GENERATORS OF VECTOR FIELDS AND TIME DEPENDENT SYMMETRIES OF EVOLUTION EQUATIONS*

MA WEN-XIU (马文秀)
(*Institute of Mathematics, Fudan University, Shanghai 200433, PRC*)

Received April 21, 1990.

ABSTRACT

In this paper, the conception of generators of vector fields with the general characteristic is introduced and the correspondence with time dependent symmetries of evolution equations is provided along with applications to special evolution equations. Furthermore, a theoretical approach for generating time polynomial dependent symmetries of hierarchies of evolution equations is proposed through hereditary symmetries.

Keywords: Lie derivative, hereditary symmetry, generator, time dependent symmetry

I. INTRODUCTION

Symmetries of evolution equations are an important topic in the field of mathematical physics. Many evolution equations posed in physics and further their corresponding hierarchies (such as KdV and AKNS hierarchies) all satisfy one common property that they possess an infinite number of symmetries. It appears that the possession of an infinite number of symmetries is a characterizing property of integrable evolution equations, which have in general soliton solutions and can often be solved via inverse scattering technique (IST). sherefore the studying of symmetries of evolution equations may improve the understanding of algebraic and geometrical properties of integrability of evolution equations.

With the further development of the study, ones have discovered[1-4] that integrable evolution equations possess not only K symmetries (old symmetries) but also τ symmetries (new symmetries mostly time polynomial dependent symmetries of the first order in t). Those K and τ symmetries often constitute a Lie subalgebra of the vector field Lie algebra. Furthermore, the evolution equations which take τ symmetries as their vector fields still possess an infinite number of symmetries and can be solved by IST[5-6]. Hence symmetries prove to be of importance in the study of evolution equations.

This paper presents the conception of generators of vector fields with general characteristics and gives the correspondence with time dependent symmetries, including time polynomial dependent symmetries of evolution equations. Section II main-

* Project supported by the National Natural Science Foundation of China and, in part, by the National Fund of Doctor Programmes of China.

ly discusses some properties of both the adjoint operators determined by elements of vector field Lie algebra $\mathscr{A}^q$ and the generators introduced in this section. Section III proposes a kind of time dependent vector fields and provides an algebraic description so that those vector fields become symmetries of a given evolution equation. The last section, through hereditary symmetries, proposes a theoretical approach for finding higher-degree time polynomial dependent symmetries of hierarchies of evolution equations. The time polynomial dependent symmetries obtained by our approach greatly extend the range of original τ symmetries in literature.

In the following, we explain some basic notions and give some related fundamental results.

Let $x=(x^1,\cdots,x^p)^T$, $u=(u^1,\cdots,u^q)^T$, in which $u^i=u^i(x,t)$, $t\in\mathbb{R}$, $1\leqslant i\leqslant q$. For $\alpha=(\alpha_1,\cdots,\alpha_p)$, $\alpha_i\geqslant 0$, $\alpha_i\in\mathbb{Z}$, $1\leqslant i\leqslant p$, we write $D^\alpha=\left(\frac{d}{dx^1}\right)^{\alpha_1}\cdots\left(\frac{d}{dx^p}\right)^{\alpha_p}$ and $u_\alpha=(u^1_\alpha,\cdots,u^q_\alpha)^T=(D^\alpha u^1,\cdots,D^\alpha u^q)^T$. By $\mathscr{A}$ denote all smooth functions with the form $P=P(x,t,u)=P(x,t,u\cdots,u_\alpha)$. Furthermore let $\mathscr{A}^q=\{(P_1,\cdots,P_q)^T|P_i\in\mathscr{A},\ 1\leqslant i\leqslant q\}$, that is to say that $\mathscr{A}^q$ consists of smooth vector fields defined over the function space to which u belongs.

Definition 1.1. Let $K=K(u)=K(x,t,u)$, $S=S(u)=S(x,t,u)\in\mathscr{A}^q$. The Gateaux derivative of $K(u)$ in the direction $S(u)$ with respect to u is defined as follows:

$$K'[S]=K'(u)[S(u)]=\frac{\partial}{\partial\varepsilon}K(u+\varepsilon S(u))|_{\varepsilon=0}. \tag{1.1}$$

In $\mathscr{A}^q$, we define the binary operation

$$[K,S]=[K(u),S(u)]=K'(u)[S(u)]-S'(u)[K(u)],\ K,S\in\mathscr{A}^q. \tag{1.2}$$

At this moment, $(\mathscr{A}^q,[\cdot,\cdot])$ constitutes a Lie algebra over the complex field indeed. If we define the following matrix differential operator[7]

$$V(G)=\begin{bmatrix}V_1(G_1) & V_2(G_1)\cdots & V_q(G_1)\\ \vdots & \vdots\ \ddots & \vdots\\ V_1(G_q) & V_2(G_q)\cdots & V_q(G_q)\end{bmatrix},\ G=(G_1,\cdots,G_q)^T\in\mathscr{A}^q, \tag{1.3a}$$

with

$$V_i(G_j)=\sum_\alpha\frac{\partial G_j}{\partial u^i_\alpha}D^\alpha,\ 1\leqslant i,\ j\leqslant q, \tag{1.3b}$$

then it is easy to show that the commutator product defined in (1.2) becomes

$$[K,S]=V(K)S-V(S)K,\ K,S\in\mathscr{A}^q. \tag{1.4}$$

We consider the following general evolution equation:

$$u_t=K(x,t,u),\ K=K(x,t,u)\in\mathscr{A}^q. \tag{1.5}$$

Definition 1.2. Let $G=G(x,t,u)\in\mathscr{A}^q$. If the infinitesimal transformation $\bar{u}=u+\varepsilon G(x,t,u)$ leaves the evolution equation (1.5) form-invariant, we call G a symmetry of (1.5).

