

## LUMP WAVES AND THEIR DYNAMICS OF A SPATIAL SYMMETRIC GENERALIZED KP MODEL

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**Abstract.** This paper aims to study lump waves formed by nonlinearity and dispersion in a spatial symmetric generalized KP model in (2+1)-dimensions. To an associated Hirota bilinear form of the model equation, positive quadratic waves are computed to generate lump waves by symbolic computation with Maple. It is shown that critical points of the positive quadratic waves are located on a straight line in the spatial space, whose coordinates travel at constant speeds. Optimal values of the corresponding lump waves are explicitly worked out, not depending on time, either. The dispersion terms and the nonlinear terms jointly create the lump waves.

**Key words:** Lump wave, Hirota bilinear form, Symbolic computation, Nonlinearity, Dispersion.

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### 1. INTRODUCTION

Closed-form solutions are nearly always desirable, because they help develop general solutions to problems. Unfortunately, closed-form solutions are not always possible. Many scientists concern themselves with finding closed-form solutions to open problems, and in lieu of that, proving whether or not a closed-form solution is possible. In soliton theory and nonlinear optics, multiple wave solutions, including solitons and lump waves, can be determined by conducting computer-based approaches. Nonlinearity and dispersion come together to form such nonlinear dispersive waves.

The Hirota direct method [1] and the inverse scattering transform [2] are among powerful mathematical techniques used in the context of soliton theory and integrable equations. The Hirota direct method can be used as a basic approach to solitons and lump waves, particularly in (2+1)- and (3+1)-dimensional nonlinear wave equations [3]-[6]. The inverse scattering technique is viewed as a generalization of the Fourier

transform to deal with nonlinear problems. It is targeted at solving Cauchy problems of nonlinear equations with Lax pairs [7] and exploring long-time asymptotics of solitonless solutions [8].

Let  $R$  be a polynomial in time  $t$  and two space variables  $x, y$ . A (2+1)-dimensional Hirota bilinear differential equation is defined by

$$R(D_t, D_x, D_y)g \cdot g = 0, \quad (1)$$

where  $D_t, D_x$  and  $D_y$  are the Hirota bilinear derivatives [1]:

$$D_t^m D_x^n D_y^k g \cdot g = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^k g(t, x, y) g(t', x', y') \Big|_{t'=t, x'=x, y'=y},$$

$m, n, k$  being nonnegative integers. From Hirota bilinear forms, nonlinear partial differential equations with a scalar dependent variable  $u$  are determined usually by taking use of the logarithmic derivative transformations

$$u = 2(\ln g)_{xx}, \quad u = 2(\ln g)_{yy}, \quad u = 2(\ln g)_{xy}, \quad u = 2(\ln g)_x, \quad u = 2(\ln g)_y. \quad (2)$$

In view of the Hirota bilinear method, an  $N$ -soliton solution (see, *e.g.*, [3, 9]) can be expressed as

$$g = \sum_{\lambda=0,1} \exp\left(\sum_{i=1}^N \lambda_i \zeta_i + \sum_{i<j} \lambda_i \lambda_j c_{ij}\right), \quad (3)$$

with  $\sum_{\lambda=0,1}$  denoting the sum over all possibilities for  $\lambda_1, \lambda_2, \dots, \lambda_N$  being either 0 or 1, and the phase shifts  $c_{ij}$  and the wave variables  $\zeta_i$  being given by

$$\exp(c_{ij}) = -\frac{R(\omega_j - \omega_i, k_i - k_j, l_i - l_j)}{R(\omega_j + \omega_i, k_i + k_j, l_i + l_j)}, \quad 1 \leq i < j \leq N, \quad (4)$$

and

$$\zeta_i = k_i x + l_i y - \omega_i t + \zeta_{i,0}, \quad 1 \leq i \leq N. \quad (5)$$

To determine an  $N$ -soliton solution, we need to impose the dispersion conditions

$$R(-\omega_i, k_i, l_i) = 0, \quad 1 \leq i \leq N, \quad (6)$$

but the constant phase shifts  $\zeta_{i,0}$  could be arbitrary. By taking the dispersion conditions (6) into consideration, an algorithm to show if such a function  $g$  in (3) solves the Hirota bilinear equation (1) is carefully formulated, together with illustrative examples, in [9, 10].

