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A LIOUVILLE INTEGRABLE HIERARCHY WITH FOUR POTENTIALS AND ITS BI-HAMILTONIAN STRUCTURE

WEN-XIU MA

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China
Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
Department of Mathematics and Statistics, University of South Florida,
Tampa, FL 33620-5700, USA
School of Mathematical and Statistical Sciences, North-West University,
Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa
Email: *mawx@cas.usf.edu*

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Abstract. We aim to construct a Liouville integrable Hamiltonian hierarchy from a specific matrix spectral problem with four potentials through the zero curvature formulation. The Liouville integrability of the resulting hierarchy is exhibited by a bi-Hamiltonian structure explored by using the trace identity. Illustrative examples of novel four-component coupled Liouville integrable nonlinear Schrödinger equations and modified Korteweg-de Vries equations are presented.

Key words: Matrix spectral problem, Zero curvature equation, Integrable hierarchy, NLS equations, mKdV equations.

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1. INTRODUCTION

Zero curvature equations play a crucial role in various fields of mathematics and physics, particularly in the study of integrable models [1, 2]. These equations are also known as the Yang-Baxter equations and were widely studied in the context of statistical mechanics and quantum field theory. Usually, an infinite sequence of zero curvature equations produces a hierarchy of integrable models, yielding a sufficient number of conserved quantities that enable the models to be solved analytically. Each zero curvature equation involves a pair of spectral matrices, which could satisfy some certain Riccati relation that ensures the existence of these conserved quantities.

The importance of zero curvature equations lies in their ability to provide a unifying framework for the study of integrable models. To construct integable models within the zero curvature formulation, it is crucial to form an appropriate infinite sequence of pairs of spatial and temporal spectral matrices. Let us take an n-dimensional potential: $u = (u_1, \cdots, u_n)^T$ and, as usual, use λ to denote the spectral parameter.

First, we use a loop matrix algebra $\tilde{g}$ to formulate a spatial spectral matrix:

$$\mathcal{M}=\mathcal{M}(u,\lambda)=f_0(\lambda)+u_1f_1(\lambda)+\cdots+u_nf_n(\lambda), \tag{1}$$

where $f_1,\cdots,f_n$ are linear independent elements in $\tilde{g}$ and f_0 is a pseudo-regular element in $\tilde{g}$. The pseudo-regular property reads

$$[\mathrm{Ker}\,\mathrm{ad}_{f_0},\mathrm{Ker}\,\mathrm{ad}_{f_0}]=0,\ \mathrm{Ker}\,\mathrm{ad}_{f_0}\oplus\mathrm{Im}\,\mathrm{ad}_{f_0}=\tilde{g}.$$

This characteristic property guarantees that we can solve the stationary zero curvature equation:

$$Z_x=i[\mathcal{M},Z], \tag{2}$$

among Laurent series matrices $Z=\sum_{s\geq 0}\lambda^{-s}Z^{[s]}$ in the loop algebra $\tilde{g}$.

Second, we take the temporal spectral matrices

$$\mathcal{N}^{[r]}=(\lambda^rZ)_++\Delta_r=\sum_{s=0}^{r}\lambda^{r-s}Z^{[s]}+\Delta_r,\ r\geq 0, \tag{3}$$

where $\Delta_r\in\tilde{g},\ r\geq 0$, to generate an integrable hierarchy through the zero curvature equations:

$$\mathcal{M}_{t_r}-\mathcal{N}^{[r]}_x+i[\mathcal{M},\mathcal{N}^{r]}]=0,\ r\geq 0. \tag{4}$$

These equations are the compatibility conditions of the spatial and temporal matrix spectral problems:

$$-i\phi_x=\mathcal{M}\phi,\ -i\phi_{t_r}=\mathcal{N}^{[r]}\phi,\ r\geq 0. \tag{5}$$

Finally, the Liouville integrability can be explored by using the trace identity [3, 4]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}\big(Z\frac{\partial\mathcal{M}}{\partial\lambda}\big)\,dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\mathrm{tr}\big(Z\frac{\partial\mathcal{M}}{\partial u}\big), \tag{6}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to u and γ is the constant determined by

$$\gamma=-\frac{\lambda}{2}\frac{\partial}{\partial\lambda}\ln|\mathrm{tr}(Z^2)|. \tag{7}$$