It is easily proved that $G = G(x,t,u) \in \mathscr{A}^q$ is a symmetry of (1.5) if and only if G satisfies the linearized equation of (1.5)[8]:

$$\frac{dG}{dt} = K'(u)[G], \tag{1.6}$$

where $\frac{d}{dt}$ denotes the total t-derivative and u satisfies the evolution equation (1.5), or equivalently satisfies the following equation[9]:

$$\frac{\partial G}{\partial t} = [K,G], \tag{1.7}$$

where $\frac{\partial}{\partial t}$ denotes the partial t-derivative and $[\cdot,\cdot]$ is defined by (1.2). Suppose that $L(\mathscr{A}^q)$ stands for the space of linear operators from $\mathscr{A}^q$ to $\mathscr{A}^q$. Let $\Phi(x,t,u) = \Phi(x,t,u,\cdots,u_\alpha) \in L(\mathscr{A}^q)$ for any $x, t, u, \cdots, u_\alpha$; by $\mathscr{U}$ (or $\mathscr{U}^q$) denote the space of this kind of Gateaux differentiable operators with respect to the variables $x,t,u,\cdots,u_\alpha$. Throughout this paper, we accept $\Phi K = \Phi(x,t,u)K$ for $\Phi \in \mathscr{U}$, $K \in \mathscr{A}^q$.

Definition 1.3[10,11]. Let $\Phi \in \mathscr{U}$. If the operator Φ satisfies

$$\Phi^2[K,S] + [\Phi K,\Phi S] - \Phi\{[K,\Phi S] + [\Phi K,S]\} = 0, \quad K,S \in \mathscr{A}^q, \tag{1.8}$$

then Φ is called a hereditary symmetry. If the operator Φ maps one symmetry of (1.5) into another symmetry of (1.5), then Φ is called a strong symmetry (or a recursion operator) of (1.5).

Definition 1.4. Let $\Phi \in \mathscr{U}$, $K \in \mathscr{A}^q$. The Lie derivative[12] $L_K\Phi \in \mathscr{U}$ of the operator Φ with respect to K is defined by

$$(L_K\Phi)S = \Phi[K,S] - [K,\Phi S], \quad S \in \mathscr{A}^q.$$

It is not difficult to show that $\Phi \in \mathscr{U}$ is a hereditary symmetry if and only if

$$L_{\Phi K}\Phi = \Phi L_K\Phi, \quad K \in \mathscr{A}^q, \tag{1.9}$$

and that $\Phi = \Phi(x,t,u) \in \mathscr{U}$ is a strong symmetry of (1.5) if and only if

$$\frac{\partial \Phi}{\partial l} + L_K\Phi = 0. \tag{1.10}$$

Besides, if we define the following Gateaux derivative $\Phi'[K]$ of $\Phi \in \mathscr{U}$ in the direction $K \in \mathscr{A}^q$

$$\Phi'[K]S = \frac{\partial}{\partial \varepsilon}\Phi(u + \varepsilon K)S|_{\varepsilon=0}, \quad S \in \mathscr{A}^q,$$

then we can obtain that

$$L_K\Phi = \Phi'[K] - [K',\Phi] = \Phi'[K] - K'\Phi + \Phi K' \tag{1.11}$$

and that (1.8) is equivalent to

$$\Phi'[\Phi K]S - \Phi'[\Phi S]K - \Phi\{\Phi'[K]S - \Phi'[S]K\} = 0,\ K,S \in \mathscr{A}^q. \tag{1.12}$$

Now we list one known result which will be used later on in the proofs.

Lemma 1.1[13,14]. *Let $\Phi = \Phi(x,t,u) \in \mathscr{U}$ be a hereditary symmetry, $K = K(x,t,u) \in \mathscr{A}^q$. If the Lie derivative $L_K\Phi = 0$, then we have*

$$[\Phi^m K, \Phi^n K] = 0,\ m,n \geqslant 0,$$

$$[\Phi^m K, \Phi^n S] = \Phi^n[\Phi^m K, S],\ S \in \mathscr{A}^q,\ m,n \geqslant 0.$$

II. Adjoint Operators and Generators

Let K be a vector field, i.e. $K \in \mathscr{A}^q$. By $\hat{K}$ denote its adjonit operator

$$\hat{K}S = [K,S],\ S \in \mathscr{A}^q. \tag{2.1}$$

Naturally $\hat{K}$ possesses the following properties of the adjoint operators corresponding to elements of general Lie algebras:

(1) The Lebnitz rule:

$$\hat{K}^n[S,T] = \sum_{i=0}^{n} \binom{n}{i} [\hat{K}^i S, \hat{K}^{n-i}T],\ S,T \in \mathscr{A}^q,\ n \geqslant 0, \tag{2.2a}$$

in particular,

$$\hat{K}[S,T] = [\hat{K}S,T] + [S,\hat{K}T],\ S,T \in \mathscr{A}^q. \tag{2.2b}$$

(2) Let $K,G \in \mathscr{A}^q$. If $[K,G] = 0$, then $\hat{K}\hat{G} = \hat{G}\hat{K}$.

Correspondingly to vector fields, we introduce the following operator:

$$\mathrm{pr}K = \sum_{\alpha,j} (D^\alpha K_j) \frac{\partial}{\partial u^j_\alpha},\ K = (K_1,\cdots,K_q)^T \in \mathscr{A}^q, \tag{2.3a}$$

and define

$$\mathrm{dpr}K(S) = (\mathrm{pr}K(S_1),\cdots,\mathrm{pr}K(S_q))^T,\ S = (S_1,\cdots,S_q)^T \in \mathscr{A}^q. \tag{2.3b}$$

Proposition 2.1. *For any K, $S \in \mathscr{A}^q$, we have $V(K)S = \mathrm{dpr}S(K)$, where $V(K)$ is determined by* (1.3).

Proof. Set $K = (K_1,\cdots,K_q)^T$, $S = (S_1,\cdots,S_q)^T$. Then the i-th component of the vector field $V(K)S$ is calculated as follows:

$$(V(K)S)_i = \sum_j V_j(K_i)S_j = \sum_j \sum_\alpha \frac{\partial K_i}{\partial u^j_\alpha} D^\alpha S_j = \mathrm{pr}S(K_i),\ 1 \leqslant i \leqslant q,$$

thus $V(K)S = \mathrm{dpr}S(K)$.

