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Trigonal curves and algebro-geometric solutions to soliton hierarchies II

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This is a continuation of a study on Riemann theta function representations of algebro-geometric solutions to soliton hierarchies. In this part, we straighten out all flows in soliton hierarchies under the Abel–Jacobi coordinates associated with Lax pairs, upon determining the Riemann theta function representations of the Baker–Akhiezer functions, and generate algebro-geometric solutions to soliton hierarchies in terms of the Riemann theta functions, through observing asymptotic behaviours of the Baker–Akhiezer functions. We emphasize that we analyse the four-component AKNS soliton hierarchy in such a way that it leads to a general theory of trigonal curves applicable to construction of algebro-geometric solutions of an arbitrary soliton hierarchy.

1. Introduction

This is a study on Riemann theta function representations of algebro-geometric solutions to soliton hierarchies. It consists of two parts. In the first part [1], we introduced a class of trigonal curves generated from linear combinations of Lax matrices in the zero curvature formulation, analysed general properties of meromorphic functions defined as ratios of the Baker–Akhiezer functions, including derivative relations

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between derivatives of the characteristic variables with respect to time and space, and determined zeros and poles of the Baker–Akhiezer functions and their Dubrovin-type dynamical equations.

This is the second part, comprising five sections. In §2, we present basic notation and background, introduced and discussed in the first part [1], on the four-component AKNS soliton hierarchy, trigonal curves and the Baker–Akhiezer functions, which will be needed in the subsequent sections of this part. In §3, we explore asymptotic properties for the three Baker–Akhiezer functions in the four-component AKNS case at the points at infinity. In §4, we straighten out all the flows of the four-component AKNS soliton hierarchy under the Abel–Jacobi coordinates, and construct algebro-geometric solutions of the whole soliton hierarchy by use of the Riemann theta functions according to the asymptotic properties of the Baker–Akhiezer functions. In the last section, we present a few concluding remarks and open questions related to lump solitons and soliton hierarchies.

2. Notation and background

(a) Four-component AKNS hierarchy

The four-component AKNS soliton hierarchy is associated with the following 3×3 matrix spectral problem:

$$\psi_x = U\psi = U(u,\lambda)\psi,\quad U=(U_{ij})_{3\times 3}=\begin{bmatrix} -2\lambda & p_1 & p_2\\ q_1 & \lambda & 0\\ q_2 & 0 & \lambda\end{bmatrix},\quad \psi=\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix}, \tag{2.1}$$

where λ is a spectral parameter and u is a four-component potential

$$u=(p,q^{\mathrm{T}})^{\mathrm{T}},\quad p=(p_1,p_2),\quad q=(q_1,q_2)^{\mathrm{T}}. \tag{2.2}$$

As usual, we solve the stationary zero curvature equation $W_x=[U,W]$, corresponding to (2.1), to obtain a formal series solution W:

$$W=\begin{bmatrix} a & b\\ c & d\end{bmatrix}=\sum_{k=0}^{\infty} W_k\lambda^{-k},\quad W_k=W_k(u)=\begin{bmatrix} a^{[k]} & b^{[k]}\\ c^{[k]} & d^{[k]}\end{bmatrix},\quad k\geq 0, \tag{2.3}$$

where $a^{[k]}$ are scalar functions, and $b^{[k]}, c^{[k]}$ are vector functions and $d^{[k]}$ are matrix functions assumed to be represented by

$$b^{[k]}=(b_1^{[k]},b_2^{[k]}),\quad c^{[k]}=(c_1^{[k]},c_2^{[k]})^{\mathrm{T}}\quad\text{and}\quad d^{[k]}=(d_{ij}^{[k]})_{2\times 2},\ k\geq 0. \tag{2.4}$$

All the involved functions above are recursively defined by

$$b^{[0]}=0,\quad c^{[0]}=0,\quad a^{[0]}=-2,\quad d^{[0]}=I_2=\mathrm{diag}(1,1), \tag{2.5a}$$

$$b^{[k+1]}=\tfrac{1}{3}(-b_x^{[k]}+pd^{[k]}-a^{[k]}p),\quad k\geq 0, \tag{2.5b}$$

$$c^{[k+1]}=\tfrac{1}{3}(c_x^{[k]}-qa^{[k]}+d^{[k]}q),\quad k\geq 0 \tag{2.5c}$$

and

$$a_x^{[k]}=pc^{[k]}-b^{[k]}q,\quad d_x^{[k]}=qb^{[k]}-c^{[k]}p,\quad k\geq 1, \tag{2.5d}$$

where we take constants of integration to be zero:

$$W_k|_{u=0}=0,\quad k\geq 1. \tag{2.6}$$

For all integers $r\geq 0$, we have introduced the following Lax matrices:

$$V^{[r]}=V^{[r]}(u,\lambda)=(V_{ij}^{[r]})_{3\times 3}=(\lambda^r W)_+=\sum_{k=0}^{r} W_k\lambda^{r-k},\quad r\geq 0, \tag{2.7}$$

to formulate the temporal spectral problems

$$\psi_{t_r}=V^{[r]}\psi=V^{[r]}(u,\lambda)\psi,\quad r\geq 0. \tag{2.8}$$

The compatibility conditions of (2.1) and (2.8), i.e. the zero curvature equations

$$U_{t_r} - V_x^{[r]} + [U, V^{[r]}] = 0, \quad r \geq 0 \tag{2.9}$$

generate the four-component AKNS soliton hierarchy

$$u_{t_r} = \begin{bmatrix} p^{\mathrm{T}} \\ q \end{bmatrix}_{t_r} = K_r = \begin{bmatrix} -3b^{[r+1]\mathrm{T}} \\ 3c^{[r+1]} \end{bmatrix}, \quad r \geq 0. \tag{2.10}$$

The Lax matrices above have a relation

$$V^{[r+1]} = \sum_{k=0}^{r+1} W_k \lambda^{r-k+1} = \lambda \sum_{k=0}^{r+1} W_k \lambda^{r-k} = \lambda V^{[r]} + W_{r+1}, \quad r \geq 0, \tag{2.11}$$

which allows us to determine asymptotic properties of the Baker–Akhiezer functions recursively in the next section. Obviously, the first two nonlinear systems in the four-component AKNS soliton hierarchy (2.10) read

$$p_{i,t_2} = -\tfrac{1}{3}[p_{i,xx} - 2(p_1q_1 + p_2q_2)p_i] \quad \text{and} \quad q_{i,t_2} = \tfrac{1}{3}[q_{i,xx} - 2(p_1q_1 + p_2q_2)q_i], \quad 1 \leq i \leq 2, \tag{2.12}$$

$$p_{i,t_3} = \tfrac{1}{9}[p_{i,xxx} - 3(p_1q_1 + p_2q_2)p_{i,x} - 3(p_{1,x}q_1 + p_{2,x}q_2)p_i], \quad 1 \leq i \leq 2 \tag{2.13a}$$

and

$$q_{i,t_3} = \tfrac{1}{9}[q_{i,xxx} - 3(p_1q_1 + p_2q_2)q_{i,x} - 3(p_1q_{1,x} + p_2q_{2,x})q_i], \quad 1 \leq i \leq 2, \tag{2.13b}$$

which are the four-component versions of the AKNS systems of nonlinear Schrödinger equations and modified Korteweg–de Vries equations, respectively.

A bi-Hamiltonian formulation of the four-component AKNS equations (2.10) is determined by

$$u_{t_r} = K_r = JG_r = J\frac{\delta \tilde{H}_{r+1}}{\delta u} = M\frac{\delta \tilde{H}_r}{\delta u}, \quad r \geq 1, \tag{2.14}$$

where the Hamiltonian functions are defined by

$$\tilde{H}_k = \frac{1}{k}\int (2a^{[k+1]} - d_{11}^{[k+1]} - d_{22}^{[k+1]})\,\mathrm{d}x, \quad k \geq 1 \tag{2.15}$$

and the Hamiltonian operators J and M, forming a Hamiltonian pair, by

$$J = \begin{bmatrix} 0 & -3I_2 \\ 3I_2 & 0 \end{bmatrix} \tag{2.16a}$$

and

$$M = \begin{bmatrix} p^{\mathrm{T}}\partial^{-1}p + (p^{\mathrm{T}}\partial^{-1}p)^{\mathrm{T}} & \left(\partial - \sum_{i=1}^{2} p_i\partial^{-1}q_i\right) I_2 - p^{\mathrm{T}}\partial^{-1}q^{\mathrm{T}} \\ \left(\partial - \sum_{i=1}^{2} p_i\partial^{-1}q_i\right) I_2 - q\partial^{-1}p & q\partial^{-1}q^{\mathrm{T}} + (q\partial^{-1}q^{\mathrm{T}})^{\mathrm{T}} \end{bmatrix}. \tag{2.16b}$$

(b) Riemann surfaces and Baker–Akhiezer functions

For each integer $n \geq 1$, we have taken a linear combination of the Lax matrices

$$W^{[n]} = W^{[n]}(u, \lambda) = (W_{ij}^{[n]})_{3\times 3} = \sum_{k=0}^{n} \alpha_k V^{[n-k]}, \tag{2.17}$$

where the Lax matrices $V^{[k]}$, $0 \leq k \leq n$, are given by (2.7) and α_k, $0 \leq k \leq n$, are arbitrary constants but $\alpha_0 \neq 0$, to introduce a trigonal curve $\mathcal{K}_g$ of degree $m = 3n$ as follows:

$$\mathcal{K}_g = \{P = (\lambda, y) \in \mathbb{C}^2 \mid \det(yI_3 - W^{[n]}) = 0\}. \tag{2.18}$$

The compactified Riemann surface, still denoted by $\mathcal{K}_g$, consists of points satisfying $\mathcal{F}_m(\lambda, y) = 0$ and the three points at infinity: $\{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$ (see [1] for details).

We have also introduced the vector of associated Baker–Akhiezer functions $\psi(P,x,x_0,t_r,t_{0,r})$ through

$$\psi_x(P,x,x_0,t_r,t_{0,r}) = U(u(x,t_r),\lambda(P))\psi(P,x,x_0,t_r,t_{0,r}), \tag{2.19}$$

$$\psi_{t_r}(P,x,x_0,t_r,t_{0,r}) = V^{[r]}(u(x,t_r),\lambda(P))\psi(P,x,x_0,t_r,t_{0,r}), \tag{2.20}$$

$$W^{[n]}(u(x,t_r),\lambda(P))\psi(P,x,x_0,t_r,t_{0,r}) = y(P)\psi(P,x,x_0,t_r,t_{0,r}) \tag{2.21}$$

and

$$\psi_i(P,x_0,x_0,t_{0,r},t_{0,r}) = 1, \quad 1 \le i \le 3, \tag{2.22}$$

where $x, t_r, x_0, t_{0,r}, \lambda(P), y(P) \in \mathbb{C}$ and $P = (\lambda, y) \in \mathcal{K}_g \backslash \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$. Associated with the Baker–Akhiezer functions, a set of meromorphic functions are defined by

$$\phi_{ij} = \phi_{ij}(P,x,x_0,t_r,t_{0,r}) = \frac{\psi_i(P,x,x_0,t_r,t_{0,r})}{\psi_j(P,x,x_0,t_r,t_{0,r})}, \quad 1 \le i,j \le 3. \tag{2.23}$$

The property (2.21) leads to

$$\phi_{ij} = \frac{yW_{ik}^{[n]} + C_{ij}^{[m]}}{yW_{jk}^{[n]} + A_{ij}^{[m]}} = \frac{F_{ij}^{[m]}}{y^2W_{ik}^{[n]} - yC_{ij}^{[m]} + D_{ij}^{[m]}} = \frac{y^2W_{jk}^{[n]} - yA_{ij}^{[m]} + B_{ij}^{[m]}}{E_{ij}^{[m]}}, \tag{2.24}$$

with

$$A_{ij}^{[m]} = W_{ji}^{[n]}W_{ik}^{[n]} - W_{jk}^{[n]}W_{ii}^{[n]}, \tag{2.25}$$

$$B_{ij}^{[m]} = W_{jk}^{[n]}(W_{jj}^{[n]}W_{kk}^{[n]} - W_{jk}^{[n]}W_{kj}^{[n]}) + W_{ji}^{[n]}(W_{jj}^{[n]}W_{ik}^{[n]} - W_{jk}^{[n]}W_{ij}^{[n]}), \tag{2.26}$$

$$C_{ij}^{[m]} = A_{ji}^{[m]}, \; D_{ij}^{[m]} = B_{ji}^{[m]}, \tag{2.27}$$

$$E_{ij}^{[m]} = (W_{jk}^{[n]})^2W_{ki}^{[n]} + W_{ji}^{[n]}W_{jk}^{[n]}(W_{ii}^{[n]} - W_{kk}^{[n]}) - (W_{ji}^{[n]})^2W_{ik}^{[n]} \tag{2.28}$$

and

$$F_{ij}^{[m]} = E_{ji}^{[m]}, \tag{2.29}$$

where $\{i,j,k\} = \{1,2,3\}$. We know from Lemma 3.1 in [1] that the meromorphic functions ϕ_{ij}, $1 \le i,j \le 3$, defined above, satisfy the following Riccati-type equations

$$\phi_{ij,x} = (U_{ii} - U_{jj})\phi_{ij} + U_{ij} + U_{ik}\phi_{kj} - U_{ji}\phi_{ij}^2 - U_{jk}\phi_{ij}\phi_{kj} \tag{2.30}$$

and

$$\phi_{ij,t_r} = (V_{ii}^{[r]} - V_{jj}^{[r]})\phi_{ij} + V_{ij}^{[r]} + V_{ik}^{[r]}\phi_{kj} - V_{ji}^{[r]}\phi_{ij}^2 - V_{jk}^{[r]}\phi_{ij}\phi_{kj}, \tag{2.31}$$

where $\{i,j,k\} = \{1,2,3\}$.