Various integrable hierarchies are constructed by the zero curvature formulation. The adopted loop algebras are generated from the special linear algebras (see, *e.g.*, [5–11]), and the special orthogonal algebras (see, *e.g.*, [12–14]). Bi-Hamiltonian structures [15] exhibit the Liouville integrability of those zero curvature equations. Among integrable hierarchies with two components, p and q, are the well-known integrable hierarchies the Ablowitz-Kaup-Newell-Segur hierarchy [5], the Kaup-Newell hierarchy [16], the Wadati-Konno-Ichikawa hierarchy [17] and the

Heisenberg hierarchy [18]. Their spectral matrices read

$$\mathcal{M}=\begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix}, \mathcal{M}=\begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix}, \mathcal{M}=\begin{bmatrix} \lambda & \lambda p \\ \lambda q & -\lambda \end{bmatrix}, \mathcal{M}=\begin{bmatrix} \lambda v & \lambda p \\ \lambda q & -\lambda v \end{bmatrix}, \tag{8}$$

where $pq+v^2=1$, respectively.

This paper aims at constructing an integrable hierarchy of four-component Hamiltonian equations within the zero curvature formulation. By the trace identity, we establish a bi-Hamiltonian structure for the resulting hierarchy. Two illustrative examples of four-component coupled integrable nonlinear Schrödinger equations and modified Korteweg-de Vries equations are presented. The last Section is devoted to a conclusion, together with some concluding remarks.

2. AN INTEGRABLE HAMILTONIAN HIERARCHY WITH FOUR COMPONENTS

Let δ_1 and δ_2 be two real numbers satisfying $\delta_1^2=\delta_2^2=1$, *i.e.*, $\delta_1,\delta_2\in\{1,-1\}$. Within the zero curvature formulation, we consider a matrix spectral problem of the form:

$$-i\phi_x=\mathcal{M}\phi=\mathcal{M}(u,\lambda)\phi,\ \mathcal{M}=\begin{bmatrix} \lambda & v_1 & v_2 & v_1 & v_2 & 0 \\ w_1 & 0 & 0 & 0 & 0 & \delta_1 v_1 \\ w_2 & 0 & 0 & 0 & 0 & \delta_2 v_2 \\ w_1 & 0 & 0 & 0 & 0 & \delta_1 v_1 \\ w_2 & 0 & 0 & 0 & 0 & \delta_2 v_2 \\ 0 & \delta_1 w_1 & \delta_2 w_2 & \delta_1 w_1 & \delta_2 w_2 & -\lambda \end{bmatrix}, \tag{9}$$

where u is the four-dimensional potential

$$u=u(x,t)=(v_1,v_2,w_1,w_2)^T. \tag{10}$$

This spectral problem cannot be reduced from the matrix Ablowitz-Kaup-Newell-Segur spectral problem (see, *e.g.*, [19] for cases of nonlocal reductions).

In order to construct an associated integrable hierarchy, we first solve the stationary zero curvature equation (2) by searching for a Laurent series solution:

$$Z=\begin{bmatrix} a & b_1 & b_2 & b_1 & b_2 & 0 \\ c_1 & 0 & d & 0 & d & \delta_1 b_1 \\ c_2 & -\delta_1\delta_2 d & 0 & -\delta_1\delta_2 d & 0 & \delta_2 b_2 \\ c_1 & 0 & d & 0 & d & \delta_1 b_1 \\ c_2 & -\delta_1\delta_2 d & 0 & -\delta_1\delta_2 d & 0 & \delta_2 b_2 \\ 0 & \delta_1 c_1 & \delta_2 c_2 & \delta_1 c_1 & \delta_2 c_2 & -a \end{bmatrix}=\sum_{s\geq 0}\lambda^{-s}Z^{[s]}, \tag{11}$$

where the basic objects are assumed to be expanded in Laurent series:

$$a=\sum_{s\ge 0}\lambda^{-s}a^{[s]},\ b_j=\sum_{s\ge 0}\lambda^{-s}b_j^{[s]},\ c_j=\sum_{s\ge 0}\lambda^{-s}c_j^{[s]},\ d=\sum_{s\ge 0}\lambda^{-s}d^{[s]},\ i=1,2. \quad (12)$$