Recent extensive studies explore lump waves (and rogue waves) in nonlinear integrable models, resembling solitons, and they describe diverse interesting nonlinear phenomena [11]. Lump waves are formulated by using rational functions, which are localized in all directions in space (see, *e.g.*, [11, 12]). The KPI model equation possesses diverse lump waves (see, *e.g.*, [4]), and its specific lump waves are derived from its solitons by conducting long wave limits [13]. On the other hand, lump waves

also exist in nonlinear nonintegrable models, which include a few generalized (2+1)-dimensional KP, BKP and KP-Boussinesq model equations [14]. While lump waves refer to coherent structures in nonlinear models, they occur in linear models in higher dimensions as well, arising from linear superposition principles (see, *e.g.*, [15]).

A powerful ansatz for lump waves is to search for positive quadratic wave solutions to bilinear equations [4, 11]. Lump waves to nonlinear model equations are derived from positive quadratic waves by the logarithmic derivative transformations. In this paper, we would like to construct lump waves in a spatial symmetric generalized (2+1)-dimensional KP model *via* such an analytical ansatz using quadratic functions. We first present a Hirota bilinear form for the considered nonlinear model equation. The introduced spatial symmetric generalized (2+1)-dimensional KP model contains two nonlinear terms and five second-order dispersion terms. The nonlinearity terms and the dispersion terms are managing forces during the formulation of lump waves. The lump waves will be computed by conducting symbolic computation with computer algebra systems. Their characteristic dynamical properties, including critical points and optimal values, will be analyzed. A few concluding remarks will be provided in the last Section.

## 2. A SPATIAL SYMMETRIC GENERALIZED KP MODEL

Let  $\alpha, \beta$  and  $\gamma_i$ ,  $1 \leq i \leq 3$ , be real constants. To study lump waves created jointly by nonlinearity and dispersion, we introduce and consider a spatial symmetric generalized KP model equation:

$$\begin{aligned} F(u) = & \alpha(6u_x v_x + 6u_{xx} v + u_{xxxx} + 6u_y w_y + 6u_{yy} w + u_{yyyy}) \\ & + \beta(4uu_{xy} + 4v_y w_x + u_x u_y + u_{yy} v + u_{xx} w + v_x w_y + u_{xxyy}) \\ & + \gamma_1(u_{tx} + u_{ty}) + \gamma_2(u_{xx} + u_{yy}) + \gamma_3 u_{xy} = 0, \end{aligned} \quad (7)$$

where  $v_y = u_x, w_x = u_y, p_x = v, q_y = w$ . The example with  $\alpha = 1, \beta = 0, \gamma_1 = -\gamma_2 = 1$  and  $\gamma_3 = 0$  of this nonlinear model contains the (2+1)-dimensional spatial symmetric KP model equation [18]:

$$6u_x v_x + 6u_{xx} v + u_{xxxx} + 6u_y w_y + 6u_{yy} w + u_{yyyy} + u_{tx} + u_{ty} - u_{xx} - u_{yy} = 0, \quad (8)$$

with  $v_y = u_x$  and  $w_x = u_y$ , which is a symmetric generalization of the (2+1)-dimensional KdV model [16, 17]. It is direct to check that the general model doesn't possess  $N$ -solitons (see, *e.g.*, [9] for examples of  $N$ -solitons).

With symbolic computation, we can show that the logarithmic derivative transformations

$$u = 2(\ln g)_{xy}, \quad v = 2(\ln g)_{xx}, \quad w = 2(\ln g)_{yy}, \quad p = 2(\ln g)_x, \quad q = 2(\ln g)_y, \quad (9)$$

transforms the above spatial symmetric generalized (2+1)-dimensional KP model

equation (7) into a Hirota bilinear equation:

$$\begin{aligned}
 R(g) &= \alpha(D_x^4 + D_y^4) + \beta D_x^2 D_y^2 + \gamma_1(D_t D_x + D_t D_y) + \gamma_2(D_x^2 + D_y^2) + \gamma_3 D_x D_y)g \cdot g \\
 &= 2[\alpha(g_{xxxx}g - 4g_{xxx}g_x + 3g_{xx}^2 + g_{yyyy}g - 4g_{yyy}g_y + 3g_{yy}^2) \\
 &\quad + \beta(g_{xxyy}g - 2g_{xxy}g_y - 2g_{xyy}g_x + g_{xx}g_{yy} + 2g_{xy}^2) \\
 &\quad + \gamma_1(g_{tx}g - g_t g_x + g_{ty}g - g_t g_y) + \gamma_2(g_{xx}g - g_x^2 + g_{yy}g - g_y^2) + \gamma_3(g_{xy}g - g_x g_y)] \\
 &= 0,
 \end{aligned} \tag{10}$$