To deal with asymptotic properties of the Baker–Akhiezer functions ψ_i, $1 \le i \le 3$, we have set

$$J_r^{(i)} = U_{i1}\phi_{1i} + U_{i2}\phi_{2i} + U_{i3}\phi_{3i} \quad \text{and} \quad I_r^{(i)} = V_{i1}^{[r]}\phi_{1i} + V_{i2}^{[r]}\phi_{2i} + V_{i3}^{[r]}\phi_{3i}, \quad 1 \le i \le 3. \tag{2.32}$$

Obviously, the properties (2.19) and (2.20) lead to

$$\frac{\psi_{i,x}(P,x,x_0,t_r,t_{0,r})}{\psi_i(P,x,x_0,t_r,t_{0,r})} = J_r^{(i)}(P,x,t_r), \quad 1 \le i \le 3 \tag{2.33}$$

and

$$\frac{\psi_{i,t_r}(P,x,x_0,t_r,t_{0,r})}{\psi_i(P,x,x_0,t_r,t_{0,r})} = I_r^{(i)}(P,x,t_r), \quad 1 \le i \le 3, \tag{2.34}$$

respectively. Then, we have the basic conservation laws

$$(I_r^{(i)})_x = \left(\frac{\psi_{i,t_r}}{\psi_i}\right)_x = \left(\frac{\psi_{i,x}}{\psi_i}\right)_{t_r} = (J_r^{(i)})_{t_r}, \quad 1 \le i \le 3, \tag{2.35}$$

from which infinitely many conservation laws can be generated by observing Laurent series of the conserved quantities $J_r^{(i)}$, $1 \le i \le 3$, and the conserved fluxes $I_r^{(i)}$, $1 \le i \le 3$, at $\lambda = \infty$ (or $\zeta = \lambda^{-1} = 0$).

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Finally, based on the basic conservation laws in (2.35), we know that (2.33) and (2.34) imply the expressions for the Baker–Akhiezer functions ψ_i, $1 \le i \le 3$:

$$\psi_i(P, x, x_0, t_r, t_{0,r}) = \exp\left(\int_{x_0}^{x} J_r^{(i)}(P, x', t_r)\,\mathrm{d}x' + \int_{t_{0,r}}^{t_r} I_r^{(i)}(P, x_0, t')\,\mathrm{d}t'\right), \quad 1 \le i \le 3. \tag{2.36}$$

3. Asymptotic behaviours

In order to generate algebro-geometric solutions in terms of the Riemann theta functions, we need to explore asymptotic properties of the three Baker–Akhiezer functions ψ_i, $1 \le i \le 3$, at the three points at infinity.

(a) Asymptotics of the first Baker–Akhiezer function

We first start with determining asymptotic properties of the meromorphic functions ϕ_{21} and ϕ_{31} at the points at infinity.

Lemma 3.1. *Let $u = (p_1, p_2, q_1, q_2)^{\mathrm{T}}$ satisfy the rth four-component AKNS equations (2.10) and $\zeta = \lambda^{-1}$. Suppose that $P \in \mathcal{K}_g \backslash \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$ and $(x, t_r) \in \mathbb{C}^2$. Then*

$$\phi_{21}(P, x, t_r) \underset{\zeta\to 0}{=} \begin{cases} \dfrac{3}{p_1}\zeta^{-1} + \dfrac{p_{1,x} - p_1p_2\chi_{1,0}}{p_1^2} + \kappa_{1,1}\zeta + O(\zeta^2), & \text{as } P \to P_{\infty_1}, \\ \kappa_{2,0} + \kappa_{2,1}\zeta + O(\zeta^2), & \text{as } P \to P_{\infty_2}, \\ -\dfrac{q_1}{3}\zeta - \dfrac{q_{1,x}}{9}\zeta^2 - \dfrac{q_{1,xx} - p_1q_1^2 - p_2q_1q_2}{27}\zeta^3 + O(\zeta^4), & \text{as } P \to P_{\infty_3} \end{cases} \tag{3.1}$$

and

$$\phi_{31}(P, x, t_r) \underset{\zeta\to 0}{=} \begin{cases} \chi_{1,0} + \chi_{1,1}\zeta + O(\zeta^2), & \text{as } P \to P_{\infty_1}, \\ \dfrac{3}{p_2}\zeta^{-1} + \dfrac{p_{2,x} - p_1p_2\kappa_{2,0}}{p_2^2} + \chi_{2,1}\zeta + O(\zeta^2), & \text{as } P \to P_{\infty_2}, \\ -\dfrac{q_2}{3}\zeta - \dfrac{q_{2,x}}{9}\zeta^2 - \dfrac{q_{2,xx} - p_2q_2^2 - p_1q_1q_2}{27}\zeta^3 + +O(\zeta^4), & \text{as } P \to P_{\infty_3}, \end{cases} \tag{3.2}$$

where

$$(p_1\chi_{1,0})_x = p_1q_2, \quad (p_1\chi_{1,1})_x = -\frac{\chi_{1,0}}{3p_1}(p_1^2p_{2,x}\chi_{1,0} - p_1p_{1,x}p_2 + p_1^3q_1 + p_1^2p_2q_2 - p_1p_{1,xx} + p_{1,x}^2),$$

$$\kappa_{1,1} = \frac{1}{3p_1^3}(p_1^2p_{2,x}\chi_{1,0} - p_1p_{1,x}p_2\chi_{1,0} - 3p_1^2p_2\chi_{1,1} + p_1^3q_1 + p_1^2p_2q_2 - p_1p_{1,xx} + p_{1,x}^2)$$

and

$$(p_2\kappa_{2,0})_x = p_2q_1, \quad (p_2\kappa_{2,1})_x = \frac{\kappa_{2,0}}{3p_2}(p_1p_2p_{2,x}\kappa_{2,0} - p_{1,x}p_2^2 - p_1p_2^2q_1 - p_2^3q_2 + p_2p_{2,xx} - p_{2,x}^2),$$

$$\chi_{2,1} = -\frac{1}{3p_2^3}(p_1p_2p_{2,x}\kappa_{2,0} - p_{1,x}p_2^2\kappa_{2,0} + 3p_1p_2^2\kappa_{2,1} - p_1p_2^2q_1 - p_2^3q_2 + p_2p_{2,xx} - p_{2,x}^2).$$

Proof. We begin with the following three ansatzes:

$$\phi_{21} \underset{\zeta\to 0}{=} \kappa_{1,-1}\zeta^{-1} + \kappa_{1,0} + \kappa_{1,2}\zeta^2 + O(\zeta^3), \quad \phi_{31} \underset{\zeta\to 0}{=} \chi_{1,0} + \chi_{1,1}\zeta + O(\zeta^2), \quad \text{as } P \to P_{\infty_1};$$

$$\phi_{21} \underset{\zeta\to 0}{=} \kappa_{2,0} + \kappa_{2,1}\zeta + O(\zeta^2), \quad \phi_{31} \underset{\zeta\to 0}{=} \chi_{2,-1}\zeta^{-1} + \chi_{2,0} + \chi_{2,1}\zeta + O(\zeta^2), \quad \text{as } P \to P_{\infty_2};$$

and $$\phi_{21} \underset{\zeta\to 0}{=} \kappa_{3,1}\zeta + \kappa_{3,2}\zeta^2 + \kappa_{3,3}\zeta^3 + O(\zeta^4),$$

$$\phi_{31} \underset{\zeta\to 0}{=} \chi_{3,1}\zeta + \chi_{3,2}\zeta^2 + \chi_{3,3}\zeta^3 + O(\zeta^4), \quad \text{as } P \to P_{\infty_3};$$

where the coefficients, κ_{ij} and χ_{ij}, are functions to be determined. Substituting those expansions into the Riccati-type equations (2.30) with $i=2,3$ and $j=1$, i.e.

$$\phi_{21,x}=q_1+3\lambda\phi_{21}-p_1\phi_{21}^2-p_2\phi_{21}\phi_{31} \quad \text{and} \quad \phi_{31,x}=q_2+3\lambda\phi_{31}-p_1\phi_{21}\phi_{31}-p_2\phi_{31}^2 \tag{3.3}$$

and comparing the three lowest powers ζ^i in each resulting equation, where i goes either from -2 to 0, or from -1 to 1, or from 0 to 2, we obtain a set of relations on the coefficient functions $\kappa_{i,j}$ and $\chi_{i,j}$, which yields the asymptotic properties in (3.1) and (3.2). The proof is completed. ■

To determine asymptotic properties of the Baker–Akhiezer function ψ_1 at the points at infinity, we now analyse

$$J_r^{(1)}=U_{11}+U_{12}\phi_{21}+U_{13}\phi_{31}=-2\lambda+p_1\phi_{21}+p_2\phi_{31} \tag{3.4}$$

and

$$I_r^{(1)}=V_{11}^{[r]}+V_{12}^{[r]}\phi_{21}+V_{13}^{[r]}\phi_{31}. \tag{3.5}$$

Lemma 3.2. *Let $u=(p_1,p_2,q_1,q_2)^{\mathrm{T}}$ satisfy the rth four-component AKNS equations (2.10) and $\zeta=\lambda^{-1}$. Suppose that $P\in\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$ and $(x,t_r)\in\mathbb{C}^2$. Then*

$$J_r^{(1)}(P,x,t_r)\underset{\zeta\to0}{=}\begin{cases}\zeta^{-1}+\dfrac{p_{1,x}}{p_1}+O(\zeta), & \text{as } P\to P_{\infty_1},\\[2mm] \zeta^{-1}+\dfrac{p_{2,x}}{p_2}+O(\zeta), & \text{as } P\to P_{\infty_2},\\[2mm] -2\zeta^{-1}+O(\zeta), & \text{as } P\to P_{\infty_3}\end{cases} \tag{3.6}$$

and

$$I_r^{(1)}(P,x,t_r)\underset{\zeta\to0}{=}\begin{cases}\zeta^{-r}+\dfrac{p_{1,t_r}}{p_1}+O(\zeta), & \text{as } P\to P_{\infty_1},\\[2mm] \zeta^{-r}+\dfrac{p_{2,t_r}}{p_2}+O(\zeta), & \text{as } P\to P_{\infty_2},\\[2mm] -2\zeta^{-r}+O(\zeta), & \text{as } P\to P_{\infty_3}.\end{cases} \tag{3.7}$$

Proof. First, based on (3.4), we obtain (3.6) directly from lemma 3.1.