Theorem 2.1. *Let $K,G \in \mathscr{A}^q$, $\Phi \in \mathscr{U}$. Then* (1) $\hat{K} = V(K) - \mathrm{dpr}\,K$ *and thus $\hat{K} \in \mathscr{U}$, where $V(K)$ is defined by* (1.3); (2) $L_G\hat{K} = \hat{K}G$; (3) $L_K\Phi = [\Phi,\hat{K}] = \Phi\hat{K} - \hat{K}\Phi$.

Proof. (1) By Proposition 2.1 and the formula (1.4), it is easy to obtain the desired expression of $\hat{K}$. Thus we also have $\hat{K} \in \mathscr{U}$.

(2) For any $S \in \mathscr{A}^q$, we have

$$(L_G\hat{K})S = \hat{K}[G,S] - [G,\hat{K}S] \stackrel{(2.2b)}{=\!=\!=} [\hat{K}G,S] = \widehat{KGS}.$$

Therefore

$$L_G\hat{K} = \widehat{KG}.$$

(3) The desired result follows directly from the definition of the Lie derivative $L_K\Phi$. The proof of Theorem 2.1 is completed.

Notice that all symmetries of the evolution equation

$$u_t = K(x,t,u) = K(x,t,u,\cdots,u_\alpha), \quad K = K(x,t,u) \in \mathscr{A}^q \tag{2.4}$$

constitute a Lie subalgebra of the vector field Lie algebra ($\mathscr{A}^q$, $[\cdot,\cdot]$). By using (1.10) and (3) of Theorem 2.1, we can obtain the following result.

Proposition 2.2. *Let* $K, G \in \mathscr{A}^q$, $\Phi = \Phi(x, u) \in \mathscr{U}$, *and* G *be a symmetry of* (2.4). *Then* (1) $\hat{G}$ *is a strong symmetry of* (2.4); (2) Φ *is a strong symmetry of* (2.4) *if and only if* $\Phi\hat{K} = \hat{K}\Phi$.

When $\dfrac{\partial K}{\partial t} = 0$, i.e. $K = K(x, u)$, since K is naturally a symmetry of (2.4), by the above proposition we see that $\hat{K}$ is a strong symmetry of (2.4) and commutes with any time independent strong symmetry of (2.4).

In the following we introduce the conception of the generators which can generate time dependent symmetries of evolution equations.

Definition 2.1. Let K, $T \in \mathscr{A}^q$. If there exists a non-negative integer r and a complex constant a so that

$$\check{K}_a^{r+1}T = 0, \text{ in which } \check{K}_a = \hat{K} - aI, \; I: \mathscr{A}^q \to \mathscr{A}^q \text{ is an identity operator,} \tag{2.5}$$

then T is called a K generator of order r with characteristic a.

From the definition we easily know that a non-zero $\check{K}_a^rT$ is the same as an eigenvector of $\hat{K}$ corresponding to an eigenvalue a. Particularly, a non-zero K generator of order 0 with characteristic a is just an eigenvector of $\hat{K}$ corresponding to an eigenvalue a. These show that the condition of existence of K generators is equivalent to that of possession of discrete spectra for $\hat{K}$. Ref. [9] introduced a kind of generators corresponding to those given here with characteristic 0.

From Definition 2.1, we easily come to the following conclusion by Proposition 2.2

Proposition 2.3. *Let* $K = K(x,u)$, $T = T(x,t,u) \in \mathscr{A}^q$ *and* T *be a* K *generator of order* r *with characteristic* 0. *Then* K *is a symmetry of the evolution equation*

$$u_t = G = \hat{K}^rT, \tag{2.6}$$

and thus $\hat{K}$ *is a strong symmetry of* (2.6).

Theorem 2.2. *Let* $K \in \mathscr{A}^q$, $\Phi \in \mathscr{U}$ *and* $L_K\Phi = 0$ *and choose* $T \in \mathscr{A}^q$ *to be a* K *generator of order* r *with characteristic* a. *Then* $\{\Phi^lT \mid l \geqslant 0\}$ *are all* K

generators of order r with characteristic a.

Proof. Since $L_K\Phi = 0$, by (3) of Theorem 2.1 we have $\Phi\hat{K} = \hat{K}\Phi$, thus $\Phi^m\hat{K}^n = \hat{K}^n\Phi^m$, $m, n \geqslant 0$. By this and (2.5) we obtain

$$\check{K}_a^{r+1}\Phi^l T = \Phi^l \check{K}_a^{r+1} T = 0, \ l \geqslant 0.$$

Hence $\{\Phi^l T \mid l \geqslant 0\}$ are all K generators of order r with characteristic a. The proof is completed.

By (1.10), we deduce immediately from the theorem the following result:

Corollary 2.1. *If $\Phi = \Phi(x, u) \in \mathscr{U}$ is a strong symmetry of (2.4) and T is a K generator of order r with characteristic a, then $\{\Phi^l T \mid l \geqslant 0\}$ are all K generators of order r with characteristic a.*

III. Time Dependent Symmetries

This section considers the following time independent evolution equation:

$$u_t = K(x, u), \ K = K(x, u) \in \mathscr{A}^q. \tag{3.1}$$

We shall give a large number of time dependent symmetries of the evolution equation (3.1), the basis of which is the generators introduced in Section II. In this section, we always suppose that k is a natural number, r, r_j ($1 \leqslant j \leqslant k$) are non-negative integers, and $\mathbb{C}$ stands for the complex field.