where  $D_t, D_x$  and  $D_y$  are the Hirota bilinear derivatives [1]. Actually, the precise relation between the nonlinear model equation and the bilinear model equation reads

$$F(u) = \left[ \frac{R(g)}{g^2} \right]_{xy}, \tag{11}$$

where  $u, v, w, p, q$  are determined through  $g$  in (9). Such links also exist in a spatial symmetric KP model [18] and a spatial symmetric HSI model [19]. It is now clear that if  $g$  is a solution to the bilinear model equation (10), then  $u, v, w, p, q$  defined by (9) solve the spatial symmetric generalized (2+1)-dimensional KP model equation (7). We would like to explore a class of lump waves in this nonlinear model below.

### 3. LUMP WAVES FORMED BY NONLINEARITY AND DISPERSION

We would now like to construct lump waves for the spatial symmetric generalized (2+1)-dimensional KP model equation (7), through conducting symbolic computations. We remark that it is direct to check that the above general nonlinear model equation does not pass the three-soliton test (see, *e.g.*, [9] for examples).

Following an ansatz on lump waves in (2+1)-dimensions [4], we compute positive quadratic wave solutions

$$g = \zeta_1^2 + \zeta_2^2 + a_9, \quad \zeta_1 = a_1x + a_2y + a_3t + a_4, \quad \zeta_2 = a_5x + a_6y + a_7t + a_8, \tag{12}$$

to the Hirota bilinear equation (10). The parameters  $a_i$ ,  $1 \leq i \leq 9$ , are real constants, which need to be determined (see, *e.g.*, [4, 11] for illustrative examples). This forms general lump waves of lower order in (2+1)-dimensions [11]. An essential step is to conduct symbolic computations to work out the involved constant parameters  $a_i$ ,  $1 \leq i \leq 9$ .

We input  $g$  by (12) into the Hirota bilinear equation (10) and then obtain a system of algebraic equations on the parameters. With computation by computer algebra systems, we solve the resulting system for  $a_3, a_7$  and  $a_9$ , and a set of solutions

reads

$$\left\{ \begin{array}{l} a_3 = -\frac{\{(a_1 + a_2)[(a_5 + a_6)^2 + a_1 a_2] + a_1(a_1^2 - 2a_6^2) + a_2(a_2^2 - 2a_5^2)\}\gamma_2}{[(a_1 + a_2)^2 + (a_5 + a_6)^2]\gamma_1} \\ \quad - \frac{[a_1(a_2^2 + a_6^2) + a_2(a_1^2 + a_5^2)]\gamma_3}{[(a_1 + a_2)^2 + (a_5 + a_6)^2]\gamma_1}, \\ a_7 = -\frac{\{(a_5 + a_6)[(a_1 + a_2)^2 + a_5 a_6] + a_5(a_5^2 - 2a_2^2) + a_6(a_6^2 - 2a_1^2)\}\gamma_2}{[(a_1 + a_2)^2 + (a_5 + a_6)^2]\gamma_1} \\ \quad - \frac{[a_5(a_2^2 + a_6^2) + a_6(a_1^2 + a_5^2)]\gamma_3}{[(a_1 + a_2)^2 + (a_5 + a_6)^2]\gamma_1}, \\ a_9 = -\frac{3\alpha[(a_1^2 + a_5^2)^2 + (a_2^2 + a_6^2)^2][(a_1 + a_2)^2 + (a_5 + a_6)^2]}{(a_1 a_6 - a_2 a_5)^2(2\gamma_2 - \gamma_3)} \\ \quad - \frac{\beta[3(a_1 a_2 + a_5 a_6)^2 + (a_1 a_6 - a_2 a_5)^2][(a_1 + a_2)^2 + (a_5 + a_6)^2]}{(a_1 a_6 - a_2 a_5)^2(2\gamma_2 - \gamma_3)}. \end{array} \right. \quad (13)$$

All other parameters can be arbitrarily taken.