Second, note that the first compatibility condition in (2.35) reads

$$I_{r,x}^{(1)}=\left(\frac{\psi_{1,t_r}}{\psi_1}\right)_x=\left(\frac{\psi_{1,x}}{\psi_1}\right)_{t_r}=J_{r,t_r}^{(1)} \tag{3.8}$$

and that from (2.11), we obtain

$$V_{11}^{[r+1]}=\lambda V_{11}^{[r]}+a^{[r+1]},\quad V_{12}^{[r+1]}=\lambda V_{12}^{[r]}+b_1^{[r+1]}\quad \text{and}\quad V_{13}^{[r+1]}=\lambda V_{13}^{[r]}+b_2^{[r+1]}$$

and thus, we have

$$I_{r+1}^{(1)}=\lambda I_r^{(1)}+a^{[r+1]}+b_1^{[r+1]}\phi_{21}+b_2^{[r+1]}\phi_{31}. \tag{3.9}$$

Now, based on (3.8) and (3.9), we can verify (3.7) from (3.6) by the mathematical induction. The proof is completed. ■

We can then show the asymptotic behaviour of the Baker–Akhiezer function ψ_1 at the points at infinity.

Theorem 3.3. *Let $u=(p_1,p_2,q_1,q_2)^{\mathrm{T}}$ satisfy the rth four-component AKNS equations (2.10) and $\zeta=\lambda^{-1}$. Suppose that $P\in\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$ and $(x,t_r)\in\mathbb{C}^2$. Then*

$$\psi_1(P,x,x_0,t_r,t_{0,r})\underset{\zeta\to0}{=}\begin{cases}\dfrac{p_1(x,t_r)}{p_1(x_0,t_{0,r})}\exp(\zeta^{-1}(x-x_0)+\zeta^{-r}(t_r-t_{0,r})+O(\zeta)), & \text{as } P\to P_{\infty_1},\\[2mm] \dfrac{p_2(x,t_r)}{p_2(x_0,t_{0,r})}\exp(\zeta^{-1}(x-x_0)+\zeta^{-r}(t_r-t_{0,r})+O(\zeta)), & \text{as } P\to P_{\infty_2},\\[2mm] \exp(-2\zeta^{-1}(x-x_0)-2\zeta^{-r}(t_r-t_{0,r})+O(\zeta)), & \text{as } P\to P_{\infty_3}.\end{cases} \tag{3.10}$$

Proof. The first formula in (2.36) on the Baker–Akhiezer function ψ_1 gives

$$\psi_1(P,x,x_0,t_r,t_{0,r}) = \exp\left(\int_{x_0}^{x} J_r^{(1)}(P,x',t_r)\,\mathrm{d}x' + \int_{t_{0,r}}^{t_r} I_r^{(1)}(P,x_0,t')\,\mathrm{d}t'\right),$$

where $J_r^{(1)}$ and $I_r^{(1)}$ are defined by (3.4) and (3.5). Based on lemma 3.2, this expression generates the asymptotic properties of ψ_1 in (3.10). The proof is completed. ∎

(b) Asymptotics of the second Baker–Akhiezer function

We now start with determining asymptotic properties of the meromorphic functions ϕ_{12} and ϕ_{32} at the points at infinity.

Lemma 3.4. *Let $u=(p_1,p_2,q_1,q_2)^{\mathrm{T}}$ satisfy the rth four-component AKNS equations (2.10) and $\zeta=\lambda^{-1}$. Suppose that $P\in\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$ and $(x,t_r)\in\mathbb{C}^2$. Then*

$$\phi_{12}(P,x,t_r) \underset{\zeta\to 0}{=} \begin{cases} \dfrac{p_1}{3}\zeta + \left(\dfrac{p_2}{3}\chi_{1,1} - \dfrac{1}{9}p_{1,x}\right)\zeta^2 + \kappa_{1,3}\zeta^3 + O(\zeta^4), & \text{as } P\to P_{\infty_1},\\ \dfrac{1}{3}p_2\chi_{2,-1} + \kappa_{2,1}\zeta + \kappa_{2,2}\zeta^2 + O(\zeta^3), & \text{as } P\to P_{\infty_2},\\ -\dfrac{3}{q_1}\zeta^{-1} + \dfrac{q_{1,x}}{q_1^2} + \dfrac{q_1q_{1,xx} - q_{1,x}^2 - p_1q_1^3 - p_2q_1^2q_2}{3q_1^3}\zeta + O(\zeta^2), & \text{as } P\to P_{\infty_3} \end{cases} \tag{3.11}$$

and

$$\phi_{32}(P,x,t_r) \underset{\zeta\to 0}{=} \begin{cases} \chi_{1,1}\zeta + \chi_{1,2}\zeta^2 + O(\zeta^3), & \text{as } P\to P_{\infty_1},\\ \chi_{2,-1}\zeta^{-1} + \chi_{2,0} + \chi_{2,1}\zeta + O(\zeta^2), & \text{as } P\to P_{\infty_2},\\ \dfrac{q_2}{q_1} + \dfrac{1}{3}\left(\dfrac{q_2}{q_1}\right)_x\zeta + \dfrac{1}{9}\left[\left(\dfrac{q_2}{q_1}\right)_{xx} + \dfrac{q_{1,x}}{q_1}\left(\dfrac{q_2}{q_1}\right)_x\right]\zeta^2 + O(\zeta^3), & \text{as } P\to P_{\infty_3}, \end{cases} \tag{3.12}$$

where

$$\chi_{1,1,x} = \tfrac{1}{3}p_1q_2,\quad \chi_{1,2,x} = \tfrac{1}{3}(p_2q_2 - p_1q_1)\chi_{1,1} - \tfrac{1}{9}p_{1,x}q_2,$$
$$\kappa_{1,3} = -\tfrac{1}{9}p_{2,x}\chi_{1,1} + \tfrac{1}{3}p_2\chi_{1,2} - \tfrac{1}{27}p_1(p_1q_1 + p_2q_2) + \tfrac{1}{27}p_{1,xx}$$

and

$$\chi_{2,-1,x} = -\tfrac{1}{3}p_2q_1\chi_{2,-1}^2,\quad \kappa_{2,1} = -\tfrac{1}{9}p_{2,x}\chi_{2,-1} + \tfrac{1}{3}p_2\chi_{2,0} + \tfrac{1}{3}p_1,$$
$$\kappa_{2,2} = -\tfrac{1}{27}p_1p_2q_1\chi_{2,-1} - \tfrac{1}{27}p_2^2q_2\chi_{2,-1} + \tfrac{1}{27}p_{2,xx}\chi_{2,-1} - \tfrac{1}{9}p_{2,x}\chi_{2,0} + \tfrac{1}{3}p_2\chi_{2,1} - \tfrac{1}{9}p_{1,x},$$
$$\chi_{2,0,x} + \tfrac{2}{3}p_2q_1\chi_{2,-1}\chi_{2,0} - \tfrac{1}{9}p_{2,x}q_1\chi_{2,-1}^2 + \tfrac{1}{3}(p_1q_1 - p_2q_2)\chi_{2,-1} = 0,$$
$$\chi_{2,1,x} + \tfrac{2}{3}p_2q_1\chi_{2,-1}\chi_{2,1} - \tfrac{1}{27}p_1p_2q_1^2\chi_{2,-1}^2 - \tfrac{1}{27}p_2^2q_1q_2\chi_{2,-1}^2 + \tfrac{1}{27}p_{2,xx}q_1\chi_{2,-1}^2 + \tfrac{1}{3}p_2q_1\chi_{2,-1}^2$$
$$- \tfrac{2}{9}p_{2,x}q_1\chi_{2,-1}\chi_{2,0} + \tfrac{1}{3}p_1q_1\chi_{2,0} - \tfrac{1}{3}p_2q_2\chi_{2,0} + \tfrac{1}{9}p_{2,x}q_2\chi_{2,-1} - \tfrac{1}{9}p_{1,x}q_1\chi_{2,-1} - \tfrac{1}{3}p_1q_2 = 0.$$

Proof. Similarly, we begin with the following three ansatzes:

$$\phi_{12} \underset{\zeta\to 0}{=} \kappa_{1,1}\zeta + \kappa_{1,2}\zeta^2 + \kappa_{1,3}\zeta^3 + O(\zeta^4),$$

$$\phi_{32} \underset{\zeta\to 0}{=} \chi_{1,1}\zeta + \chi_{1,2}\zeta^2 + O(\zeta^3),\quad \text{as } P\to P_{\infty_1};$$

$$\phi_{12} \underset{\zeta\to 0}{=} \kappa_{2,0} + \kappa_{2,1}\zeta + \kappa_{2,2}\zeta^2 + O(\zeta^3),$$

$$\phi_{32} \underset{\zeta\to 0}{=} \chi_{2,-1}\zeta^{-1} + \chi_{2,0} + \chi_{2,1}\zeta + O(\zeta^2), \quad \text{as } P \to P_{\infty_2};$$

$$\phi_{12} \underset{\zeta\to 0}{=} \kappa_{3,-1}\zeta^{-1} + \kappa_{3,0} + \kappa_{3,1}\zeta + O(\zeta^2)$$

and

$$\phi_{32} \underset{\zeta\to 0}{=} \chi_{3,0} + \chi_{3,1}\zeta + \chi_{3,2}\zeta^2 + O(\zeta^3), \quad \text{as } P \to P_{\infty_3};$$

where the coefficients, κ_{ij} and χ_{ij}, are functions to be determined. Substituting those expansions into the Riccati-type equations (2.30) with $i=1,3$ and $j=2$, i.e.

$$\phi_{12,x} = -3\lambda\phi_{12} + p_1 + p_2\phi_{32} - q_1\phi_{12}^2 \quad \text{and} \quad \phi_{32,x} = q_2\phi_{12} - q_1\phi_{12}\phi_{32}, \tag{3.13}$$

and comparing the three lowest powers ζ^i in each resulting equation, where i goes either from -2 to 0, or from -1 to 1, or from 0 to 2, we obtain a set of relations on the coefficient functions $\kappa_{i,j}$ and $\chi_{i,j}$, which leads to the asymptotic properties in (3.11) and (3.12). This proves the lemma. ■

To determine asymptotic properties of the Baker–Akhiezer function ψ_2 at the points at infinity, we now analyse

$$J_r^{(2)} = U_{21}\phi_{12} + U_{22} + U_{23}\phi_{32} = q_1\phi_{12} + \lambda \tag{3.14}$$

and

$$I_r^{(2)} = V_{21}^{[r]}\phi_{12} + V_{22}^{[r]} + V_{23}^{[r]}\phi_{32}. \tag{3.15}$$

Lemma 3.5. *Let* $u = (p_1, p_2, q_1, q_2)^{\mathrm{T}}$ *satisfy the rth four-component AKNS equations (2.10) and* $\zeta = \lambda^{-1}$. *Suppose that* $P \in \mathcal{K}_g \backslash \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$ *and* $(x, t_r) \in \mathbb{C}^2$. *Then*

$$J_r^{(2)}(P, x, t_r) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{-1} + O(\zeta), & \text{as } P \to P_{\infty_1}, \\ \zeta^{-1} + \rho_r^{(2)} + O(\zeta), & \text{as } P \to P_{\infty_2}, \\ -2\zeta^{-1} + \dfrac{q_{1,x}}{q_1} + O(\zeta), & \text{as } P \to P_{\infty_3} \end{cases} \tag{3.16}$$

and

$$I_r^{(2)}(P, x, t_r) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{-r} + O(\zeta), & \text{as } P \to P_{\infty_1}, \\ \zeta^{-r} + \sigma_r^{(2)} + O(\zeta), & \text{as } P \to P_{\infty_2}, \\ -2\zeta^{-r} + \dfrac{q_{1,t_r}}{q_1} + O(\zeta), & \text{as } P \to P_{\infty_3}, \end{cases} \tag{3.17}$$

where $\rho_r^{(2)} = \frac{1}{3}p_2q_1\chi_{2,-1}$ *and* $\sigma_{r,x}^{(2)} = \rho_{r,t_r}^{(2)}$, *with* $\chi_{2,-1}$ *being defined in lemma 3.4.*

Proof. The proof is similar. First, based on (3.14), we obtain (3.16) directly from lemma 3.4. Second, note that the second compatibility condition in (2.35) reads

$$I_{r,x}^{(2)} = \left(\frac{\psi_{2,t_r}}{\psi_2}\right)_x = \left(\frac{\psi_{2,x}}{\psi_2}\right)_{t_r} = J_{r,t_r}^{(2)} \tag{3.18}$$

and that from (2.11), we get

$$V_{21}^{[r+1]} = \lambda V_{21}^{[r]} + c_1^{[r+1]}, \quad V_{22}^{[r+1]} = \lambda V_{22}^{[r]} + d_{11}^{[r+1]} \quad \text{and} \quad V_{23}^{[r+1]} = \lambda V_{23}^{[r]} + d_{12}^{[r+1]}$$

and this leads to

$$I_{r+1}^{(2)} = \lambda I_r^{(2)} + c_1^{[r+1]}\phi_{12} + d_{11}^{[r+1]} + d_{12}^{[r+1]}\phi_{32}. \tag{3.19}$$

Now, based on (3.18) and (3.19), we can prove (3.17) from (3.16) by mathematical induction. This completes the proof. ■

We can then prove the asymptotic behaviour of the Baker–Akhiezer function ψ_2 at the points at infinity as follows.