Theorem 3.1. *Let $T_{ij} = T_{ij}(x, u) \in \mathscr{A}^q$, $0 \leqslant i \leqslant r_j, 1 \leqslant j \leqslant k$, and $a_j \in \mathbb{C}$ ($1 \leq j \leqslant k$) be distinct. Then*

$$\tau = \sum_{j=1}^{k} e^{a_j t} \sum_{i=0}^{r_j} t^i T_{ij} \tag{3.2}$$

is a symmetry of the evolution Eq. (3.1) if and only if for any $1 \leqslant j \leqslant k$, T_{0j} is a K generator of order r_j with characteristic a_j, and

$$T_{ij} = \frac{1}{i!} \check{K}_{a_j}^i T_{0j}, \ 1 \leqslant i \leqslant r_j, \ 1 \leqslant j \leqslant k. \tag{3.3}$$

Proof. By the expression (3.2) of τ we have

$$\begin{aligned} \frac{\partial \tau}{\partial t} &= \sum_{j=1}^{k} a_j e^{a_j t} \sum_{i=0}^{r_j} t^i T_{ij} + \sum_{j=1}^{k} e^{a_j t} \sum_{i=1}^{r_j} i t^{i-1} T_{ij} \\ &= \sum_{j=1}^{k} e^{a_j t} \sum_{i=0}^{r_j - 1} t^i [a_j T_{ij} + (i+1) T_{(i+1),j}] \ \sum_{j=1}^{k} a_j e^{a_j t} t^{r_j} T_{r_j, j} \end{aligned}$$

and

$$[K, \tau] = \sum_{j=1}^{k} e^{a_j t} \sum_{i=0}^{r_j} t^i [K, T_{ij}].$$

Using (1.7), we know that τ is a symmetry of (3.1) if and only if

$$\check{K}_{a_j} T_{r_j, j} = 0, \quad 1 \leqslant j \leqslant k,$$

and

$$T_{i+1,j}=\frac{1}{i+1}\check{K}_{a_j}T_{ij},\quad 0\leqslant i\leqslant r_j-1,\ 1\leqslant j\leqslant k,$$

from which the desired result follows.

When $k=1$, we obtain the following corollary from the above theorem at once.

Corollary 3.1. *Let $T_i=T_i(x,u)\in\mathscr{A}^q$, $0\leqslant i\leqslant r$, $a\in\mathbb{C}$. Then $\tau=e^{at}\sum_{i=0}^{r}t^iT_i$ is a symmetry of (3.1) iff T_0 is a K generator of order r with characteristic a, and $T_i=\frac{1}{i!}\check{K}_a^iT_0$, $1\leqslant i\leqslant r$.*

More particularly, when $a=0$, we arrive at

Corollary 3.2[9]. *Let $T_i=T_i(x,u)\in\mathscr{A}^q$, $0\leqslant i\leqslant r$. Then $\tau=\sum_{i=0}^{r}t^iT_i$ is a symmetry of (3.1) iff T_0 is a K generator of order r with characteristic 0 and $T_i=\frac{1}{i!}\hat{K}^iT_0$, $1\leqslant i\leqslant r$.*

By this corollary, we know that the evolution Eq. (3.1) has and only has the time polynomial dependent symmetries generated by K generaators of any order with characteristic 0.

Lemma 3.1. *Let s be a non-negative integer, $a\in\mathbb{C}$, $T_i=T_i(x,u)\in\mathscr{A}^q$, $-s\leqslant i\leqslant r$, and $T_{-s}\neq 0$. If $\tau=e^{at}\sum_{i=-s}^{r}t^iT_i$ is a symmetry of (3.1), then $s=0$.*

Proof. By a simple calculation, we see that $\frac{\partial\tau}{\partial t}=[K,\tau]$ becomes

$$e^{at}\sum_{i=-s}^{r}at^iT_i+e^{at}\sum_{i=-s-1}^{s-1}(i+1)t^iT_{i+1}=e^{at}\sum_{i=-s}^{r}t^i[K,T_i].\tag{3.4}$$

If $s>0$, comparing the terms containing t^{-s-1}, we can obtain $T_{-s}=0$, which contradicts the assumption of the lemma. Thus the result of the lemma is true.

When $a=0$, the above result shows that time Laurent polynomial dependent symmetries of one time independent evolution equation must be of time polynomial forms.

Lemma 3.2. *Let s be a non-negative integer, l a non-zero integer, $a\in\mathbb{C}$ non-zero, $T_i=T_i(x,u)\in\mathscr{A}^q$, $-s\leqslant i\leqslant r$. If $\tau=e^{at^l}\sum_{i=-s}^{r}t^iT_i$ is a non-zero symmetry of (3.1), then $l=1$.*

Proof. By a direct calculation, we easily obtain that $\frac{\partial\tau}{\partial t}=[K,\tau]$ is equivalent tothe following equality:

$$\sum_{i=-s+l-1}^{r+l-1}alt^iT_{i-l+1}+\sum_{-i=s-1}^{r-1}(i+1)t^iT_{i+1}=\sum_{i=-s}^{r}t^i[K,T_i].\tag{3.5}$$

Let $l>1$. By comparing the coefficients of $r+s$ highest degree terms of t, i.e. t^i $(-s+l-1\leqslant i\leqslant r+l-1)$ in (3.5), it is not difficult to obtain $T_i=0$, $-s\leqslant i\leqslant r$, which contradicts the assumption $\tau\neq 0$. Let $l<-1$. Similarly by comparing the coefficients of $r+s$ lowest degree terms of t, i.e. t^i $(-s+l-1\leqslant i\leqslant r+-1)$ in (3.5), we can obtain that $T_i=0$, $-s\leqslant i\leqslant r$, which also contradicts the assumption of the lemma. Therefore $l=1$, which completes the proof.

Theorem 3.2. *Let s, s' and l, l' be all non-negative integers, $a_{ij}\in\mathbb{C}$, $T_{ij}=T_{ij}(x,u)\in\mathscr{A}^q$, $-s'\leqslant i\leqslant s$, $-l'\leqslant j\leqslant l$. Then for the evolution Eq.* (3.1), *the time dependent symmetry τ with the form*

$$\tau=\sum_{i=-s'}^{s}\sum_{j=-l'}^{l}t^ie^{a_{ij}t^j}T_{ij} \tag{3.6}$$

must possess the form (3.2).