Obviously, the corresponding stationary zero curvature equation yields the initial conditions:

$$a_x^{[0]}=0,\ b_1^{[0]}=b_2^{[0]}=c_1^{[0]}=c_2^{[0]}=0,\ d_x^{[0]}=0, \quad (13)$$

and the recursion relations:

$$\begin{cases} b_1^{[s+1]}=-ib_{1,x}^{[s]}+v_1a^{[s]}+2\delta_1\delta_2v_2d^{[s]},\\ b_2^{[s+1]}=-ib_{2,x}^{[s]}+v_2a^{[s]}-2v_1d^{[s]}, \end{cases} \quad (14)$$

$$\begin{cases} c_1^{[s+1]}=ic_{1,x}^{[s]}+w_1a^{[s]}-2w_2d^{[s]},\\ c_2^{[s+1]}=ic_{2,x}^{[s]}+w_2a^{[s]}+2\delta_1\delta_2w_1d^{[s]}, \end{cases} \quad (15)$$

and

$$\begin{cases} d_x^{[s+1]}=i(w_1b_2^{[s+1]}-\delta_1\delta_2w_2b_1^{[s+1]}+\delta_1\delta_2v_1c_2^{[s+1]}-v_2c_1^{[s+1]}),\\ a_x^{[s+1]}=-2i(w_1b_1^{[s+1]}+w_2b_2^{[s+1]}-v_1c_1^{[s+1]}-v_2c_2^{[s+1]})\\ \qquad\quad =-2(w_1b_{1,x}^{[s]}+w_2b_{2,x}^{[s]}+v_1c_{1,x}^{[s]}+v_2c_{2,x}^{[s]}), \end{cases} \quad (16)$$

where $s\ge 0$. To have a unique Laurent series solution, we take the initial values,

$$a^{[0]}=1,\ d^{[0]}=0, \quad (17)$$

and choose the constant of integration as zero,

$$a^{[s]}|_{u=0}=0,\ d^{[s]}|_{u=0}=0,\ s\ge 1. \quad (18)$$

Then, we can work out that

$$b_1^{[1]}=v_1,\ b_2^{[1]}=v_2,\ c_1^{[1]}=w_1,\ c_2^{[1]}=w_2,\ a^{[1]}=0,\ d^{[1]}=0;$$

$$\begin{cases} b_1^{[2]}=-iv_{1,x},\ b_2^{[2]}=-iv_{2,x},\ c_1^{[2]}=iw_{1,x},\ c_2^{[2]}=iw_{2,x},\\ a^{[2]}=-2v_1w_1-2v_2w_2,\ d^{[2]}=-\delta_1\delta_2v_1w_2+v_2w_1; \end{cases}$$

$$\begin{cases} b_1^{[3]}=-v_{1,xx}-2v_1^2w_1-4v_1v_2w_2+2\delta_1\delta_2v_2^2w_1,\\ b_2^{[3]}=-v_{2,xx}+2\delta_1\delta_2v_1^2w_2-4v_1v_2w_1-2v_2^2w_2, \end{cases}$$

$$\begin{cases} c_1^{[3]}=-w_{1,xx}-2v_1w_1^2+2\delta_1\delta_2v_1w_2^2-4v_2w_1w_2,\\ c_2^{[3]}=-w_{2,xx}-4v_1w_1w_2+2\delta_1\delta_2v_2w_1^2-2v_2w_2^2, \end{cases}$$

$$\begin{cases} a^{[3]} = 2i(v_{1,x}w_1 - v_1w_{1,x} + v_{2,x}w_2 - v_2w_{2,x}), \\ d^{[3]} = -i(\delta_1\delta_2 v_1 w_{2,x} - v_2 w_{1,x} - \delta_1\delta_2 v_{1,x} w_2 + v_{2,x} w_1); \end{cases}$$

and

$$\begin{cases} b_1^{[4]} = i(v_{1,xxx} + 6v_1v_{1,x}w_1 + 6v_1v_{2,x}w_2 - 6\delta_1\delta_2 v_2 v_{2,x} w_1 + 6v_{1,x}v_2w_2), \\ b_2^{[4]} = i(v_{2,xxx} + 6v_1v_{2,x}w_1 - 6\delta_1\delta_2 v_1 v_{1,x} w_2 + 6v_{1,x}v_2w_1 + 6v_2v_{2,x}w_2), \end{cases}$$

$$\begin{cases} c_1^{[4]} = -i(w_{1,xxx} + 6v_1w_1w_{1,x} - 6\delta_1\delta_2 v_1 w_2 w_{2,x} + 6v_2w_1w_{2,x} + 6v_2w_{1,x}w_2), \\ c_2^{[4]} = -i(w_{2,xxx} + 6v_1w_1w_{2,x} + 6v_1w_{1,x}w_2 - 6\delta_1\delta_2 v_2 w_1 w_{1,x} + 6v_2w_2w_{2,x}), \end{cases}$$

