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A combined Kaup–Newell type integrable hierarchy with four potentials and its bi-Hamiltonian formulation

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This paper aims to introduce a Kaup–Newell type matrix eigenvalue problem with four potentials, based on a specific matrix Lie algebra, and construct its associated Liouville integrable Hamiltonian hierarchy, through the zero curvature formulation. The Liouville integrability of the resulting hierarchy is shown by determining its recursion operator and bi-Hamiltonian formulation. An illustrative example of combined derivative nonlinear Schrödinger equations with two arbitrary constants is explicitly presented.

Keywords: Matrix eigenvalue problem; zero curvature equation; integrable hierarchy; derivate nonlinear Schrödinger equations.

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1. Introduction

Integrable models comes in hierarchies with hereditary recursion operators [1] and they are associated with Lax pairs of matrix eigenvalue problems [2]. Matrix eigenvalue problems can also be used to establish inverse scattering transforms, which solve Cauchy problems, and Hamiltonian structures, which connect symmetries with conserved quantities. Integrable models have diverse applications in physical sciences and engineering, such as fluid dynamics, nonlinear optics and quantum mechanics.

Among well-known examples of integrable hierarchies are the Ablowitz–Kaup–Newell–Segur hierarchy [3] and its various hierarchies of integrable couplings [5]. Matrix Lie algebras provide a strong basis for studying integrable models through the zero curvature formulation [4–6]. It is always intriguing to see what kind of Lax pairs will engender integrable models. In this paper, we would like to propose a novel 4×4 matrix eigenvalue problem and compute an associated integrable hierarchy, on the basis of a specific matrix Lie algebra.

The zero curvature formulation paves the way for constructing integrable models (see [6, 7] for details). As usual, we denote a column potential vector by $u = (u_1, \ldots, u_q)^T$ and the spectral parameter by λ. A matrix F_0 in a given loop matrix algebra $\tilde{g}$ with the loop parameter λ is called to be pseudo-regular, if it satisfies

$$\mathrm{Im}\,\mathrm{ad}_{F_0} \oplus \mathrm{Ker}\,\mathrm{ad}_{F_0} = \tilde{g}, \quad [\mathrm{Ker}\,\mathrm{ad}_{F_0}, \mathrm{Ker}\,\mathrm{ad}_{F_0}] = 0, \tag{1.1}$$

where ad_{F_0} denotes the adjoint action of F_0 on $\tilde{g}$. We take one pseudo-regular matrix F_0 and q linear independent matrices $F_1, \ldots, F_q$ in $\tilde{g}$ to formulate a spatial spectral matrix:

$$\mathcal{M} = \mathcal{M}(u, \lambda) = F_0(\lambda) + u_1 F_1(\lambda) + \cdots + u_q F_q(\lambda). \tag{1.2}$$

Then try to determine a Laurent series solution $Y = \sum_{n\geq 0} \lambda^{-n} Y^{[n]}$ to the stationary zero curvature equation

$$Y_x = [\mathcal{M}, Y], \tag{1.3}$$

in the underlying loop algebra $\tilde{g}$.

The next step is to determine an infinite sequence of temporal spectral matrices

$$\mathcal{N}^{[m]} = (\lambda^m Y)_+ + \Delta_m = \sum_{n=0}^{m} \lambda^{m-n} Y^{[n]} + \Delta_m, \quad m \geq 0, \tag{1.4}$$

where $\Delta_m \in \tilde{g}$, $m \geq 0$, which provide the other parts of Lax pairs, such that the zero curvature equations:

$$\mathcal{M}_{t_m} - \mathcal{N}^{[m]}_x + [\mathcal{M}, \mathcal{N}^{[m]}] = 0, \quad m \geq 0, \tag{1.5}$$

produce a hierarchy of integrable models:

$$u_{t_m} = X^{[m]} = X^{[m]}(u), \quad m \geq 0. \tag{1.6}$$

The equations in (1.5) actually represent the solvability conditions of the spatial and temporal matrix eigenvalue problems:

$$\varphi_x = \mathcal{M}\varphi, \quad \varphi_{t_m} = \mathcal{N}^{[m]}\varphi, \quad m \geq 0. \tag{1.7}$$

During this process, one often needs the trial and error strategy.