Proof. We rewrite τ in (3.6) as the following form

$$\tau=\sum_{j=1}^{k}\tau_j,\quad \tau_j=e^{a_jt^{l_j}}\sum_{i=-s_j}^{r_j}t^iS_{ij}, \tag{3.7}$$

where l_j, $1\leqslant j\leqslant k$, are integers; $s_j, 1\leqslant j\leqslant k$, denote non-negative integers, $a_j\in\mathbb{C}$, $1\leqslant j\leqslant k$, $S_{ij}=S_{ij}(x,u)\in\mathscr{A}^q$, $-s_j\leqslant i\leqslant r_j$, $1\leqslant j\leqslant k$, and further the condition $(a_j,l_j)\neq(a_{j'},l_{j'})$, $j\neq j'$, may be satisfied. Therefore τ is a symmetry of (3.1) if and only if τ_j, $1\leqslant j\leqslant k$, are all symmetries of (3.1). From this point, by using Lemmas 3.1 and 3.2 we can deduce the result desired in the theorem.

Example 3.1. We consider the evolution equation

$$u_t=\Phi\left(\frac{1}{2}\right)=2u+xu_x,\quad x,t\in\mathbb{R},$$

where

$$\Phi=\partial^2+4u+2u_x\partial^{-1},\quad \partial=\frac{d}{dx}.$$

We can prove that (see [3,14])

$$\left[\Phi^mu_x,\Phi^n\left(\frac{1}{2}\right)\right]=(2m+1)\Phi^{m+n-1}u_x,\ m,n\geqslant 0,\ m+n\geqslant 1, \tag{3.8}$$

$$\left[\Phi^m\left(\frac{1}{2}\right),\ \Phi^n\left(\frac{1}{2}\right)\right]=2(m-n)\Phi^{m+n-1}\left(\frac{1}{2}\right),\quad m,n\geqslant 0. \tag{3.9}$$

When $n=1$, the relations (3.8), (3.9) show that the equation under consideration possesses two hierarchies of generators of order 0: $\{\Phi^mu_x|m\geqslant 0\}$ and $\left\{\Phi^m\left(\frac{1}{2}\right)\middle| m\geqslant 0\right\}$. Thus by Corollary 3.1 we know that $u_t=2u+xu_x$ possesses the following two hierarchies of time dependent symmetries:

$$\tau_m=e^{-(2m+1)t}\Phi^mu_x,\quad m\geqslant 0,$$

$$\tau'_m=e^{-2(m-1)t}\Phi^m\left(\frac{1}{2}\right),\ m\geqslant 0.$$

Example 3.2. We consider the evolution equation $u_t=u_x$, x, $t\in\mathbb{R}$. Set $\Phi=\partial^2+$

$4u+2u_x\partial^{-1}, \partial=\frac{d}{dx}$ and choose $T=\Phi\left(\frac{1}{2}x\right)=2xu+\frac{1}{2}x^2u_x$. It is easy to show that T is a u_x generator of the second order with characteristic 0. Noticing that Φ is a strong symmetry of the evolution equation $u_t=u_x$, we obtain by Corollary 2.1 that $\{\Phi^nT|n\geqslant 0\}$ are all u_x generators of the second order with characteristic 0. Hence according to Corollary 3.2, we obtain a hierarchy of time polynomial dependent symmetries of the second degree of $u_t=u_x$:

$$\tau_n=\frac{1}{2}t^2[u_x,[u_x,\Phi^nT]]+t[u_x,\Phi^nT]+\Phi^nT,\ n\geqslant 0.$$

Further, since

$$[u_x,\Phi^nT]=\Phi^n[u_x,T]=\Phi^n(2u+xu_x)=\Phi^{n+1}\left(\frac{1}{2}\right),\ n\geqslant 0,$$

$$[u_x,[u_x,\Phi^nT]]=\left[u_x,\Phi^{n+1}\left(\frac{1}{2}\right)\right]\overset{(3.8)}{=\!=}\Phi^nu_x,\ n\geqslant 0,$$

we have

$$\tau_n=\frac{1}{2}t^2\Phi^nu_x+t\Phi^{n+1}\left(\frac{1}{2}\right)+\Phi^{n+1}\left(\frac{1}{2}x\right),\ n\geqslant 0.$$

In addition, $\tau_{-1}=\frac{1}{2}t+\frac{1}{2}x$ is also a symmetry of $u_t=u_x$. We point out that $\{\tau_n\}_{n=-1}^{\infty}$ can generate time polynomial dependent symmetries of any degree of $u_t=u_x$. For example, when $m\neq n, [\tau_m,\tau_n]$ is a time polynomial dependent symmetry of the third degree of $u_t=u_x$.

IV. Time Polynomial Dependent Symmetries of Equation Hierarchies

In this section, we consider time polynomial dependent symmetries corresponding to generators with characteristic 0 for equation hierarchies generated by hereditary symmetries.

Definition 4.1[9]. Let $\mathscr{L}_1$ be a Lie subalgebra of $\mathscr{A}^q$. If for any $K\in\mathscr{L}_1$ either (1) $K^{\perp}(\mathscr{L}_1)=\{A\in\mathscr{L}_1|\hat{K}A=0\}$ is Abelian or (2) $\hat{K}S=0$ for any $S\in\mathscr{L}_1$, then $\mathscr{L}_1$ is called a beautiful Lie algebra. An element in $\mathscr{L}_1$ satisfying the condition (2) is said to be trivial.

Henceforth in this section, we always suppose that $\mathscr{L}_1$ is a beautiful Lie subalgebra of $\mathscr{A}^q$.