Theorem 3.6. *Let $u=(p_1,p_2,q_1,q_2)^{\mathrm{T}}$ satisfy the rth four-component AKNS equations (2.10) and $\zeta=\lambda^{-1}$. Suppose that $P\in\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$ and $(x,t_r)\in\mathbb{C}^2$. Then*

$$\psi_2(P,x,x_0,t_r,t_{0,r}) \underset{\zeta\to 0}{=} \begin{cases} \exp(\zeta^{-1}(x-x_0)+\zeta^{-r}(t_r-t_{0,r})+O(\zeta)), & \text{as } P\to P_{\infty_1},\\ \exp\left(\left(\int_{x_0}^{x}\rho_r^{(2)}(P,x',t_r)\,\mathrm{d}x'+\int_{t_{0,r}}^{t_r}\sigma_r^{(2)}(P,x_0,t')\,\mathrm{d}t'\right.\right. & \\ \quad\left.\left.+\zeta^{-1}(x-x_0)+\zeta^{-r}(t_r-t_{0,r})+O(\zeta)\right)\right), & \text{as } P\to P_{\infty_2},\\ \dfrac{q_1(x,t_r)}{q_1(x_0,t_{0,r})}\exp(-2\zeta^{-1}(x-x_0)-2\zeta^{-r}(t_r-t_{0,r})+O(\zeta)), & \text{as } P\to P_{\infty_3}, \end{cases} \tag{3.20}$$

where $\rho_r^{(2)}$ and $\sigma_r^{(2)}$ are defined in lemma 3.5.

Proof. Similarly, the second formula in (2.36) presents

$$\psi_2(P,x,x_0,t_r,t_{0,r})=\exp\left(\left(\int_{x_0}^{x}J_r^{(2)}(P,x',t_r)\,\mathrm{d}x'+\int_{t_{0,r}}^{t_r}I_r^{(2)}(P,x_0,t')\,\mathrm{d}t'\right)\right),$$

where $J_r^{(2)}$ and $I_r^{(2)}$ are given by (3.14) and (3.15). This expression generates the asymptotic properties of the Baker–Akhiezer function ψ_2 in (3.20), based on lemma 3.5. The proof is completed. ∎

(c) Asymptotics of the third Baker–Akhiezer function

We thirdly start with determining asymptotic properties of the meromorphic functions ϕ_{13} and ϕ_{23} at the points at infinity.

Lemma 3.7. *Let $u=(p_1,p_2,q_1,q_2)^{\mathrm{T}}$ satisfy the rth four-component AKNS equations (2.10) and $\zeta=\lambda^{-1}$. Suppose that $P\in\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$ and $(x,t_r)\in\mathbb{C}^2$. Then*

$$\phi_{13}(P,x,t_r) \underset{\zeta\to 0}{=} \begin{cases} \dfrac{1}{3}p_1\chi_{1,-1}+\kappa_{1,1}\zeta+\kappa_{1,2}\zeta^2+O(\zeta^3), & \text{as } P\to P_{\infty_1},\\ \dfrac{p_2}{3}\zeta+\left(\dfrac{p_1}{3}\chi_{2,1}-\dfrac{1}{9}p_{2,x}\right)\zeta^2+\kappa_{2,3}\zeta^3+O(\zeta^4), & \text{as } P\to P_{\infty_2},\\ -\dfrac{3}{q_2}\zeta^{-1}+\dfrac{q_{2,x}}{q_2^2}+\dfrac{q_2q_{2,xx}-q_{2,x}^2-p_2q_2^3-p_1q_1q_2^2}{3q_2^3}\zeta+O(\zeta^2), & \text{as } P\to P_{\infty_3} \end{cases} \tag{3.21}$$

and

$$\phi_{23}(P,x,t_r) \underset{\zeta\to 0}{=} \begin{cases} \chi_{1,-1}\zeta^{-1}+\chi_{1,0}+\chi_{1,1}\zeta+O(\zeta^2), & \text{as } P\to P_{\infty_1},\\ \chi_{2,1}\zeta+\chi_{2,2}\zeta^2+O(\zeta^3), & \text{as } P\to P_{\infty_2},\\ \dfrac{q_1}{q_2}+\dfrac{1}{3}\left(\dfrac{q_1}{q_2}\right)_x\zeta+\dfrac{1}{9}\left[\left(\dfrac{q_1}{q_2}\right)_{xx}+\dfrac{q_{2,x}}{q_2}\left(\dfrac{q_1}{q_2}\right)_x\right]\zeta^2+O(\zeta^3), & \text{as } P\to P_{\infty_3}, \end{cases} \tag{3.22}$$

where

$$\chi_{2,1,x}=\tfrac{1}{3}p_2q_1,\quad \chi_{2,2,x}=\tfrac{1}{3}(p_1q_1-p_2q_2)\chi_{2,1}-\tfrac{1}{9}p_{2,x}q_1,$$
$$\kappa_{2,3}=-\tfrac{1}{9}p_{1,x}\chi_{2,1}+\tfrac{1}{3}p_1\chi_{2,2}-\tfrac{1}{27}p_2(p_1q_1+p_2q_2)+\tfrac{1}{27}p_{2,xx},$$

and

$$\chi_{1,-1,x}=-\tfrac{1}{3}p_1q_2\chi_{1,-1}^2,\quad \kappa_{1,1}=-\tfrac{1}{9}p_{1,x}\chi_{1,-1}+\tfrac{1}{3}p_1\chi_{1,0}+\tfrac{1}{3}p_2,$$
$$\kappa_{1,2}=-\tfrac{1}{27}p_1p_2q_2\chi_{1,-1}-\tfrac{1}{27}p_1^2q_1\chi_{1,-1}+\tfrac{1}{27}p_{1,xx}\chi_{1,-1}-\tfrac{1}{9}p_{1,x}\chi_{1,0}+\tfrac{1}{3}p_1\chi_{1,1}-\tfrac{1}{9}p_{2,x},$$
$$\chi_{1,0,x}+\tfrac{2}{3}p_1q_2\chi_{1,-1}\chi_{1,0}-\tfrac{1}{9}p_{1,x}q_2\chi_{1,-1}^2+\tfrac{1}{3}(p_2q_2-p_1q_1)\chi_{1,-1}=0,$$
$$\chi_{1,1,x}+\tfrac{2}{3}p_1q_2\chi_{1,-1}\chi_{1,1}-\tfrac{1}{27}p_1p_2q_2^2\chi_{1,-1}^2-\tfrac{1}{27}p_1^2q_1q_2\chi_{1,-1}^2+\tfrac{1}{27}p_{1,xx}q_2\chi_{1,-1}^2+\tfrac{1}{3}p_1q_2\chi_{1,-1}^2$$
$$-\tfrac{2}{9}p_{1,x}q_2\chi_{1,-1}\chi_{1,0}+\tfrac{1}{3}p_2q_2\chi_{1,0}-\tfrac{1}{3}p_1q_1\chi_{1,0}+\tfrac{1}{9}p_{1,x}q_1\chi_{1,-1}-\tfrac{1}{9}p_{2,x}q_2\chi_{1,-1}-\tfrac{1}{3}p_2q_1=0.$$

Proof. Similarly, we begin with the following three ansatzes:

$$\phi_{13}\underset{\zeta\to 0}{=}\kappa_{1,0}+\kappa_{1,1}\zeta+\kappa_{1,2}\zeta^2+O(\zeta^3),$$
$$\phi_{23}\underset{\zeta\to 0}{=}\chi_{1,-1}\zeta^{-1}+\chi_{1,0}+\chi_{1,1}\zeta+O(\zeta^2),\quad \text{as } P\to P_{\infty_1};$$
$$\phi_{13}\underset{\zeta\to 0}{=}\kappa_{2,1}\zeta+\kappa_{2,2}\zeta^2+\kappa_{2,3}\zeta^3+O(\zeta^4),$$
$$\phi_{23}\underset{\zeta\to 0}{=}\chi_{2,1}\zeta+\chi_{2,2}\zeta^2+O(\zeta^3),\quad \text{as } P\to P_{\infty_2};$$

and

$$\phi_{13}\underset{\zeta\to 0}{=}\kappa_{3,-1}\zeta^{-1}+\kappa_{3,0}+\kappa_{3,1}\zeta+O(\zeta^2),$$
$$\phi_{23}\underset{\zeta\to 0}{=}\chi_{3,0}+\chi_{3,1}\zeta+\chi_{3,2}\zeta^2+O(\zeta^3),\quad \text{as } P\to P_{\infty_3};$$

where the coefficients, κ_{ij} and χ_{ij}, are functions to be determined. Substituting those expansions into the Riccati-type equations (2.30) with $i=1,2$ and $j=3$, i.e.

$$\phi_{13,x}=-3\lambda\phi_{13}+p_1\phi_{23}+p_2-q_2\phi_{13}^2\quad\text{and}\quad \phi_{23,x}=q_1\phi_{13}-q_2\phi_{13}\phi_{23},\tag{3.23}$$

and comparing the three lowest powers ζ^i in each resulting equation, where i goes either from -2 to 0, or from -1 to 1, or from 0 to 2, we get a set of relations on the coefficient functions $\kappa_{i,j}$ and $\chi_{i,j}$, which engenders the asymptotic properties in (3.21) and (3.22). The proof is completed. ■

In order to determine asymptotic properties of the Baker–Akhiezer function ψ_3 at the points at infinity, we similarly analyse

$$J_r^{(3)}=U_{31}\phi_{13}+U_{32}\phi_{23}+U_{33}=q_2\phi_{13}+\lambda\tag{3.24}$$

and

$$I_r^{(3)}=V_{31}^{[r]}\phi_{13}+V_{32}^{[r]}\phi_{23}+V_{33}^{[r]}.\tag{3.25}$$

Lemma 3.8. *Let $u=(p_1,p_2,q_1,q_2)^{\mathrm{T}}$ satisfy the rth four-component AKNS equations (2.10) and $\zeta=\lambda^{-1}$. Suppose that $P\in\mathcal{K}_g\backslash\{P_{\infty_1},P_{\infty_2},P_{\infty_3}\}$ and $(x,t_r)\in\mathbb{C}^2$. Then*

$$J_r^{(3)}(P,x,t_r)\underset{\zeta\to 0}{=}\begin{cases}\zeta^{-1}+\rho_r^{(3)}+O(\zeta), & \text{as } P\to P_{\infty_1},\\ \zeta^{-1}+O(\zeta), & \text{as } P\to P_{\infty_2},\\ -2\zeta^{-1}+\dfrac{q_{2,x}}{q_2}+O(\zeta), & \text{as } P\to P_{\infty_3}\end{cases}\tag{3.26}$$

and

$$I_r^{(3)}(P,x,t_r)\underset{\zeta\to 0}{=}\begin{cases}\zeta^{-r}+\sigma_r^{(3)}+O(\zeta), & \text{as } P\to P_{\infty_1},\\ \zeta^{-r}+O(\zeta), & \text{as } P\to P_{\infty_2},\\ -2\zeta^{-r}+\dfrac{q_{2,t_r}}{q_2}+O(\zeta), & \text{as } P\to P_{\infty_3},\end{cases}\tag{3.27}$$

where $\rho_r^{(3)}=\frac{1}{3}p_1q_2\chi_{1,-1}$ and $\sigma_{r,x}^{(3)}=\rho_{r,t_r}^{(3)}$, with $\chi_{1,-1}$ being defined in lemma 3.7.

Proof. Similarly, first based on (3.24), we obtain (3.26) directly from lemma 3.7.

