable suggestions.

REFERENCES

1. Hirota, R., 2004. The direct method in soliton theory. Cambridge University Press, Cambridge.
2. Ma, W.X., Huang, T.W. and Zhang, Y., 2010. A multiple exp-function method for nonlinear differential equations and its application, Phys. Scripta, 82:065003 (8pp).
3. Ma, W.X. and Fan, E.G., 2011. Linear superposition principle applying to Hirota bilinear equations, Comput. Math. Appl., 61:950-959.
4. Hietarinta, J., 2005. Hirota's bilinear method and soliton solutions, Phys. AUC, 15(part 1):31-37.
5. Ma, W.X. and You, Y., 2005. Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions, Trans. Amer. Math. Soc., 357:1753-1778.
6. Ma, W.X., Li, C.X. and He, J.S., 2009. A second Wronskian formulation of the Boussinesq equation, Nonlinear Anal., 70:4245-4258.
7. Yu, G.F. and Hu, X.B., 2009. Extended Gram-type determinant solutions to the Kadomtsev-Petviashvili equation, Math. Comput. Simulation, 80:184-191.
8. Zhang, Y. and Yan, J.J., 2010. Soliton resonance of the NI-BKP equation, In: Nonlinear and modern mathematical physics, ed. W. X. Ma, X. B. Hu and Q. P. Liu. AIP CP 1212, American Institute of Physics, Melville, NY, 231-242.
9. Hirota, R., 1974. A new form of Bäcklund transformations and its relation to the inverse scattering problem, Progr. Theoret. Phys., 52:1498-1512.
10. Hirota, R. and Ito, M., 1983. Resonance of solitons in one dimension, J. Phys. Soc. Jpn., 52:744-748.
11. Ma, W.X., Zhang, Y., Tang, Y.N. and Tu, J.Y., 2012. Hirota bilinear equations with linear subspaces of solutions, Appl. Math. Comput., 218:7174-7183.
12. Ma, W.X., 2005. Integrability, In: Encyclopedia of nonlinear science, ed. A. Scott, Taylor & Francis, New York, 450-453.
13. Ma, W.X. and Fuchssteiner, B., 1996. Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation, Internat. J. Non-Linear Mech., 31:329-338.
14. Ma, W.X., 2003. Diversity of exact solutions to a restricted Boiti-Leon-Pempinelli dispersive long-wave system, Phys. Lett. A, 319:325-333.
15. Mohyud-Din, S.T., Noor, M.A. and Waheed, A., 2009. Variation of parameters method for solving sixth-order boundary value problems, Commun. Korean Math. Soc., 24:605-615.
16. Mohyud-Din, S.T. and Yildirim, A., 2010. Ma's variation of parameters method for Fisher's equations, Adv. Appl. Math. Mech., 2:379-388.
17. Hu, H.C., Tong, B. and Lou, S.Y., 2006. Nonsingular positon and complexiton solutions for the coupled KdV system, Phys. Lett. A, 351:403-412.
18. Tang, Y.N., Xu, W., Gao, L. and Shen, J.W., 2007. An algebraic method with computerized symbolic computation for the one-dimensional generalized BBM equation of any order, Chaos Solitons & Fractals, 32:1846-1852.
19. Zhang, S. and Xia, T.C., 2007. A generalized new auxiliary equation method and its applications to nonlinear partial differential equations, Phys. Lett. A, 363:356-360.