Theorem 4.1. *Let $K\in\mathscr{L}_1$ be non-trivial, $T\in\mathscr{A}^q$ a K generator of order r with characteristic a and s a natural number. Choose arbitrarily s vector fields $G_i\in K^{\perp}(\mathscr{L}_1)$, $1\leqslant i\leqslant s$. If G_i, $1\leqslant i\leqslant s$, satisfy*

$$\hat{K}^i\hat{G}_1^{j_1}\cdots\hat{G}_s^{j_s}T\in\mathscr{L}_1,\ i,j_1,\cdots,j_s\geqslant 0,\ i+j_1+\cdots+j_s=r, \tag{4.1}$$

then we have

$$\hat{K}^i\hat{G}_1^{j_1}\cdots\hat{G}_s^{j_s}T=0,\ i,j_1,\cdots,j_s\geqslant 0,\ i+j_1+\cdots+j_s=r+1. \tag{4.2}$$

Proof. First noticing that $K \in \mathscr{L}_1$ is a non-trivial element, we see that $K^{\perp}(\mathscr{L}_1)$ is Abelian. And then we begin to prove the theorem by induction for order r. When $r = 0$, by the condition (4.1) we have $T \in \mathscr{L}_1$. Thus $T \in K^{\perp}(\mathscr{L}_1)$ and particularly $\hat{G}_i T = 0$, $1 \leqslant i \leqslant s$, which shows that the equality (4.2) is true for $r = 0$.

Now assume that the result of the theorem is also true for $r = l$. Let $r = l+1$ in the following. For any given $i, j_1, \cdots, j_s \geqslant 0$ satisfying $i + j_1 + \cdots + j_s = l+2$, we want to prove that $\hat{K}^i \hat{G}_1^{j_1} \cdots \hat{G}_s^{j_s} T = 0$.

If $j \geqslant 1$, set $T_1 = \hat{K}T$. This moment T_1 is a K generator of order l with characteristic 0. Notice that $\hat{K}\hat{G}_j = \hat{G}_j \hat{K}$, $1 \leqslant j \leqslant s$ (see the property (2) of adjoint operators in Section II). By (4.1) we have

$$\hat{K}^j \hat{G}_1^{i_1} \cdots \hat{G}_s^{i_s} T_1 = \hat{K}^{j+1} \hat{G}_1^{i_1} \cdots \hat{G}_s^{i_s} T \in \mathscr{L}_1, \ j, i_1, \cdots, i_s \geqslant 0, \ j + i_1 + \cdots + i_s = l.$$

Therefore by the inductive hypothesis, we obtain

$$\hat{K}^{j+1} \hat{G}_1^{i_1} \cdots \hat{G}_s^{i_s} T = \hat{K}^j \hat{G}_1^{i_1} \cdots \hat{G}_s^{i_s} T_1 = 0, \ j, i_1, \cdots, i_s \geqslant 0, \ j + i_1 + \cdots + i_s = l + 1,$$

in particular,

$$\hat{K}^i \hat{G}_1^{j_1} \cdots \hat{G}_s^{j_s} T = 0.$$

If some $j_k \geqslant 1$, let $j_1 \geqslant 1$ without loss of generality. Set $T_1 = \hat{G}_1 T$. Since $\hat{K}^{l+2} T = 0$ and $\hat{K}^{l+1} \in \mathscr{L}_1$ by (4.1), we have $\hat{K}^{l+1} T \in K^{\perp}(\mathscr{L}_1)$. Hence $\hat{K}^{l+1} T_1 = \hat{G}_1 \hat{K}^{l+1} T = 0$ which shows that T_1 is a K generator of order l with characteristic 0. Also, by (4.1) we have

$$\hat{K}^j \hat{G}_1^{i_1} \cdots \hat{G}_s^{i_s} T_1 = \hat{K}^j \hat{G}_1^{i_1+1} \cdots \hat{G}_s^{i_s} T \in \mathscr{L}_1, \ j, i_1, \cdots, i_s \geqslant 0, \ j + i_1 + \cdots + i_s = l.$$

In this case, again by the inductive hypothesis, we have

$$\hat{K}^j \hat{G}_1^{i_1+1} \cdots \hat{G}_s^{i_s} T = \hat{K}^j \hat{G}_1^{i_1} \cdots \hat{G}_s^{i_s} T_1 = 0, \ j, i_1, \cdots, i_s \geqslant 0, \ j + i_1 + \cdots + i_s = l + 1,$$

in particular,

$$\hat{K}^i \hat{G}_1^{j_1} \cdots \hat{G}_s^{j_s} T = 0.$$

Summing up, we see that the result of the theorem is indeed true for $r = l + 1$. Until now, we complete the proof by induction.

Corollary 4.1. *Under the assumption of Theorem* 4.1, $\hat{G}_1^{j_1} \cdots \hat{G}_s^{j_s} T$, $j_1, \cdots, j_s \geqslant 0$, $j_1 + \cdots + j_s \leqslant r$, *are* K *generators of order* $r - (j_1 + \cdots + j_s)$ *with characteristic* 0.

Corollary 4.2. *Let* $K \in \mathscr{L}_1$ *be non-trivial*, $T \in \mathscr{A}^q$ *a* K *generator of order* r *with characteristic* 0, *and* $G \in K^{\perp}(\mathscr{L}_1)$. *If* $\hat{K}^i \hat{G}^j T \in \mathscr{L}_1$, $i, j \geqslant 0$, $i + j = r$, *then* $\hat{K}^{r-s} T$, $0 \leqslant s \leqslant r$, *are* G *generators of order* s *with characteristic* 0.

Theorem 4.2. *Let* $K \in \mathscr{L}_1$ *be non-trivial*, $\Phi \in \mathscr{U}$ *a hereditary symmetry and* $L_K \Phi = 0$, $T \in \mathscr{A}^q$ *a* K *generator of order* r *with characteristic* 0, *and* l *a non-negative integer. If* $G = K_l = \Phi^l K \in \mathscr{L}_1$ *and* $\hat{K}^i \hat{G}^j T \in \mathscr{L}_1$, $i, j \geqslant 0$, $i + j = r$, *then* T *is also a* G *generator of order* r *with characteristic* 0.

Proof. By Lemma 1.1, we have $[\Phi^m K, \Phi^n K] = 0$, $m, n \geqslant 0$. In particular, we have $[K, G] = 0$. Also, we have $G \in \mathscr{L}_1$. Thus $G \in K^{\perp}(\mathscr{L}_1)$. At this moment, by using Corollary 4.2 we can obtain that T is also a G generator of order r with characteristic 0.