The two frequency parameters,  $a_3$  and  $a_7$ , represent a class of dispersion relations in nonlinear dispersive waves in (2+1)-dimensions, and the constant term parameter,  $a_9$ , shows a complicated relation with the wave numbers, which is crucial in generating lump waves within the Hirota bilinear formulation. A kind of higher-order dispersion relations appearing in lump waves has also been presented for the second model of the integrable KP hierarchy [20], and specific nonlinear dynamical properties have been explored in various generalized KP models (see, *e.g.*, [21, 22]).

All the above expressions for the wave frequencies and the constant term in (13) are simplified by computations with computer algebra systems. A direct observation is that if

$$a_1 + a_2 = a_5 + a_6 = 0, \quad (14)$$

then

$$a_1 a_6 - a_2 a_5 = 0. \quad (15)$$

Moreover, based on (13), the parameter  $a_9$  is positive if and only if

$$a_{10}(2\gamma_2 - \gamma_3) < 0, \quad (16)$$

where  $a_{10}$  is defined by

$$a_{10} = 3\alpha[(a_1^2 + a_5^2)^2 + (a_2^2 + a_6^2)^2] + \beta[3(a_1 a_2 + a_5 a_6)^2 + (a_1 a_6 - a_2 a_5)^2]. \quad (17)$$

Therefore, to formulate lump waves by means of the logarithmic derivative transformations, we need to impose the basic conditions:

$$\gamma_1 \neq 0, \quad a_{10}(2\gamma_2 - \gamma_3) < 0, \quad (18)$$

and

$$a_1 a_6 - a_2 a_5 \neq 0, \quad (19)$$

The two conditions (18) and (19) guarantee the analyticity of the resulting solutions of  $u, v, w$  in the whole spatial and temporal space, and the condition (19) ensures the localness of the solutions of  $u, v, w$  in all spatial directions. Therefore, such solutions  $u, v, w$  present lump waves, under the two basic conditions (18) and (19).

The condition (18) can be satisfied, when we impose either

$$\gamma_1 \neq 0, \alpha(2\gamma_2 - \gamma_3) < 0, \beta(2\gamma_2 - \gamma_3) \leq 0. \quad (20)$$

or

$$\gamma_1 \neq 0, \alpha(2\gamma_2 - \gamma_3) \leq 0, \beta(2\gamma_2 - \gamma_3) < 0. \quad (21)$$

The conditions determined by (20) and (21) involves the coefficients,  $\alpha, \beta$ , of the nonlinear terms and the coefficients,  $\gamma_1, \gamma_2, \gamma_3$ , of the dispersion terms. If  $2\gamma_2 - \gamma_3 > 0$ , then we require  $\alpha < 0$  and  $\beta \leq 0$  or  $\alpha \leq 0$  and  $\beta < 0$ , and if  $2\gamma_2 - \gamma_3 < 0$ , we require  $\alpha > 0$  and  $\beta \geq 0$  or  $\alpha \geq 0$  and  $\beta > 0$ , to satisfy (20) or (21). Therefore, the nonlinearity and the dispersion jointly govern lump waves for the model equation (7), but the nonlinearity does not affect the speeds of the two single waves in the lumps, based on (13). Moreover, the condition on the dispersion terms

$$\gamma_1(2\gamma_2 - \gamma_3) \neq 0 \quad (22)$$

is always required in the formulating process of the lump waves.

There are two special cases, which still possess lump waves. One corresponds to  $\beta = 0$  and the other,  $\alpha = 0$ . Recently, various multi-component integrable nonlinear Schrödinger models or modified Korteweg-de Vries models have been explored (see, *e.g.*, [23–25]). It would be of great interest to check if there exist lump waves in (2+1)-dimensional generalizations of those integrable models in (1+1)-dimensions.

#### 4. DYNAMICAL CHARACTERISTICS

Let us consider dynamical characteristic properties of the lump waves computed in the previous Section.