$$\begin{cases} a^{[4]} = 6v_1^2w_1^2 - 6\delta_1\delta_2 v_1^2 w_2^2 + 24v_1v_2w_1w_2 - 6\delta_1\delta_2 v_2^2 w_1^2 + 6v_2^2w_2^2 \\ \qquad +2v_1w_{1,xx} + 2v_{1,x,x}w_1 + 2v_2w_{2,xx} + 2v_{2,xx}w_2 - 2v_{1,x}w_{1,x} - 2v_{2,x}w_{2,x}, \\ d^{[4]} = -6(v_1w_1 + v_2w_2)(\delta_1\delta_2 v_1 w_2 - v_2 w_1) + \delta_1\delta_2 v_{1,xx} w_2 - v_{2,xx} w_1 \\ \qquad -v_2w_{1,xx} + \delta_1\delta_2 v_1 w_{2,xx} - \delta_1\delta_2 v_{1,x} w_{2,x} + v_{2,x} w_{1,x}. \end{cases}$$

Based on these computations, we can take $\Delta_r = 0,\ r \geq 0$, to introduce the temporal matrix spectral problems:

$$-i\phi_{t_r} = \mathcal{N}^{[r]}\phi = \mathcal{N}^{[r]}(u,\lambda)\phi,\ \mathcal{N}^{[r]} = (\lambda^r Z)_+ = \sum_{s=0}^{r} \lambda^s Z^{[r-s]},\ r \geq 0, \tag{19}$$

which are the other parts of Lax pairs of matrix spectral problems in the zero curvature formulation. The compatibility conditions of the spatial and temporal matrix spectral problems in (9) and (19) are the zero curvature equations in (4). Those equations generate a four-component integrable hierarchy:

$$u_{t_r} = X^{[r]} = (ib_1^{[r+1]}, ib_2^{[r+1]}, -ic_1^{[r+1]}, -ic_2^{[r+1]})^T,\ r \geq 0, \tag{20}$$

or more concretely,

$$v_{1,t_r} = ib_1^{[r+1]},\ v_{2,t_r} = ib_2^{[r+1]},\ w_{1,t_r} = -ic_1^{[r+1]},\ w_{2,t_r} = -ic_2^{[r+1]},\ r \geq 0. \tag{21}$$

As particular examples, this integrable hierarchy contains the coupled integrable nonlinear Schrödinger equations:

$$\begin{cases} iv_{1,t_2} = v_{1,xx} + 2v_1^2w_1 + 4v_1v_2w_2 - 2\delta_1\delta_2 v_2^2 w_1, \\ iv_{2,t_2} = v_{2,xx} - 2\delta_1\delta_2 v_1^2 w_2 + 4v_1v_2w_1 + 2v_2^2w_2, \\ iw_{1,t_2} = -w_{1,xx} - 2v_1w_1^2 + 2\delta_1\delta_2 v_1 w_2^2 - 4v_2w_1w_2, \\ iw_{2,t_2} = -w_{2,xx} - 4v_1w_1w_2 + 2\delta_1\delta_2 v_2 w_1^2 - 2v_2w_2^2, \end{cases} \tag{22}$$

and the coupled integrable modified Korteweg-de Vries equations:

$$\begin{cases} v_{1,t_3} = -v_{1,xxx} - 6v_1v_{1,x}w_1 - 6v_1v_{2,x}w_2 + 6\delta_1\delta_2v_2v_{2,x}w_1 - 6v_{1,x}v_2w_2, \\ v_{2,t_3} = -v_{2,xxx} - 6v_1v_{2,x}w_1 + 6\delta_1\delta_2v_1v_{1,x}w_2 - 6v_{1,x}v_2w_1 - 6v_2v_{2,x}w_2, \\ w_{1,t_3} = -w_{1,xxx} - 6v_1w_1w_{1,x} + 6\delta_1\delta_2v_1w_2w_{2,x} - 6v_2w_1w_{2,x} - 6v_2w_{1,x}w_2, \\ w_{2,t_3} = -w_{2,xxx} - 6v_1w_1w_{2,x} - 6v_1w_{1,x}w_2 + 6\delta_1\delta_2v_2w_1w_{1,x} - 6v_2w_2w_{2,x}. \end{cases} \tag{23}$$

These two models extend the set of coupled integrable nonlinear Schrödinger equations and modified Korteweg-de Vries equations.