The last step is to find a bi-Hamiltonian formulation for the resulting hierarchy (1.6), via computing a recursion operator and applying the so-called trace identity:

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(Y\frac{\partial \mathcal{M}}{\partial \lambda}\right)dx = \lambda^{-\kappa}\frac{\partial}{\partial \lambda}\lambda^{\kappa}\operatorname{tr}\left(Y\frac{\partial \mathcal{M}}{\partial u}\right), \tag{1.8}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to u, and κ is a constant, which is independent of the spectral parameter λ. It then follows that every equation in the hierarchy has a bi-Hamiltonian formulation and thus Liouville integrability (see, e.g., [6]).

There exist various hierarchies of Liouville integrable models, studied in the literature [3–18]. One-component integrable hierarchies include the Korteg–de Vries hierarchy, the nonlinear Schrödinger hierarchy and the modified Korteg–de Vries hierarchy [1]. The case of two components is very popular and the well-known examples include the Ablowitz–Kaup–Newell–Segur integrable hierarchy [3], the Heisenberg hintegrable hierarchy [19], the Kaup–Newell integrable hierarchy [20] and the Wadati–Konno–Ichikawa integrable hierarchy [21]. All those hierarchies are associated with 2×2 spectral matrices. The case of higher-order spectral matrices has a high degree of difficulty.

In this paper, we aim to propose a specific 4×4 spectral matrix and construct a hierarchy of four-component Liouville integrable models through the zero curvature formulation, on the basis of a special matrix Lie algebra. A recursion operator and a bi-Hamiltonian formulation are determined to show the Liouville integrability for the resulting hierarchy. An illustrative example, consisting of generalized combined integrable nonlinear Schrödinger equations, is presented. A conclusion and a few concluding remarks are given in the final section.

2. A Four-Component Integrable Hierarchy

Let δ be an arbitrary real number, and T be a square matrix of order $r\in\mathbb{N}$ such that

$$T^2 = I_r, \tag{2.1}$$

where I_r denotes the identity matrix of order r. We define a set $\tilde{g}$ of block matrices to be

$$\tilde{g} = \left\{ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}_{2r\times 2r} \middle| \, A_4 = TA_1T^{-1},\ A_3 = \delta T A_2 T^{-1} \right\}. \tag{2.2}$$

Obviously, this forms a matrix Lie algebra under the matrix commutator $[A,B] = AB - BA$. We will use this Lie algebra with $r=2$, $\delta=1$ and

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \tag{2.3}$$

to formulate a specific spectral matrix below.

Let α_1 and α_2 be two arbitrary real numbers, and $u = u(x,t) = (u_1, u_2, u_3, u_4)^T$, a column vector with four potentials. Assume that

$$\alpha = \alpha_1 - \alpha_2 \neq 0. \tag{2.4}$$

Based on recent studies on matrix eigenvalue problems involving four potentials (see, e.g., [22–26] for examples of matrix eigenvalue problems of both fourth-order and higher-order), we would like to introduce a matrix eigenvalue problem of the form:

$$\begin{aligned} &\varphi_x = \mathcal{M}\varphi = \mathcal{M}(u,\lambda)\varphi, \\ &\mathcal{M} = \begin{bmatrix} \alpha_1\lambda^2 & \lambda u_1 & \lambda u_2 & 0 \\ \lambda u_3 & \alpha_2\lambda^2 & 0 & \lambda u_4 \\ \lambda u_4 & 0 & \alpha_2\lambda^2 & \lambda u_3 \\ 0 & \lambda u_2 & \lambda u_1 & \alpha_1\lambda^2 \end{bmatrix}, \end{aligned} \tag{2.5}$$

where λ is again the spectral parameter. This spectral matrix $\mathcal{M}$ is built from the matrix Lie algebra $\tilde{g}$, mentioned previously, and it is a generalization of the Kaup–Newell eigenvalue problem [20]. Interestingly, starting from this eigenvalue problem, an associated integrable hierarchy of bi-Hamiltonian equations can be generated. All equations in the hierarchy possess particular combined structures.

To construct an associated integrable hierarchy, we usually start out on solving the corresponding stationary zero curvature equation (1.3). Let us take

$$Y = \begin{bmatrix} a & b & e & f \\ c & -a & -f & g \\ g & -f & -a & c \\ f & e & b & a \end{bmatrix} = \sum_{n\geq 0} \lambda^{-n} Y^{[n]}. \tag{2.6}$$