Second, note that the third compatibility condition in (2.35) reads

$$I_{r,x}^{(3)} = \left(\frac{\psi_{3,t_r}}{\psi_3}\right)_x = \left(\frac{\psi_{3,x}}{\psi_3}\right)_{t_r} = J_{r,t_r}^{(3)}, \tag{3.28}$$

and that from (2.11), we obtain

$$V_{31}^{[r+1]} = \lambda V_{31}^{[r]} + c_2^{[r+1]}, \quad V_{32}^{[r+1]} = \lambda V_{32}^{[r]} + d_{21}^{[r+1]} \quad \text{and} \quad V_{33}^{[r+1]} = \lambda V_{33}^{[r]} + d_{22}^{[r+1]}$$

and this tells us

$$I_{r+1}^{(3)} = \lambda I_r^{(3)} + c_2^{[r+1]}\phi_{13} + d_{21}^{[r+1]}\phi_{23} + d_{22}^{[r+1]}. \tag{3.29}$$

Finally, based on (3.28) and (3.29), we can verify (3.27) from (3.26) by mathematical induction. This completes the proof. ∎

We can then show the following asymptotic behaviour of the Baker–Akhiezer function ψ_3 at the points at infinity.

Theorem 3.9. *Let* $u = (p_1, p_2, q_1, q_2)^{\mathrm{T}}$ *satisfy the rth four-component AKNS equations (2.10) and* $\zeta = \lambda^{-1}$. *Suppose that* $P \in \mathcal{K}_g \backslash \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$ *and* $(x, t_r) \in \mathbb{C}^2$. *Then*

$$\psi_3(P, x, x_0, t_r, t_{0,r}) \underset{\zeta\to 0}{=} \begin{cases} \exp\left(\displaystyle\int_{x_0}^{x} \rho_r^{(3)}(P, x', t_r)\,\mathrm{d}x' + \int_{t_{0,r}}^{t_r} \sigma_r^{(3)}(P, x_0, t')\,\mathrm{d}t' \right. \\ \quad \left. + \zeta^{-1}(x - x_0) + \zeta^{-r}(t_r - t_{0,r}) + O(\zeta)\right), & \text{as } P \to P_{\infty_1}, \\ \exp(\zeta^{-1}(x - x_0) + \zeta^{-r}(t_r - t_{0,r}) + O(\zeta)), & \text{as } P \to P_{\infty_2}, \\ \dfrac{q_2(x, t_r)}{q_2(x_0, t_{0,r})} \exp(-2\zeta^{-1}(x - x_0) - 2\zeta^{-r}(t_r - t_{0,r}) + O(\zeta)), & \text{as } P \to P_{\infty_3}, \end{cases} \tag{3.30}$$

where $\rho_r^{(3)}$ *and* $\sigma_r^{(3)}$ *are defined in lemma 3.8.*

Proof. Similarly, the third formula in (2.36) reads

$$\psi_3(P, x, x_0, t_r, t_{0,r}) = \exp\left(\int_{x_0}^{x} J_r^{(3)}(P, x', t_r)\,\mathrm{d}x' + \int_{t_{0,r}}^{t_r} I_r^{(3)}(P, x_0, t')\,\mathrm{d}t'\right),$$

where $J_r^{(3)}$ and $I_r^{(3)}$ are determined by (3.24) and (3.25). Based on lemma 3.8, this expression generates the asymptotic properties of the Baker–Akhiezer function ψ_3 in (3.30). The proof is completed. ∎

Now, note that a meromorphic function on a compact Riemann surface has the same number of zeros and poles. Therefore, in view of lemma 3.1, lemma 3.4 and lemma 3.7, and from the expressions in (2.24) for the meromorphic functions ϕ_{ij}, $1 \le i, j \le 3$, we can assume that their divisors are given by

$$(\phi_{21}(P, x, t_r)) = \mathcal{D}_{P_{\infty_3}, \hat{\nu}_{h_1}(x,t_r), \ldots, \hat{\nu}_g(x,t_r)} - \mathcal{D}_{P_{\infty_1}, \hat{\mu}_{h_1}(x,t_r), \ldots, \hat{\mu}_g(x,t_r)}, \tag{3.31}$$

$$(\phi_{31}(P, x, t_r)) = \mathcal{D}_{P_{\infty_3}, \hat{\xi}_{h_2}(x,t_r), \ldots, \hat{\xi}_g(x,t_r)} - \mathcal{D}_{P_{\infty_2}, \hat{\mu}_{h_2}(x,t_r), \ldots, \hat{\mu}_g(x,t_r)}, \tag{3.32}$$

$$(\phi_{12}(P, x, t_r)) = \mathcal{D}_{P_{\infty_1}, \hat{\mu}_{h_1}(x,t_r), \ldots, \hat{\mu}_g(x,t_r)} - \mathcal{D}_{P_{\infty_3}, \hat{\nu}_{h_1}(x,t_r), \ldots, \hat{\nu}_g(x,t_r)}, \tag{3.33}$$

$$(\phi_{32}(P, x, t_r)) = \mathcal{D}_{P_{\infty_1}, \hat{\xi}_{h_3}(x,t_r), \ldots, \hat{\xi}_g(x,t_r)} - \mathcal{D}_{P_{\infty_2}, \hat{\nu}_{h_3}(x,t_r), \ldots, \hat{\nu}_g(x,t_r)}, \tag{3.34}$$

$$(\phi_{13}(P, x, t_r)) = \mathcal{D}_{P_{\infty_2}, \hat{\mu}_{h_2}(x,t_r), \ldots, \hat{\mu}_g(x,t_r)} - \mathcal{D}_{P_{\infty_3}, \hat{\xi}_{h_2}(x,t_r), \ldots, \hat{\xi}_g(x,t_r)} \tag{3.35}$$

and

$$(\phi_{23}(P, x, t_r)) = \mathcal{D}_{P_{\infty_2}, \hat{\nu}_{h_3}(x,t_r), \ldots, \hat{\nu}_g(x,t_r)} - \mathcal{D}_{P_{\infty_1}, \hat{\xi}_{h_3}(x,t_r), \ldots, \hat{\xi}_g(x,t_r)}, \tag{3.36}$$

for some natural numbers h_i, $1 \le i \le 3$. The case of $h_i > 1$ for some $1 \le i \le 3$ could happen, particularly when $y = -A_{ij}^{[m]}/W_{jk}^{[n]}$, and $E_{ij}^{[m]}$ and $2(A_{ij}^{[m]})^2 + W_{jk}^{[n]} B_{ij}^{[m]}$ have common zeros, or when $y = -C_{ij}^{[m]}/W_{ik}^{[n]}$, and $F_{ij}^{[m]}$ and $2(C_{ij}^{[m]})^2 + W_{ik}^{[n]} D_{ij}^{[m]}$ have common zeros, where $\{i, j, k\} = \{1, 2, 3\}$.

4. Algebro-geometric solutions

In order to straighten out the corresponding flows in the soliton hierarchy (2.10), we equip $\mathcal{K}_g$ with a homology basis of **a**-cycles: $\mathbf{a}_1, \ldots, \mathbf{a}_g$, and **b**-cycles: $\mathbf{b}_1, \ldots, \mathbf{b}_g$, which are independent and have intersection numbers as follows:

$$\mathbf{a}_j \circ \mathbf{a}_k = 0, \quad \mathbf{b}_j \circ \mathbf{b}_k = 0 \quad \text{and} \quad \mathbf{a}_j \circ \mathbf{b}_k = \delta_{jk}, \quad 1 \le j, k \le g.$$

In what follows, we will choose the following set as our basis for the space of holomorphic differentials on $\mathcal{K}_g$ [2,3]:

$$\tilde{\omega}_l = \frac{1}{3y^2(P) + S_m} \begin{cases} \lambda^{l-1}\,\mathrm{d}\lambda, & 1 \le l \le \deg(S_m) - 1, \\ y(P)\lambda^{l-\deg(S_m)}\,\mathrm{d}\lambda, & \deg(S_m) \le l \le g, \end{cases} \tag{4.1}$$

which are g linearly independent holomorphic differentials on $\mathcal{K}_g$. By using the above homology basis, the period matrices $A = (A_{jk})$ and $B = (B_{jk})$ can be constructed as

$$A_{kj} = \int_{\mathbf{a}_j} \tilde{\omega}_k \quad \text{and} \quad B_{kj} = \int_{\mathbf{b}_j} \tilde{\omega}_k, \quad 1 \le j, k \le g. \tag{4.2}$$

It is possible to show that matrices A and B are invertible [4]. So, we can define the matrices C and τ by $C = A^{-1}$ and $\tau = A^{-1}B$. The matrix τ can be shown to be symmetric ($\tau_{kj} = \tau_{jk}$), and it has a positive-definite imaginary part ($\mathrm{Im}\,\tau > 0$) [5–7]. If we normalize $\tilde{\omega}_l$, $1 \le l \le g$, into a new basis $\underline{\omega} = (\omega_1, \ldots, \omega_g)$:

$$\omega_j = \sum_{l=1}^{g} C_{jl} \tilde{\omega}_l, \quad 1 \le j \le g, \tag{4.3}$$

where $C = (C_{ij})_{g \times g}$, then we obtain

$$\int_{\mathbf{a}_k} \omega_j = \sum_{l=1}^{g} C_{jl} \int_{\mathbf{a}_k} \tilde{\omega}_l = \delta_{jk} \quad \text{and} \quad \int_{\mathbf{b}_k} \omega_j = \tau_{jk}, \quad 1 \le j, k \le g. \tag{4.4}$$

To compute the **b**-periods of Abelian differentials of the second kind, we assume that

$$\omega_k \underset{\zeta \to 0}{=} \sum_{l=0}^{\infty} \varrho_{k,l}(P_{\infty_j}) \zeta^l \,\mathrm{d}\zeta, \quad \text{as } P \to P_{\infty_j}, \quad 1 \le k \le g, \ 1 \le j \le 3, \tag{4.5}$$

where $\varrho_{k,l}(P_{\infty_j})$, $l \ge 0$, are constants.

Now, let $\mathcal{T}_g$ be the period lattice $\mathcal{T}_g = \{\underline{z} \in \mathbb{C}^g \mid \underline{z} = \underline{N} + \underline{L}\tau,\ \underline{N}, \underline{L} \in \mathbb{Z}^g\}$. The complex torus $\mathscr{T}_g = \mathbb{C}^g/\mathcal{T}_g$ is called the Jacobian variety of $\mathcal{K}_g$. The Abel map $\underline{\mathcal{A}} : \mathcal{K}_g \to \mathscr{T}_g$ is defined as follows:

$$\underline{\mathcal{A}}(P) = \left(\int_{Q_0}^{P} \omega_1, \ldots, \int_{Q_0}^{P} \omega_g \right) (\mathrm{mod}\ \mathcal{T}_g), \tag{4.6}$$

where $Q_0 \in \mathcal{K}_g$ is a fixed base point. We take the natural linear extension of the Abel map to the space of divisors $\mathrm{Div}(\mathcal{K}_g)$:

$$\underline{\mathcal{A}}\left(\sum n_k P_k \right) = \sum n_k \underline{\mathcal{A}}(P_k), \tag{4.7}$$

where $P, P_k \in \mathcal{K}_g$.