3. BI-HAMILTONIAN STRUCTURE

To furnish a bi-Hamiltonian structure for the integrable hierarchy (21), we apply the trace identity (6) to the spatial matrix spectral problem (9). Based on the solution Z defined by (11), one can easily work out

$$\text{tr}\big(Z\frac{\partial \mathcal{M}}{\partial \lambda}\big) = 2a,\ \text{tr}\big(Z\frac{\partial \mathcal{M}}{\partial u}\big) = 4(c_1,c_2,b_1,b_2)^T, \tag{24}$$

and consequently, by the trace identity, one has

$$\frac{\delta}{\delta u}\int \lambda^{-(s+1)}a^{[s+1]}\,dx = 2\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}(c_1^{[s]},c_2^{[s]},b_1^{[s]},b_2^{[s]})^T,\ s\ge 0. \tag{25}$$

A check with $s=2$ leads to $\gamma = 0$, and therefore, one obtains

$$\frac{\delta}{\delta u}\mathcal{H}^{[s]} = 2(c_1^{[s+1]},c_2^{[s+1]},b_1^{[s+1]},b_2^{[s+1]})^T,\ s\ge 0, \tag{26}$$

where these Hamiltonian functionals are determined by

$$\mathcal{H}^{[s]} = -\int \frac{a^{[s+2]}}{s+1}\,dx,\ s\ge 0. \tag{27}$$

This allows us to present a Hamiltonian structure for the integrable hierarchy (21):

$$u_{t_r} = X^{[r]} = J\frac{\delta\mathcal{H}^{[r]}}{\delta u},\ J = \left[\begin{array}{cc|cc} 0 & 0 & \frac{1}{2}i & 0 \\ 0 & 0 & 0 & \frac{1}{2}i \\ \hline -\frac{1}{2}i & 0 & 0 & 0 \\ 0 & -\frac{1}{2}i & 0 & 0 \end{array}\right],\ r\ge 0, \tag{28}$$

where J is the Hamiltonian and $\mathcal{H}^{[r]}$ are the functionals given by (27). This Hamiltonian structure also tells a relation $S = J\frac{\delta\mathcal{H}}{\delta u}$ from a conserved functional $\mathcal{H}$ to a symmetry S of the same model. These vector fields satisfy a characteristic property:

$$[\![X^{[s_1]},X^{[s_2]}]\!] = X^{[s_1]\prime}(u)[X^{[s_2]}] - X^{[s_2]\prime}(u)[X^{[s_1]}] = 0,\ s_1,s_2\ge 0, \tag{29}$$

which can bee seen from a Lax operator algebra:

$$[\![\mathcal{N}^{[s_1]},\mathcal{N}^{[s_2]}]\!] = \mathcal{N}^{[s_1]\prime}(u)[X^{[s_2]}] - \mathcal{N}^{[s_2]\prime}(u)[X^{[s_1]}] + [\mathcal{N}^{[s_1]},\mathcal{N}^{[s_2]}] = 0,\ s_1,s_2 \geq 0. \tag{30}$$

This is a direct consequence of the isospectral zero curvature equations (see [20] for details).

On the other hand, from the recursion relation $X^{[r+1]} = \Phi X^{[r]}$, we can compute a hereditary recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$ for the hierarchy (21):

$$\begin{cases} \Phi_{11} = i(-\partial_x - 2v_1\partial^{-1}w_1 - 2v_2\partial^{-1}w_2),\ \Phi_{12} = i(-2v_1\partial^{-1}w_2 + 2\delta_1\delta_2 v_2\partial^{-1}w_1), \\ \Phi_{13} = i(-2v_1\partial^{-1}v_1 + 2\delta_1\delta_2 v_2\partial^{-1}w_2),\ \Phi_{14} = i(-2v_1\partial^{-1}v_2 - 2v_2\partial^{-1}v_1); \end{cases} \tag{31}$$

$$\begin{cases} \Phi_{21} = i(-2v_2\partial^{-1}w_1 + 2\delta_1\delta_2 v_1\partial^{-1}w_2),\ \Phi_{22} = i(-\partial_x - 2v_2\partial^{-1}w_2 - 2v_1\partial^{-1}w_1), \\ \Phi_{23} = i(-2v_2\partial^{-1}v_1 - 2v_1\partial^{-1}v_2),\ \Phi_{24} = i(2v_2\partial^{-1}v_2 + 2\delta_1\delta_2 v_1\partial^{-1}v_1); \end{cases} \tag{32}$$