##### 4.1. LINE OF CRITICAL POINTS

Let us first determine critical points of the quadratic function  $f$  defined by (12). To this end, we solve a system of two equations:

$$\frac{\partial g}{\partial x}(t, x(t), y(t)) = 0, \quad \frac{\partial g}{\partial y}(t, x(t), y(t)) = 0.$$

Upon considering that  $f$  is quadratic, this leads precisely to

$$a_1 \zeta_1 + a_5 \zeta_2 = 0, \quad a_2 \zeta_1 + a_6 \zeta_2 = 0.$$

Therefore, because of the condition by (19), we have

$$\zeta_1 = a_1x + a_2y + a_3t + a_4 = 0, \quad \zeta_2 = a_5x + a_6y + a_7t + a_8 = 0. \quad (23)$$

Solving this linear system for  $x$  and  $y$ , we obtain all critical points of the multivariate quadratic function  $f$ :

$$x(t) = \frac{[(a_1 + a_2)^2 + (a_5 + a_6)^2 - 2(a_2^2 + a_6^2)]\gamma_2 + (a_2^2 + a_6^2)\gamma_3}{[(a_1 + a_2)^2 + (a_5 + a_6)^2]\gamma_1}t + \frac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \quad (24)$$

$$y(t) = -\frac{[(a_1 - a_2)^2 + (a_5 - a_6)^2 - 2(a_2^2 + a_6^2)]\gamma_2 - (a_1^2 + a_5^2)\gamma_3}{[(a_1 + a_2)^2 + (a_5 + a_6)^2]\gamma_1}t - \frac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}, \quad (25)$$

at an arbitrarily fixed time  $t$ .

Evidently, those critical points form a characteristic line, whose two spatial coordinates travel at constant speeds. Moreover, we can show that all those points  $(x(t), y(t))$  determined above are also critical points of the three lump waves  $u, v$  and  $w$  by (9).

#### 4.2. ANALYTICITY CONDITION

Taking (23) into consideration, we find that the sum of two squares, namely, the function  $g - a_9$  vanishes at all critical points determined by (24) and (25). As a consequence, we see that  $g > 0$  if and only if  $a_9 > 0$ . The sufficiency is obvious, and the necessity is due to that  $g = 0$  at the critical points if  $a_9 = 0$  and  $g = 0$  at any point determined by the circle  $\zeta_1^2 + \zeta_2^2 = -a_9$  if  $a_9 < 0$ .

Consequently,  $u, v, w$  defined by (9) are analytic in  $\mathbb{R}^3$  if and only if the parameter  $a_9$  must be positive,  $\gamma_1 \neq 0$  and (19) is satisfied. The necessary and sufficient condition to guarantee the positiveness of  $a_9$  was presented in Section 3, which can be satisfied by either

$$\alpha(2\gamma_2 - \gamma_3) < 0, \beta(2\gamma_2 - \gamma_3) \leq 0, \quad (26)$$

or

$$\alpha(2\gamma_2 - \gamma_3) \leq 0, \beta(2\gamma_2 - \gamma_3) < 0. \quad (27)$$

## 4.3. OPTIMAL VALUES

By applying the second partial derivative test in calculus, the both lump waves,  $v$  and  $w$ , have a peak at the critical points  $(x(t), y(t))$ , because we can work out that

$$\begin{cases} v_{xx} = -\frac{24(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)^4(2\gamma_2 - \gamma_3)^2}{a_{10}^2[(a_1 + a_2)^2 + (a_5 + a_6)^2]^2} < 0, \\ v_{xx}v_{yy} - v_{xy}^2 = \frac{192(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)^{10}(2\gamma_2 - \gamma_3)^4}{a_{10}^4[(a_1 + a_2)^2 + (a_5 + a_6)^2]^4} > 0, \end{cases} \quad (28)$$

and

$$\begin{cases} w_{xx} = -\frac{8[(a_1a_6 - a_2a_5)^2 + 3(a_1a_2 + a_5a_6)^2](a_1a_6 - a_2a_5)^4(2\gamma_2 - \gamma_3)^2}{a_{10}^2[(a_1 + a_2)^2 + (a_5 + a_6)^2]^2} < 0, \\ w_{xx}w_{yy} - w_{xy}^2 = \frac{192(a_2^2 + a_6^2)^2(a_1a_6 - a_2a_5)^{10}(2\gamma_2 - \gamma_3)^4}{a_{10}^4[(a_1 + a_2)^2 + (a_5 + a_6)^2]^4} > 0, \end{cases} \quad (29)$$

where  $a_{10}$  is defined by (17).