Let $\omega^{(2)}_{\infty_j,l}(P)$, $1\le j\le 3$ and $l\ge 2$, denote the normalized Abelian differential of the second kind, being holomorphic on $\mathcal{K}_g\backslash\{P_{\infty_j}\}$ and possessing the asymptotic property:

$$\omega^{(2)}_{\infty_j,l}(P) \underset{\zeta\to 0}{=} (\zeta^{-l}+O(1))\,\mathrm{d}\zeta, \quad \text{as } P\to P_{\infty_j}, \quad 1\le j\le 3,\ l\ge 2. \tag{4.8}$$

The adopted normalization condition is

$$\int_{\mathbf{a}_k}\omega^{(2)}_{\infty_j,l}=0, \quad 1\le k\le g,\ 1\le j\le 3,\ l\ge 2 \tag{4.9}$$

and (4.8) implies that the residues of $\omega^{(2)}_{\infty_j,l}$ at P_{∞_j} are all zero. Based on the asymptotic properties of the Baker–Akhiezer functions ψ_j, $1\le j\le 3$, we introduce the following Abelian differentials of the second kind

$$\Omega^{(2)}_2=\omega^{(2)}_{P_{\infty_1},2}+\omega^{(2)}_{P_{\infty_2},2}-2\omega^{(2)}_{P_{\infty_3},2} \tag{4.10}$$

and

$$\tilde{\Omega}^{(2)}_r=r\omega^{(2)}_{P_{\infty_1},r+1}+r\omega^{(2)}_{P_{\infty_2},r+1}-2r\omega^{(2)}_{P_{\infty_3},r+1}. \tag{4.11}$$

Then for $\Omega^{(2)}_2$, we have the asymptotic expansions

$$\int_{Q_0}^{P}\Omega^{(2)}_2 \underset{\zeta\to 0}{=} \begin{cases} -\zeta^{-1}+e^{(2)}_{2,1}(Q_0)+O(\zeta), & \text{as } P\to P_{\infty_1},\\ -\zeta^{-1}+e^{(2)}_{2,2}(Q_0)+O(\zeta), & \text{as } P\to P_{\infty_2},\\ 2\zeta^{-1}+e^{(2)}_{2,3}(Q_0)+O(\zeta), & \text{as } P\to P_{\infty_3} \end{cases} \tag{4.12}$$

and for $\tilde{\Omega}^{(2)}_r$, we have the asymptotic expansions:

$$\int_{Q_0}^{P}\tilde{\Omega}^{(2)}_r \underset{\zeta\to 0}{=} \begin{cases} -\zeta^{-r}+\tilde{e}^{(2)}_{r,1}(Q_0)+O(\zeta), & \text{as } P\to P_{\infty_1},\\ -\zeta^{-r}+\tilde{e}^{(2)}_{r,2}(Q_0)+O(\zeta), & \text{as } P\to P_{\infty_2},\\ 2\zeta^{-r}+\tilde{e}^{(2)}_{r,3}(Q_0)+O(\zeta), & \text{as } P\to P_{\infty_3}, \end{cases} \tag{4.13}$$

where the paths of integration are chosen to be the same as the one in the Abel map (4.6). Define the $\mathbf{b}$-periods of the differentials $\Omega^{(2)}_2$ and $\tilde{\Omega}^{(2)}_r$, respectively, by

$$\underline{U}^{(2)}_2=(U^{(2)}_{2,1},\dots,U^{(2)}_{2,g}), \quad U^{(2)}_{2,k}=\frac{1}{2\pi\mathrm{i}}\int_{\mathbf{b}_k}\Omega^{(2)}_2, \quad 1\le k\le g \tag{4.14}$$

and

$$\underline{\tilde{U}}^{(2)}_r=(\tilde{U}^{(2)}_{r,1},\dots,\tilde{U}^{(2)}_{r,g}),\ \tilde{U}^{(2)}_{r,k}=\frac{1}{2\pi\mathrm{i}}\int_{\mathbf{b}_k}\tilde{\Omega}^{(2)}_r, \quad 1\le k\le g. \tag{4.15}$$

Through the relationship between the normalized meromorphic differential of the second kind and the normalized holomorphic differentials ω_k, $1\le k\le g$, we can derive that

$$U^{(2)}_{2,k}=\varrho_{k,0}(P_{\infty_1})+\varrho_{k,0}(P_{\infty_2})-2\varrho_{k,0}(P_{\infty_3}), \quad 1\le k\le g \tag{4.16}$$

and

$$\tilde{U}^{(2)}_{r,k}=\varrho_{k,r}(P_{\infty_1})+\varrho_{k,r}(P_{\infty_2})-2\varrho_{k,r}(P_{\infty_3}), \quad 1\le k\le g. \tag{4.17}$$

Let $\omega^{(3)}_{Q_1,Q_2}$ stand for the normalized Abelian differential of the third kind, holomorphic on $\mathcal{K}_g\backslash\{Q_1,Q_2\}$ and with simple poles at Q_l with residues $(-1)^{l+1}$, $l=1,2$. The adopted normalization condition reads

$$\int_{\mathbf{a}_k} \omega^{(3)}_{Q_1,Q_2} = 0, \quad 1 \le k \le g \tag{4.18}$$

and, thus,

$$\int_{\mathbf{b}_k} \omega^{(3)}_{Q_1,Q_2} = 2\pi \mathrm{i} \int_{Q_2}^{Q_1} \omega_k, \quad 1 \le k \le g, \tag{4.19}$$

where the path of integration from Q_2 to Q_1 does not intersect the cycles $\mathbf{a}_1, \ldots, \mathbf{a}_g, \mathbf{b}_1, \ldots, \mathbf{b}_g$. We then set

$$e^{(3)}_{2,j}(Q_0) = e^{(3)}_{2,j}(Q_0, x, x_0, t_r, t_{0,r}) = \int_{Q_0}^{P_{\infty_j}} \omega^{(3)}_{\hat{\nu}_0(x_0,t_{0,r}),\hat{\nu}_0(x,t_r)}, \quad 1 \le j \le 3 \tag{4.20}$$

and

$$e^{(3)}_{3,j}(Q_0) = e^{(3)}_{3,j}(Q_0, x, x_0, t_r, t_{0,r}) = \int_{Q_0}^{P_{\infty_j}} \omega^{(3)}_{\hat{\xi}_0(x_0,t_{0,r}),\hat{\xi}_0(x,t_r)}, \quad 1 \le j \le 3, \tag{4.21}$$

where the paths of integration are chosen to be the same as the one in the Abel map (4.6).

Denote by $\theta(\underline{z})$ the Riemann theta function associated with $\mathcal{K}_g$ equipped with the above homology basis [6]:

$$\theta(\underline{z}) = \sum_{\underline{N} \in \mathbb{Z}^g} \exp(\pi \mathrm{i} \langle \underline{N}\tau, \underline{N} \rangle + 2\pi \mathrm{i} \langle \underline{N}, \underline{z} \rangle), \tag{4.22}$$

where $\underline{z} = (z_1, \ldots, z_g) \in \mathbb{C}^g$ is a complex vector, and $\langle \cdot, \cdot \rangle$ stands for the Hermitian inner product on $\mathbb{C}^g$:

$$\langle \underline{z}, \underline{w} \rangle = \sum_{j=1}^{g} z_i \bar{w}_j, \ \underline{z} = (z_1, \ldots, z_g) \in \mathbb{C}^g, \quad \underline{w} = (w_1, \ldots, w_g) \in \mathbb{C}^g. \tag{4.23}$$

The Riemann theta function is even and quasi-periodic. More precisely, it satisfies

$$\theta(z_1, \ldots, z_{j-1}, -z_j, z_{j+1}, \ldots, z_g) = \theta(\underline{z}), \quad 1 \le j \le g \tag{4.24}$$

and

$$\theta(\underline{z} + \underline{N} + \underline{L}\tau) = \exp(-\pi \mathrm{i} \langle \underline{L}\tau, \underline{L} \rangle - 2\pi \mathrm{i} \langle \underline{L}, \underline{z} \rangle)\theta(\underline{z}), \tag{4.25}$$

where $\underline{z} = (z_1, \ldots, z_g) \in \mathbb{C}^g$, $\underline{N} = (N_1, \ldots, N_g) \in \mathbb{Z}^g$ and $\underline{L} = (L_1, \ldots, L_g) \in \mathbb{Z}^g$. For brevity, define the function $\underline{z} : \mathcal{K}_g \times \sigma^g \mathcal{K}_g \to \mathbb{C}^g$ by

$$\underline{z}(P, \underline{Q}) = \underline{M} - \underline{\mathcal{A}}(P) + \sum_{j=1}^{g} \mathcal{D}_{Q_1,\ldots,Q_g}(Q_j) \underline{\mathcal{A}}(Q_j), \tag{4.26}$$

where $P \in \mathcal{K}_g$, $\underline{Q} = (Q_1, \ldots, Q_g) \in \sigma^g \mathcal{K}_g$, $\sigma^g \mathcal{K}_g$ denotes the gth symmetric power of $\mathcal{K}_g$ [7], and $\underline{M} = (M_1, \ldots, M_g)$ is a vector of Riemann constants [6,8]:

$$M_j = \frac{1}{2}(1 + \tau_{jj}) - \sum_{l=1,\, l \ne j}^{g} \int_{\mathbf{a}_l} \omega_l(P) \int_{Q_0}^{P} \omega_j, \quad 1 \le j \le g. \tag{4.27}$$

By Riemann's vanishing theorem [8,9], the function $\theta(\underline{z}(P, \underline{Q}))$ has exactly g zeros $Q_1, \ldots, Q_g$ if the divisor $\mathcal{D} = Q_1 + \cdots Q_g$ is non-special.

Introduce three particular points in the gth symmetric power $\sigma^g \mathcal{K}_g$:

$$\underline{\hat{\mu}}(x, t_r) = (\hat{\mu}_1(x, t_r), \ldots, \hat{\mu}_g(x, t_r)), \tag{4.28}$$

$$\underline{\hat{\nu}}(x, t_r) = (\hat{\nu}_1(x, t_r), \ldots, \hat{\nu}_g(x, t_r)), \tag{4.29}$$

$$\underline{\hat{\xi}}(x, t_r) = (\hat{\xi}_1(x, t_r), \ldots, \hat{\xi}_g(x, t_r)) \tag{4.30}$$

and denote the corresponding three particular divisors in $\mathrm{Div}(\mathcal{K}_g)$ by

$$\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)} = \sum_{j=1}^{g} \hat{\mu}_j(x, t_r), \quad \mathcal{D}_{\underline{\hat{\nu}}(x,t_r)} = \sum_{j=1}^{g} \hat{\nu}_j(x, t_r) \quad \text{and} \quad \mathcal{D}_{\underline{\hat{\xi}}(x,t_r)} = \sum_{j=1}^{g} \hat{\xi}_j(x, t_r). \tag{4.31}$$

Theorem 4.1 (Theta function representations of the Baker–Akhiezer functions). *Let $\Omega_\mu \subset \mathbb{C}^2$ be an open and connected set, $(x_0, t_{0,r}), (x, t_r) \in \Omega_\mu$ and $P = (\lambda, y) \in \mathcal{K}_g \backslash \{P_{\infty_i}, 1 \le i \le 3\}$. Suppose that $\mathcal{K}_g$ is non-singular and $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ or $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ or $\mathcal{D}_{\underline{\hat{\xi}}(x,t_r)}$ is non-special for $(x, t_r) \in \Omega_\mu$. Then, the Baker–Akhiezer functions have the following theta function representations*:

$$\begin{aligned}\psi_1(P, x, x_0, t_r, t_{0,r}) &= \frac{\theta(\underline{z}(P, \underline{\hat{\mu}}(x, t_r)))\theta(\underline{z}(P_{\infty_3}, \underline{\hat{\mu}}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_3}, \underline{\hat{\mu}}(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x_0, t_{0,r})))} \exp\left(\left(e_{2,3}^{(2)}(Q_0) - \int_{Q_0}^{P} \Omega_2^{(2)}\right)(x - x_0)\right.\\ &\quad \left. + \left(\tilde{e}_{r,3}^{(2)}(Q_0) - \int_{Q_0}^{P} \tilde{\Omega}_r^{(2)}\right)(t_r - t_{0,r})\right),\end{aligned} \tag{4.32}$$

$$\begin{aligned}\psi_2(P, x, x_0, t_r, t_{0,r}) &= \frac{\theta(\underline{z}(P, \underline{\hat{\nu}}(x, t_r)))\theta(\underline{z}(P_{\infty_1}, \underline{\hat{\nu}}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_1}, \underline{\hat{\nu}}(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\nu}}(x_0, t_{0,r})))} \exp\left(\left(e_{2,1}^{(2)}(Q_0) - \int_{Q_0}^{P} \Omega_2^{(2)}\right)(x - x_0)\right.\\ &\quad \left. + \left(\tilde{e}_{r,1}^{(2)}(Q_0) - \int_{Q_0}^{P} \tilde{\Omega}_r^{(2)}\right)(t_r - t_{0,r}) + \left(e_{2,1}^{(3)}(Q_0) - \int_{Q_0}^{P} \omega^{(3)}_{\hat{\nu}_0(x_0,t_{0,r}),\hat{\nu}_0(x,t_r)}\right)\right)\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}\psi_3(P, x, x_0, t_r, t_{0,r}) &= \frac{\theta(\underline{z}(P, \underline{\hat{\xi}}(x, t_r)))\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\xi}}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\xi}}(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\xi}}(x_0, t_{0,r})))} \exp\left(\left(e_{2,2}^{(2)}(Q_0) - \int_{Q_0}^{P} \Omega_2^{(2)}\right)(x - x_0)\right.\\ &\quad \left. + \left(\tilde{e}_{r,2}^{(2)}(Q_0) - \int_{Q_0}^{P} \tilde{\Omega}_r^{(2)}\right)(t_r - t_{0,r}) + \left(e_{3,2}^{(3)}(Q_0) - \int_{Q_0}^{P} \omega^{(3)}_{\hat{\xi}_0(x_0,t_{0,r}),\hat{\xi}_0(x,t_r)}\right)\right),\end{aligned} \tag{4.34}$$

where the paths of integration are the same as the one in the Abel map (4.6).