Theorem 4.3. *Let $K=K(x,u)\in\mathscr{L}_1$ be non-trivial, $\Phi=\Phi(x,u)\in\mathscr{U}$ a hereditary symmetry and a strong symmtery of the evolution equation $u_t=K(x,u)$, and $T=T(x,u)\in\mathscr{A}^q$ a K generator of order r with characteristic 0. If for some integer $l\geqslant 0$, we have $K_l=\Phi^l K\in\mathscr{L}_1$ and $\hat{K}^i\hat{K}_l^j T\in\mathscr{L}_1$, $i,j\geqslant 0$, $i+j=r$, then the evolution equation*

$$u_t=K_l=\Phi^l K \tag{4.3}$$

possesses a hierarchy of K symmetries and a hierarchy of time polynomial dependent symmetries

$$K_m=\Phi^m K,\ m\geqslant 0, \tag{4.4}$$

$$\tau_{nj}^l=\sum_{i=0}^{r-j}\frac{t^i}{i!}\hat{K}_l^{i+j}\Phi^n T,\ 0\leqslant j\leqslant r,\ n\geqslant 0. \tag{4.5}$$

Furthermore

$$\tau_{nj}^l=\Phi^n\tau_{0j}^l,\ 0\leqslant j\leqslant r,\ n\geqslant 0. \tag{4.6}$$

Proof. Since $\dfrac{\partial\Phi}{\partial t}=0$, we obtain $L_K\Phi=0$ by (1.10). By Lemma 1.1,

$$[\Phi^m K,\Phi^n K]=0,\ m,n\geqslant 0, \tag{4.7}$$

$$[\Phi^m K,\Phi^n S]=\Phi^n[\Phi^m K,S],\ S\in\mathscr{A}^q,\ m,n\geqslant 0. \tag{4.8}$$

The equality (4.7) shows that $\{K_m|m\geqslant 0\}$ are all time independent symmetries of the evolution equation (4.3). In the following, we prove that τ_{nj}^l are also symmetries of (4.3), but are time polynomial dependent. By Theorem 2.2 we see that $\{\Phi^n T|n\geqslant 0\}$ are all K generators of order r with characteristic 0. From this we can deduce, by using Theorem 4.2, that all $\{\Phi^n T|n\geqslant 0\}$ are also K_l generators of order r with characteristic 0. Naturally for $0\leqslant j\leqslant r$, $n\geqslant 0$ $\hat{K}_l^j\Phi^n T$ is a K_l generator of order $r-j$ with characteristic 0. Thus by Corollary 3.2 we obtain that τ_{nj}^l are time polynomial dependent symmetries of degree $r-j$ of (4.3). Besides, (4.6) is a direct corollary of (4.8). The proof is completed.

Noticing (4.7) and (4.8), we can deduce the following result by the above theorem.

Corollary 4.3. *Under the assumption of Theorem 4.3, the evolution equation (4.3) possesses the following time polynomial dependent symmetries:*

$$\tau_n^l(q_1,\cdots,q_s;j_1,\cdots,j_s)=\sum_{i=0}^{r-(j_1+\cdots+j_s)}\frac{t^i}{i!}\hat{K}_l^i\hat{K}_{q_1}^{j_1}\cdots\hat{K}_{q_s}^{j_s}\Phi^n T,$$

where $s\geqslant 1$, $q_1,\cdots,q_s\geqslant 0, j_1,\cdots,j_s\geqslant 0$, $j_1+\cdots j_s\leqslant r$, $n\geqslant 0$, and these symmetries satisfy

$$\tau_n^l(q_1,\cdots,q_s;j_1,\cdots,j_s)=\Phi^n\tau_0^l(q_1,\cdots,q_s;j_1,\cdots,j_s).$$

Theorem 4.3 with $r=1$ gives rise to the following corollary.

Corollary 4.4. *Let $K = K(x, u) \in \mathscr{L}_1$ be non-trivial, $\Phi = \Phi(x, u) \in \mathscr{U}$ be hereditary and $L_K\Phi = 0$, $T = T(x, u) \in \mathscr{A}^q$ and $[K, T] \in K^{\perp}(\mathscr{L}_1)$. If for some integer $l \geqslant 0$, we have $K_l = \Phi^l K \in \mathscr{L}_1$ and $[K_l, T] \in \mathscr{L}_1$, then the evolution equation* (4.3) *possesses symmetries*

$$K_m = \Phi^m K, \quad m \geqslant 0, \tag{4.9}$$

$$\tau^l_{n0} = t[K_l, \Phi^n T] + \Phi^n T, \quad n \geqslant 0, \tag{4.10}$$

$$\tau^l_{n1} = [K_l, \Phi^n T], \quad n \geqslant 0. \tag{4.11}$$

Note that Theorem 4.3, in fact, gives a theoretical method for generating, based-upon hereditary symmetries, time polynomial dependent symmetries of hierarchies of evolution equations. A key point of the method is to find generators with characteristic 0 corresponding to the first equation $u_t = K(x,u)$ in one hierarchy.