$$\begin{cases} \Phi_{31} = i(2w_1\partial^{-1}w_1 - 2\delta_1\delta_2 w_2\partial^{-1}w_2),\ \Phi_{32} = i(2w_1\partial^{-1}w_2 + 2w_2\partial^{-1}w_1), \\ \Phi_{33} = i(\partial_x + 2w_1\partial^{-1}v_1 + 2w_2\partial^{-1}v_2),\ \Phi_{34} = i(2w_1\partial^{-1}v_2 - 2\delta_1\delta_2 w_2\partial^{-1}v_1); \end{cases} \tag{33}$$

$$\begin{cases} \Phi_{41} = i(2w_2\partial^{-1}w_1 + 2w_1\partial^{-1}w_2),\ \Phi_{42} = i(2w_2\partial^{-1}w_2 - 2\delta_1\delta_2 w_1\partial^{-1}w_1), \\ \Phi_{43} = i(2w_2\partial^{-1}v_1 - 2\delta_1\delta_2 w_1\partial^{-1}v_2),\ \Phi_{44} = i(\partial_x + 2w_2\partial^{-1}v_2 + 2w_1\partial^{-1}v_1). \end{cases} \tag{34}$$

It is easy to see that the operator $M = \Phi J$ is skew-symmetric, and thus, the Hamiltonian functionals commute under the corresponding Poisson bracket [3]:

$$\{\mathcal{H}^{[s_1]},\mathcal{H}^{[s_2]}\}_J = \int \Big(\frac{\delta \mathcal{H}^{[s_1]}}{\delta u}\Big)^T J \frac{\delta \mathcal{H}^{[s_2]}}{\delta u}\, dx = 0,\ s_1,s_2 \geq 0. \tag{35}$$

Therefore, all models in the hierarchy (21) possess infinitely many commuting symmetries $\{X^{[s]}\}_{s=0}^{\infty}$ and conserved functionals $\{\mathcal{H}^{[s]}\}_{s=0}^{\infty}$.

Finally, for the hierarchy (21), combing J with the hereditary recursion operator Φ [21] yields a bi-Hamiltonian structure:

$$u_{t_r} = X^{[r]} = J\frac{\delta \mathcal{H}^{[r]}}{\delta u} = M\frac{\delta \mathcal{H}^{[r-1]}}{\delta u},\ \ r \geq 1, \tag{36}$$

where J and $M = \Phi J$ constitute a Hamiltonian pair [15]. Consequently, each model in the hierarchy (21) is Liouville integrable and has two Abelian algebras of symmetries and conserved functionals, (29) and (35). Particularly, (22) and (23) present two specific examples of nonlinear Liouville integrable Hamiltonian models.

4. CONCLUDING REMARKS

A Liouville integrable hierarchy of Hamiltonian equations with four potentials has been generated from a specific special matrix spectral problem within the zero curvature formulation. It was crucial to determine a Laurent series solution to the corresponding stationary zero curvature equation. The resulting integrable models possess a bi-Hamiltonian structure, explored by an application of the trace identity to the underlying matrix spectral problem.

It is possible to generalize the considered spatial matrix spectral problem by taking more copies of v_1 and v_2. Another way is to introduce more potentials in a spatial spectral matrix to generate bigger integrable models (see, *e.g.*, [22]). Higher-order integrable models and local integrable reductions of the resulting hierarchy could be worked out as well (see, [23–25] for the case of the matrix Ablowitz-Kaup-Newell-Segur spectral problem).

It should be particularly interesting to explore structures of solitons to the resulting integrable equations by powerful approaches in soliton theory, such as the Riemann-Hilbert technique [26], the Zakharov-Shabat dressing method [27], the Darboux transformation [28, 29] and the determinant approach [30]. Other types of important solutions can be computed from wave number reductions of solitons (see, *e.g.*, [31–34]), and there are also many recent works on the dynamics of different types of localized waveforms in a variety of physical systems (see, *e.g.*, [35–43]). Moreover, nonlocal reduced integrable equations can be generated by conducting nonlocal group reductions of matrix spectral problems (see, *e.g.*, [44, 45]). Understanding the structures of integrable models can lead to the discovery of new types of solitons and other localized waveforms that are spatially confined and exhibit a well-defined shape, and can advance our understanding of the fundamental laws of physics.

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