Proof. Let Ψ_1, Ψ_2 and Ψ_3 denote the right-hand sides of (4.32), (4.33) and (4.34), respectively. By theorem 4.4 in [1], ψ_1 has the simple zeros $\hat{\mu}_1(x, t_r), \ldots, \hat{\mu}_g(x, t_r)$ and the simple poles $\hat{\mu}_1(x_0, t_{0,r}), \ldots, \hat{\mu}_g(x_0, t_{0,r})$, ψ_2 has the simple zeros $\hat{\nu}_0(x, t_r), \hat{\nu}_1(x, t_r), \ldots, \hat{\nu}_g(x, t_r)$ and the simple poles $\hat{\nu}_0(x_0, t_{0,r}), \hat{\nu}_1(x_0, t_{0,r}), \ldots, \hat{\nu}_g(x_0, t_{0,r})$, and ψ_3 has the simple zeros $\hat{\xi}_0(x, t_r), \hat{\xi}_1(x, t_r), \ldots, \hat{\xi}_g(x, t_r)$ and the simple poles $\hat{\xi}_0(x_0, t_{0,r}), \hat{\xi}_1(x_0, t_{0,r}), \ldots, \hat{\xi}_g(x_0, t_{0,r})$. They all have three essential singularities at $P_{\infty_1}, P_{\infty_2}, P_{\infty_3}$. By Riemann's vanishing theorem [8], we know that Ψ_i, $1 \le i \le 3$, have the same properties as ψ_i, $1 \le i \le 3$, respectively. Thus, the Riemann-Roch theorem tells us that $\Psi_i/\psi_i = \gamma_i$, $1 \le i \le 3$, where γ_i, $1 \le i \le 3$, are constants depending on P. Using the asymptotic properties of ψ_i and Ψ_i, $1 \le i \le 3$, one has

$$\frac{\Psi_1}{\psi_1} \underset{\zeta \to 0}{=} \frac{\exp(-2\zeta^{-1}(x - x_0) - 2\zeta^{-r}(t_r - t_{0,r}) + O(\zeta))(1 + O(\zeta))}{\exp(-2\zeta^{-1}(x - x_0) - 2\zeta^{-r}(t_r - t_{0,r}) + O(\zeta))} \underset{\zeta \to 0}{=} 1 + O(\zeta) \quad \text{as } P \to P_{\infty_3},$$

$$\frac{\Psi_2}{\psi_2} \underset{\zeta \to 0}{=} \frac{\exp(\zeta^{-1}(x - x_0) + \zeta^{-r}(t_r - t_{0,r}) + O(\zeta))(1 + O(\zeta))}{\exp(\zeta^{-1}(x - x_0) + \zeta^{-r}(t_r - t_{0,r}) + O(\zeta))} \underset{\zeta \to 0}{=} 1 + O(\zeta) \quad \text{as } P \to P_{\infty_1}$$

and

$$\frac{\Psi_3}{\psi_3} \underset{\zeta \to 0}{=} \frac{\exp(\zeta^{-1}(x - x_0) + \zeta^{-r}(t_r - t_{0,r}) + O(\zeta))(1 + O(\zeta))}{\exp(\zeta^{-1}(x - x_0) + \zeta^{-r}(t_r - t_{0,r}) + O(\zeta))} \underset{\zeta \to 0}{=} 1 + O(\zeta) \quad \text{as } P \to P_{\infty_2}.$$

These show that $\gamma_i = 1$, $1 \le i \le 3$. Therefore, $\Psi_i = \psi_i$, $1 \le i \le 3$. This completes the proof of the theorem. ■

Using the linear equivalences [9,10]

$$\mathcal{D}_{P_{\infty_3}, \hat{\nu}_{h_1}(x,t_r), \ldots, \hat{\nu}_g(x,t_r)} \sim \mathcal{D}_{P_{\infty_1}, \hat{\mu}_{h_1}(x,t_r), \ldots, \hat{\mu}_g(x,t_r)},$$

$$\mathcal{D}_{P_{\infty_3}, \hat{\xi}_{h_2}(x,t_r), \ldots, \hat{\xi}_g(x,t_r)} \sim \mathcal{D}_{P_{\infty_2}, \hat{\mu}_{h_2}(x,t_r), \ldots, \hat{\mu}_g(x,t_r)}$$

and

$$\mathcal{D}_{P_{\infty_1}, \hat{\xi}_{h_3}(x,t_r), \ldots, \hat{\xi}_g(x,t_r)} \sim \mathcal{D}_{P_{\infty_2}, \hat{\nu}_{h_3}(x,t_r), \ldots, \hat{\nu}_g(x,t_r)},$$

which are due to (3.31), (3.32) and (3.34), we obtain

$$\underline{A}(P_{\infty_3}) + \sum_{j=h_1}^{g} \underline{A}(\hat{\nu}_j(x,t_r)) = \underline{A}(P_{\infty_1}) + \sum_{j=h_1}^{g} \underline{A}(\hat{\mu}_j(x,t_r)),$$

$$\underline{A}(P_{\infty_3}) + \sum_{j=h_2}^{g} \underline{A}(\hat{\xi}_j(x,t_r)) = \underline{A}(P_{\infty_2}) + \sum_{j=h_2}^{g} \underline{A}(\hat{\mu}_j(x,t_r))$$

and

$$\underline{A}(P_{\infty_1}) + \sum_{j=h_3}^{g} \underline{A}(\hat{\xi}_j(x,t_r)) = \underline{A}(P_{\infty_2}) + \sum_{j=h_3}^{g} \underline{A}(\hat{\nu}_j(x,t_r)),$$

respectively. Define the Abel–Jacobi coordinates

$$\underline{\rho}^{(1)}(x,t_r) = \underline{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \sum_{j=1}^{g} \int_{Q_0}^{\hat{\mu}_j(x,t_r)} \underline{\omega}, \tag{4.35}$$

$$\underline{\rho}^{(2)}(x,t_r) = \underline{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) = \sum_{j=1}^{g} \int_{Q_0}^{\hat{\nu}_j(x,t_r)} \underline{\omega}, \tag{4.36}$$

$$\underline{\rho}^{(3)}(x,t_r) = \underline{A}(\mathcal{D}_{\underline{\hat{\xi}}(x,t_r)}) = \sum_{j=1}^{g} \int_{Q_0}^{\hat{\xi}_j(x,t_r)} \underline{\omega} \tag{4.37}$$

and then we have

$$\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r))) = \theta(\underline{M} - \underline{A}(P) + \underline{\rho}^{(1)}(x,t_r)),$$

$$\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t_r))) = \theta(\underline{M} - \underline{A}(P) + \underline{\rho}^{(2)}(x,t_r))$$

and

$$\theta(\underline{z}(P,\underline{\hat{\xi}}(x,t_r))) = \theta(\underline{M} - \underline{A}(P) + \underline{\rho}^{(3)}(x,t_r)).$$

The Abel–Jacobi coordinates can be linearized on the Riemann surface $\mathcal{K}_g$ as follows.

Theorem 4.2 (Straightening out of the flows). *Let $(x,t_r),(x_0,t_{0,r}) \in \mathbb{C}^2$ and $u=(p_1,p_2,q_1,q_2)^{\mathrm{T}}$ solve the rth four-component AKNS equations (2.10). Suppose that $\mathcal{K}_g$ is non-singular and $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ or $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ or $\mathcal{D}_{\underline{\hat{\xi}}(x,t_r)}$ is non-special. Then, we have*

$$\underline{\rho}^{(1)}(x,t_r) = \underline{\rho}^{(1)}(x_0,t_{0,r}) + \underline{U}_2^{(2)}(x-x_0) + \underline{\tilde{U}}_{2,r}^{(2)}(t-t_{0,r}) \pmod{\mathcal{T}_g}, \tag{4.38}$$

$$\underline{A}(\hat{\nu}_0(x,t_r)) + \underline{\rho}^{(2)}(x,t_r) = \underline{A}(\hat{\nu}_0(x_0,t_{0,r})) + \underline{\rho}^{(2)}(x_0,t_{0,r}) + \underline{U}_2^{(2)}(x-x_0) + \underline{\tilde{U}}_{2,r}^{(2)}(t-t_{0,r}) \pmod{\mathcal{T}_g} \tag{4.39}$$

and

$$\underline{A}(\hat{\xi}_0(x,t_r)) + \underline{\rho}^{(3)}(x,t_r) = \underline{A}(\hat{\xi}_0(x_0,t_{0,r})) + \underline{\rho}^{(3)}(x_0,t_{0,r}) + \underline{U}_2^{(2)}(x-x_0) + \underline{\tilde{U}}_{2,r}^{(2)}(t-t_{0,r}) \pmod{\mathcal{T}_g}. \tag{4.40}$$

Proof. In order to prove the theorem, we introduce three meromorphic differentials

$$\Omega_j(x,x_0,t_r,t_{0,r}) = \frac{\partial}{\partial\lambda}\ln(\psi_j(P,x,x_0,t_r,t_{0,r}))\,\mathrm{d}\lambda, \quad 1 \le j \le 3. \tag{4.41}$$

Let us first prove (4.38). From the theta function representation (4.32) for ψ_1, one infers

$$\Omega_1(x,x_0,t_r,t_{0,r}) = -(x-x_0)\Omega_2^{(2)} - (t_r-t_{0,r})\tilde{\Omega}_r^{(2)} + \sum_{j=1}^{g} \omega^{(3)}_{\hat{\mu}_j(x,t_r),\hat{\mu}_j(x_0,t_{0,r})} + \tilde{\omega}, \tag{4.42}$$

where $\tilde{\omega}$ is a holomorphic differential on $\mathcal{K}_g$, which can be expressed by

$$\tilde{\omega} = \sum_{j=1}^{g} h_j \omega_j, \tag{4.43}$$

$h_j \in \mathbb{C}$ being constants, $1 \le j \le g$.

Since $\psi_1(P, x, x_0, t_r, t_{0,r})$ is single-valued on $\mathcal{K}_g$, all **a**- and **b**-periods of Ω_1 are integer multiples of $2\pi\mathrm{i}$ and thus

$$2\pi \mathrm{i} l_k = \int_{\mathbf{a}_k} \Omega_1(x, x_0, t_r, t_{0,r}) = \int_{\mathbf{a}_k} \tilde{\omega} = h_k, \quad 1 \le k \le g,$$

for some $l_k \in \mathbb{Z}$. Similarly, for some $n_k \in \mathbb{Z}$, we have

$$\begin{aligned}
2\pi \mathrm{i} n_k &= \int_{\mathbf{b}_k} \Omega_1(x, x_0, t_r, t_{0,r}) \\
&= -(x - x_0)\int_{\mathbf{b}_k} \Omega_2^{(2)} - (t_r - t_{0,r})\int_{\mathbf{b}_k} \tilde{\Omega}_r^{(2)} + \sum_{j=1}^{g}\int_{\mathbf{b}_k} \omega^{(3)}_{\hat{\mu}_j(x,t_r),\hat{\mu}_j(x_0,t_{0,r})} + \int_{\mathbf{b}_k} \tilde{\omega} \\
&= -(x - x_0)\int_{\mathbf{b}_k} \Omega_2^{(2)} - (t_r - t_{0,r})\int_{\mathbf{b}_k} \tilde{\Omega}_r^{(2)} + 2\pi\mathrm{i}\sum_{j=1}^{g}\int_{\hat{\mu}_j(x_0,t_{0,r})}^{\hat{\mu}_j(x,t_r)} \omega_k + 2\pi\mathrm{i}\sum_{j=1}^{g} l_j \int_{\mathbf{b}_k} \omega_j \\
&= -2\pi\mathrm{i}(x - x_0)U_{2,k}^{(2)} - 2\pi\mathrm{i}(t_r - t_{0,r})\tilde{U}_{r,k}^{(2)} \\
&\quad + 2\pi\mathrm{i}\left(\sum_{j=1}^{g}\int_{Q_0}^{\hat{\mu}_j(x,t_r)} \omega_k - \sum_{j=1}^{g}\int_{Q_0}^{\hat{\mu}_j(x_0,t_{0,r})} \omega_k\right) + 2\pi\mathrm{i}\sum_{j=1}^{g} l_j \tau_{jk}, \quad 1 \le k \le g.
\end{aligned}$$

Thus, we arrive at

$$\underline{N} = -(x - x_0)\underline{U}_2^{(2)} - (t_r - t_{0,r})\underline{\tilde{U}}_r^{(2)} + \sum_{j=1}^{g}\int_{Q_0}^{\hat{\mu}_j(x,t_r)} \underline{\omega} - \sum_{j=1}^{g}\int_{Q_0}^{\hat{\mu}_j(x_0,t_{0,r})} \underline{\omega} + \underline{L}\tau, \tag{4.44}$$

where $\underline{N} = (n_1, \ldots, n_g) \in \mathbb{Z}^g$ and $\underline{L} = (l_1, \ldots, l_g) \in \mathbb{Z}^g$. The equation (4.44) exactly tells the first equality in (4.38).