Example 4.1. Consider KdV hierarchy

$$u_t = K_l = \Phi^l K = \Phi^{l+1} u_x, \quad x, t \in \mathrm{R}, \quad l \geqslant 0, \tag{4.12a}$$

with

$$\Phi = \partial^2 + 4u + 2u_x\partial^{-1}, \quad K = \Phi u_x = u_{xxx} + 6uu_x. \tag{4.12b}$$

Let $\mathscr{L}_1$ consist of constant coefficient polynomials in $1, u, u_x, \cdots$. It is known that the Lie algebra $\mathscr{L}_1$ is beautiful and $K = u_{xxx} + 6uu_x$ is nontrivial[15]. We easily show that $T = \frac{1}{2}$ is a K generator of the first order with characteristic 0. Noticing (3.8), we can obtain, according to the skeleton of Corollary 4.4, a hierarchy of time polynomial dependent symmetries of (4.12)[2,3]:

$$\tau^l_n = t[K_l, \Phi^n T] + \Phi^n T = (2l+3)tK_{l+n} + \Phi^n\left(\frac{1}{2}\right), \quad n \geqslant 0.$$

Example 4.2. Consider the following hiearachy of evolution equations[16,17]:

$$u_t = \Phi^l u_x, \quad u = (u^1, u^2, u^3)^T, \quad x, t \in \mathrm{R}, \quad l \geqslant 0, \tag{4.13a}$$

where the operator Φ reads as

$$\Phi = \begin{bmatrix} 0 & 0 & -\frac{1}{4}\partial^2 + u^1 + \frac{1}{2}u^1_x\partial^{-1} \\ 1 & 0 & u^2 + \frac{1}{2}u^2_x\partial^{-1} \\ 0 & 1 & u^3 + \frac{1}{2}u^3_x\partial^{-1} \end{bmatrix}. \tag{4.13b}$$

Choosing $T = \left(-\frac{1}{3}u^2, -\frac{2}{3}u^3, 1\right)^T$, we have that $L_T\Phi = \frac{1}{3}$ and $[u_x, \Phi T] = \frac{1}{2}u_x$. Also, it has been proved that Φ is a hereditary symmetry and $L_K\Phi = 0$[16]. Hence by an approach in Ref. [14] we can obtain

$$[\Phi^m u_x, \Phi^n u_x] = 0, \quad m, n \geqslant 0, \tag{4.14}$$

$$[\Phi^m u_x, \Phi^n T] = \left(\frac{1}{3}m + \frac{1}{2}\right)\Phi^{m+n-1}u_x, \quad m,n \geqslant 0, m+n \geqslant 1, \tag{4.15}$$

$$[\Phi^m T, \Phi^n T] = \frac{1}{3}(m-n)\Phi^{m+n-1}T, \quad m,n \geqslant 0. \tag{4.16}$$

From (4.14) and (4.15), we know that $\{\Phi^n T \mid n \geqslant 0\}$ are all K_l $(=\Phi^l u_x,\ l \geqslant 0)$ generators of the first order with characteristic 0. Therefore we can obtain a hierarchy of time polynomial dependent symmetries of (4.13):

$$\tau_n^l = t[K_l, \Phi^n T] + \Phi^n T = \left(\frac{1}{3}l + \frac{1}{2}\right)tK_{l+n-1} + \Phi^n T, \quad K_{-1} = 0, \quad n \geqslant 0.$$

From (4.14—4.16), it is easy to see that this hierarchy of symmetries constitutes one Lie algebra with K symmetries $\{K_m = \Phi^m u_x\}_{m=0}^{\infty}$:

$$[K_m, K_n] = 0, \quad m,n \geqslant 0,$$

$$[K_m, \tau_n^l] = \left(\frac{1}{3}m + \frac{1}{2}\right)K_{m+n-1}, \quad K_{-1} = 0, \quad m,n \geqslant 0,$$

$$[\tau_m^l, \tau_n^l] = \frac{1}{3}(m-n)\tau_{m+n-1}^l, \quad \tau_{-1} = 0, \quad m,n \geqslant 0.$$

For a lot of hierarchies of soliton equations, the generators of the first order with characteristic 0 similar to those in the above examples have been found out[2,18]. This kind of generators is the simplest among the generators to generate time polynomial dependent symmetries of evolution equations. But it is more difficult to search for the higher-order generators with characteristic 0.

The author wishes to express his sincere thanks to Prof. Tu Gui-zhang for enthusiastic encouragement and continuing guidance. The author is also grateful to Meng Da-zhi, Hu Xing-biao and Chen Zhi-xiong for helpful discussions.

References

[1] Olver, P. J., *Math. Proc. Camb. Phil. Soc.*, **88**(1980), 71.
[2] Chen, H. H., Lee, Y. C. & Lin, J. E., in *Advances in Nonlinear Waves* II, *Research Notes in Mathematics*, 111 (Ed. Debnath, L.), Pitman Advanced, Pub., Boston, 1985, pp. 233—239.
[3] 李翊神、朱国城，中国科学(A辑), 1987, 3: 235—241.
[4] Li, Y. S. & Zhu, G. C., *J. Phys. A: Math. Gen.*, **19**(1986), 3713.
[5] 李翊神、程 艺，中国科学(A辑), 1988, 1: 1.
[6] Cheng, Y. & Li, Y. S., *J. Phys. A: Math. Gen.*, **20**(1987), 1951.
[7] Tu, G. Z., *J. Math. Anal. Appl.*, **94**(1983), 348.
[8] Magri, F., *J. Math. Phys.* **19**(1978), 1156.
[9] Fuchssteiner, B., *Progr. Theoret. Phys.*, **70**(1983), 1508.
[10] Fuchssteiner, B., *Nonlinear Anal. Theor. Math. Appl.*, **3**(1979), 849.
[11] Fuchssteiner, B., *Progr. Theoret. Phys.*, **65**(1981), 861.
[12] Magri, F., in *Nonlinear Evolution Equations and Dynamical Systems*, *Lecture Notes in Physics*, Vol. 120 (Eds. Boiti, M., Pempinelli, F. & Soliani, G.), Springer-Verlag, Berlin, 1980, pp. 233—263.
[13] Tu, G. Z., *J. Phys. A: Math. Gen.*, **21**(1988), 1951.

[14] Ma, W. X., *J. Phys. A: Math. Gen.*, **23**(1990), 2707.
[15] Olver, P. J., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New-York, 1986, Chapter 5.
[16] 马文秀，非线性演化方程的广义 Hamilton 结构之研究，硕士论文，中国科学院计算中心，1985.
[17] Antonowicz, M. & Fordy, A. P., *Physica*, D, **28**(1987), 345.
[18] 李翊神、程 艺，1+1 维可积系统中的对称、其代数结构及守恒量（即将发表于偏微分方程杂志）.