In an analogous way, we can determine that

$$\begin{cases} u_{xx} = -\frac{2(a_1^2 + a_5^2)(a_1a_6 - a_2a_5)^4(a_1a_2 + a_5a_6)(2\gamma_2 - \gamma_3)^2}{a_{10}^2[(a_1 + a_2)^2 + (a_5 + a_6)^2]^2}, \\ u_{xx}u_{yy} - u_{xy}^2 = \frac{64a_{11}(a_1a_6 - a_2a_5)^{10}(2\gamma_2 - \gamma_3)^4}{a_{10}^4[(a_1 + a_2)^2 + (a_5 + a_6)^2]^4}, \end{cases} \quad (30)$$

where  $a_{10}$  is given by (17) and  $a_{11}$  is defined by

$$a_{11} = 3(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2. \quad (31)$$

Accordingly, the lump wave  $u$  has the minimum (or maximum) points  $(x(t), y(t))$ , when  $a_1a_2 + a_5a_6 < 0$  (or  $a_1a_2 + a_5a_6 > 0$ ) and  $a_{11} > 0$ ;  $u$  has the saddle points  $(x(t), y(t))$ , when  $a_{11} < 0$ ; and the second partial derivative test is inconclusive, when  $a_{11} = 0$ .

Moreover, we can compute the optimal values of  $v, w$  and  $u$  at the critical points  $(x(t), y(t))$  as follows:

$$v_{maximum} = -\frac{4(a_1^2 + a_5^2)(a_1a_6 - a_2a_5)^2(2\gamma_2 - \gamma_3)}{a_{10}[(a_1 + a_2)^2 + (a_5 + a_6)^2]}, \quad (32)$$

$$w_{maximum} = -\frac{4(a_2^2 + a_6^2)(a_1a_6 - a_2a_5)^2(2\gamma_2 - \gamma_3)}{a_{10}[(a_1 + a_2)^2 + (a_5 + a_6)^2]}, \quad (33)$$

$$u_{optimum} = -\frac{4(a_1a_6 - a_2a_5)^2(2\gamma_2 - \gamma_3)(a_1a_2 + a_5a_6)}{a_{10}[(a_1 + a_2)^2 + (a_5 + a_6)^2]}, \quad (34)$$



where  $a_{10}$  is given by (17). Consequently, we observe that all optimal values are all constants along the characteristic line of critical points, which do not depend on time  $t$  (see also, *e.g.*, [18, 19] for more examples). Therefore, the peak and the valley of the lump waves along the characteristic line continue to be the same. On the other hand, upon observing those three formulas for the optimal values, we find that any of the lump waves of  $u, v, w$  may not decay, when the two directions  $(a_1, a_2)$  and  $(a_5, a_6)$  becomes parallel to each other, namely,  $a_1 a_6 - a_2 a_5$  goes to zero.

## 5. CONCLUDING REMARKS

A spatial symmetric generalized (2+1)-dimensional KP model was analyzed and its lump waves were computed by conducting symbolic computations with computer algebra systems. Dynamical characteristic properties of the resulting lump waves were explored, including critical points and optimal values, along with an exploration on the effects of the nonlinear terms and the dispersion terms.

Various studies exhibit a remarkable richness of lump waves across a wide range of disciplines, reflecting their significance in understanding physical phenomena in linear wave models [15], and nonlinear (2+1)-dimensional nonintegrable models (see, *e.g.*, [26]-[30]) and (3+1)-dimension nonintegrable models (see, *e.g.*, [21, 31]). Both the Hirota bilinear forms and the generalized bilinear forms are used as the basis for formulations of lump waves [11]. Moreover, there are abundant interaction solutions between lump waves and other interesting waves, both homoclinic and heteroclinic, in (2+1)-dimensional integrable models (see, *e.g.*, [32–34]).

It is known that  $N$ -solitons have been systematically studied by Riemann-Hilbert problems for local and nonlocal integrable models obtained by groups reductions (see, *e.g.*, [35]-[39]). It is of much interest to explore if there exist lump waves in (2+1)-dimensional generalizations of reduced integrable models (see, *e.g.*, [40–42]), both local and nonlocal.

The richness of lump waves underscores their importance across diverse fields of study, offering insights into fundamental principles of wave propagation, coherence, and nonlinear dynamics. Understanding and harnessing the properties of lump waves will advance our understanding of nonlinear dispersive waves in physical and engineering sciences.

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