Similarly, we can prove (4.39) and (4.40) by using the other two meromorphic differentials Ω_2 and Ω_3, respectively. The only difference is to change $\sum_{j=1}^{g} \omega^{(3)}_{\hat{\mu}_j(x,t_r),\hat{\mu}_j(x_0,t_{0,r})}$ into $\sum_{j=0}^{g} \omega^{(3)}_{\hat{\nu}_j(x,t_r),\hat{\nu}_j(x_0,t_{0,r})}$ or $\sum_{j=0}^{g} \omega^{(3)}_{\hat{\xi}_j(x,t_r),\hat{\xi}_j(x_0,t_{0,r})}$ on the right-hand side of (4.42), which brings the terms $\underline{A}(\hat{\nu}_0(x, t_r))$ and $\underline{A}(\hat{\nu}_0(x_0, t_{0,r}))$ in (4.39), and $\underline{A}(\hat{\xi}_0(x, t_r))$ and $\underline{A}(\hat{\xi}_0(x_0, t_{0,r}))$ in (4.40). The proof is completed. ■

Now, we are able to present theta function representations of solutions of the rth four-component AKNS equations (2.10).

Theorem 4.3 (Theta function representations of solutions). *Let $\Omega_\mu \subset \mathbb{C}^2$ be an open and connected set, $(x_0, t_{0,r}), (x, t_r) \in \Omega_\mu$ and $P = (\lambda, y) \in \mathcal{K}_g \backslash \{P_{\infty_i}, 1 \le i \le 3\}$. Suppose that $\mathcal{K}_g$ is non-singular and $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ or $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ or $\mathcal{D}_{\underline{\hat{\xi}}(x,t_r)}$ is non-special for $(x, t_r) \in \Omega_\mu$. Then, the solution $u = (p_1, p_2, q_1, q_2)^{\mathrm{T}}$ of the rth four-component AKNS equations (2.10) has the following theta function representations:*

$$\begin{aligned}
p_1(x, t_r) &= p_1(x_0, t_{0,r})\frac{\theta(\underline{z}(P_{\infty_1}, \underline{\hat{\mu}}(x, t_r)))\theta(\underline{z}(P_{\infty_3}, \underline{\hat{\mu}}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_3}, \underline{\hat{\mu}}(x, t_r)))\theta(\underline{z}(P_{\infty_1}, \underline{\hat{\mu}}(x_0, t_{0,r})))} \\
&\quad \times \exp((e_{2,3}^{(2)}(Q_0) - e_{2,1}^{(2)}(Q_0))(x - x_0) + (\tilde{e}_{r,3}^{(2)}(Q_0) - \tilde{e}_{r,1}^{(2)}(Q_0))(t_r - t_{0,r})),
\end{aligned} \tag{4.45}$$

$$\begin{aligned}
p_2(x, t_r) &= p_2(x_0, t_{0,r})\frac{\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\mu}}(x, t_r)))\theta(\underline{z}(P_{\infty_3}, \underline{\hat{\mu}}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_3}, \underline{\hat{\mu}}(x, t_r)))\theta(\underline{z}(P_{\infty_2}, \underline{\hat{\mu}}(x_0, t_{0,r})))} \\
&\quad \times \exp((e_{2,3}^{(2)}(Q_0) - e_{2,2}^{(2)}(Q_0))(x - x_0) + (\tilde{e}_{r,3}^{(2)}(Q_0) - \tilde{e}_{r,2}^{(2)}(Q_0))(t_r - t_{0,r}))
\end{aligned} \tag{4.46}$$

and

$$q_1(x,t_r)=q_1(x_0,t_{0,r})\frac{\theta(\underline{z}(P_{\infty_3},\underline{\hat{\nu}}(x,t_r)))\theta(\underline{z}(P_{\infty_1},\underline{\hat{\nu}}(x_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\nu}}(x,t_r)))\theta(\underline{z}(P_{\infty_3},\underline{\hat{\nu}}(x_0,t_{0,r})))}$$
$$\times\exp((e^{(2)}_{2,1}(Q_0)-e^{(2)}_{2,3}(Q_0))(x-x_0)+(\tilde{e}^{(2)}_{r,1}(Q_0)-\tilde{e}^{(2)}_{r,3}(Q_0))(t_r-t_{0,r})$$
$$+e^{(3)}_{2,1}(Q_0,x,x_0,t_r,t_{0,r})-e^{(3)}_{2,3}(Q_0,x,x_0,t_r,t_{0,r})), \tag{4.47}$$

$$q_2(x,t_r)=q_2(x_0,t_{0,r})\frac{\theta(\underline{z}(P_{\infty_3},\underline{\hat{\xi}}(x,t_r)))\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(x_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(x,t_r)))\theta(\underline{z}(P_{\infty_3},\underline{\hat{\xi}}(x_0,t_{0,r})))}$$
$$\times\exp((e^{(2)}_{2,2}(Q_0)-e^{(2)}_{2,3}(Q_0))(x-x_0)+(\tilde{e}^{(2)}_{r,2}(Q_0)-\tilde{e}^{(2)}_{r,3}(Q_0))(t_r-t_{0,r})$$
$$+e^{(3)}_{3,2}(Q_0,x,x_0,t_r,t_{0,r})-e^{(3)}_{3,3}(Q_0,x,x_0,t_r,t_{0,r})). \tag{4.48}$$

Proof. Based on the asymptotic properties of $\Omega_2^{(2)}$ and $\tilde{\Omega}_r^{(2)}$ in (4.12) and (4.13), and following theorem 4.1, we can expand the Baker–Akhiezer functions near the indicated points at infinity as follows:

$$\psi_1\underset{\zeta\to 0}{=}\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(x,t_r)))\theta(\underline{z}(P_{\infty_3},\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_3},\underline{\hat{\mu}}(x,t_r)))\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(x_0,t_{0,r})))}$$
$$\times\exp((e^{(2)}_{2,3}(Q_0)-e^{(2)}_{2,1}(Q_0))(x-x_0)+(\tilde{e}^{(2)}_{r,3}(Q_0)-\tilde{e}^{(2)}_{r,1}(Q_0))(t_r-t_{0,r})$$
$$+\zeta^{-1}(x-x_0)+\zeta^{-r}(t_r-t_{0,r})+O(\zeta))(1+O(\zeta)),\quad \text{as } P\to P_{\infty_1},$$
$$\psi_1\underset{\zeta\to 0}{=}\frac{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(x,t_r)))\theta(\underline{z}(P_{\infty_3},\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_3},\underline{\hat{\mu}}(x,t_r)))\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(x_0,t_{0,r})))}$$
$$\times\exp((e^{(2)}_{2,3}(Q_0)-e^{(2)}_{2,2}(Q_0))(x-x_0)+(\tilde{e}^{(2)}_{r,3}(Q_0)-\tilde{e}^{(2)}_{r,2}(Q_0))(t_r-t_{0,r})$$
$$+\zeta^{-1}(x-x_0)+\zeta^{-r}(t_r-t_{0,r})+O(\zeta))(1+O(\zeta)),\quad \text{as } P\to P_{\infty_2}$$

and

$$\psi_2\underset{\zeta\to 0}{=}\frac{\theta(\underline{z}(P_{\infty_3},\underline{\hat{\nu}}(x,t_r)))\theta(\underline{z}(P_{\infty_1},\underline{\hat{\nu}}(x_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\nu}}(x,t_r)))\theta(\underline{z}(P_{\infty_3},\underline{\hat{\nu}}(x_0,t_{0,r})))}\exp((e^{(2)}_{2,1}(Q_0)-e^{(2)}_{2,3}(Q_0))(x-x_0)$$
$$+(\tilde{e}^{(2)}_{r,1}(Q_0)-\tilde{e}^{(2)}_{r,3}(Q_0))(t_r-t_{0,r})+(e^{(3)}_{2,1}(Q_0)-e^{(3)}_{2,3}(Q_0))$$
$$-2\zeta^{-1}(x-x_0)-2\zeta^{-r}(t_r-t_{0,r})+O(\zeta))(1+O(\zeta)),\quad \text{as } P\to P_{\infty_3},$$
$$\psi_3\underset{\zeta\to 0}{=}\frac{\theta(\underline{z}(P_{\infty_3},\underline{\hat{\xi}}(x,t_r)))\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(x_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_2},\underline{\hat{\xi}}(x,t_r)))\theta(\underline{z}(P_{\infty_3},\underline{\hat{\xi}}(x_0,t_{0,r})))}\exp((e^{(2)}_{2,2}(Q_0)-e^{(2)}_{2,3}(Q_0))(x-x_0)$$
$$+(\tilde{e}^{(2)}_{r,2}(Q_0)-\tilde{e}^{(2)}_{r,3}(Q_0))(t_r-t_{0,r})+(e^{(3)}_{3,2}(Q_0)-e^{(3)}_{3,3}(Q_0))$$
$$-2\zeta^{-1}(x-x_0)-2\zeta^{-r}(t_r-t_{0,r})+O(\zeta))(1+O(\zeta)),\quad \text{as } P\to P_{\infty_3}.$$

Now, comparing with the asymptotic behaviours of ψ_1 and ψ_2 and ψ_3 established in (3.10), (3.20) and (3.30), respectively, we obtain the Riemann theta function presentations of p_1, p_2, q_1 and q_2 in (4.45)–(4.48) immediately. This completes the proof. ■

5. Concluding remarks

The present study, consisting of two parts, is dedicated to the development of explicit Riemann theta function representations of algebro-geometric solutions to entire soliton hierarchies. This is the second part. In this part, we straightened out all soliton flows under the Abel–Jacobi coordinates through determining zeros and poles of the Baker–Akhiezer functions, and constructed the Riemann theta function representations for algebro-geometric solutions to the four-component AKNS equations from checking asymptotic behaviours of the Baker–Akhiezer functions at the points at infinity.

We point out that we can similarly construct algebro-geometric solutions to a linear combination of different AKNS equations in the four-component AKNS soliton hierarchy, which just increases asymptotic complexity [11–13]. Various choices of linear combinations of Lax matrices lead to different algebro-geometric solutions to soliton hierarchies. However, it needs further investigation how to apply higher-order algebraic curves in finding algebro-geometric solutions to soliton equations. Higher-order matrix spectral problems [14–16] generate tremendous difficulty in computing algebro-geometric solutions. More components in the vector of eigenfunctions will cause complicated situations while deriving asymptotic expansions for the Baker–Akhiezer functions.

Two other interesting directions for further study are reductions and a density property of algebro-geometric solutions. Reducing algebro-geometric solutions tells various classes of exact solutions to soliton equations [9]. Two such classes of analytical solutions on the real field are quasi-periodic wave solutions [17] and lump solutions [18–20]. The study of lump solutions by bilinear techniques also brings us to an important question in multilinear algebra: how to determine positive definiteness (or positive semidefiniteness) for hypermatrices of even orders? For example, when does a real fourth-order hypermatrix, $A=(a_{ijkl})_{n\times n\times n\times n}$, satisfy $\sum_{i,j,k,l=1}^{n} a_{ijkl}x_i x_j x_k x_l > 0$ (or ≥ 0) for all non-zero vectors $(x_1,\ldots,x_n)\in\mathbb{R}^n$? The density property tells us about the computability of exact solutions to soliton equations via approximations. Commuting Lie symmetries, inherited from a recursion operator of a soliton hierarchy, yields an infinite number of one-parameter Lie groups of solutions to each equation in the hierarchy [21]. We conjecture that those infinitely many one-parameter Lie groups of solutions, starting from equilibria and algebro-geometric solutions, form a dense subset of solutions in the solution set of each equation in the underlying soliton hierarchy, under the uniform norm [22].

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