The reason to take this form is that with $\mathcal{M}$ in (2.5), an arbitrary matrix in $\tilde{g}$ will lead to a commutator matrix of the above mentioned form. At this moment, the corresponding stationary zero curvature equation (1.3) leads equivalently to

$$\begin{cases} a_x = \lambda c u_1 + \lambda g u_2 - \lambda b u_3 - \lambda e u_4, \\ b_x = \alpha\lambda^2 b - 2\lambda a u_1 - 2\lambda f u_2, \\ c_x = -\alpha\lambda^2 c + 2\lambda a u_3 + 2\lambda f u_4, \end{cases} \tag{2.7}$$

$$\begin{cases} e_x = \alpha\lambda^2 e - 2\lambda f u_1 - 2\lambda f u_1 - 2\lambda a u_2, \\ g_x = -\alpha\lambda^2 g + 2\lambda f u_3 + 2\lambda a u_4, \\ f_x = \lambda g u_1 + \lambda c u_2 - \lambda e u_3 - \lambda b u_4. \end{cases} \tag{2.8}$$

Based on these equations, the basic objects of Y are assumed to be given as follows:

$$\begin{cases} a = \sum_{n\geq 0} \lambda^{-2n} a^{[n]}, & b = \sum_{n\geq 0} \lambda^{-2n-1} b^{[n]}, \quad c = \sum_{n\geq 0} \lambda^{-2n-1} c^{[n]}, \\ e = \sum_{n\geq 0} \lambda^{-2n-1} e^{[n]}, & f = \sum_{n\geq 0} \lambda^{-2n} f^{[n]}, \quad g = \sum_{n\geq 0} \lambda^{-2n-1} g^{[n]}. \end{cases} \tag{2.9}$$

Obviously, we can have two important equations:

$$\begin{cases} -\alpha\lambda a_x = u_3 b_x + u_1 c_x + u_4 e_x + u_2 g_x, \\ -\alpha\lambda f_x = u_4 b_x + u_2 c_x + u_3 e_x + u_1 g_x, \end{cases} \tag{2.10}$$

which enable us to get the recursion relations for determining the solution Y. Therefore, we can see that the above equations in (2.7) and (2.8) yield the two initial equations

$$\begin{cases} a_x^{[0]} = u_1 c^{[0]} + u_2 g^{[0]} - u_3 b^{[0]} - u_4 e^{[0]}, \\ f_x^{[0]} = u_1 g^{[0]} + u_2 c^{[0]} - u_3 e^{[0]} - u_4 b^{[0]} \end{cases} \tag{2.11}$$

and the recursion relations which determine the Laurent series solution:

$$\begin{cases} a_x^{[n+1]} = -\dfrac{1}{\alpha}(u_3 b_x^{[n]} + u_1 c_x^{[n]} + u_4 e_x^{[n]} + u_2 g_x^{[n]}), \\ f_x^{[n+1]} = -\dfrac{1}{\alpha}(u_4 b_x^{[n]} + u_2 c_x^{[n]} + u_3 e_x^{[n]} + u_1 g_x^{[n]}), \end{cases} \tag{2.12}$$

$$\begin{cases} b^{[n+1]} = \dfrac{1}{\alpha}(b_x^{[n]} + 2u_1 a^{[n+1]} + 2u_2 f^{[n+1]}), \\ c^{[n+1]} = \dfrac{1}{\alpha}(-c_x^{[n]} + 2u_3 a^{[n+1]} + 2u_4 f^{[n+1]}), \end{cases} \tag{2.13}$$

$$\begin{cases} e^{[n+1]} = \dfrac{1}{\alpha}(e_x^{[n]} + 2u_1 f^{[n+1]} + 2u_2 a^{[n+1]}), \\ g^{[n+1]} = \dfrac{1}{\alpha}(-g_x^{[n]} + 2u_3 f^{[n+1]} + 2u_4 a^{[n+1]}), \end{cases} \tag{2.14}$$

where $n \geq 0$. To achieve the uniqueness of the Laurent series solution, we just need to fix the initial data,

$$\begin{cases} b^{[0]} = \beta u_1 + \gamma u_2, & c^{[0]} = \beta u_3 + \gamma u_4, \\ e^{[0]} = \beta u_2 + \gamma u_1, & g^{[0]} = \beta u_4 + \gamma u_3, \\ a^{[0]} = \text{const.}, & f^{[0]} = \text{const.}, \end{cases} \tag{2.15}$$

where β and γ are two arbitrary constants, and select the constants of integration to be zero,

$$a^{[n]}|_{u=0} = 0, \quad f^{[n]}|_{u=0} = 0, \quad n \geq 1. \tag{2.16}$$

The initial values for $a^{[0]}$ and $f^{[0]}$ don't affect all other coefficients in the Laurent series solution, but the two constants β and γ bring the diversity of associated integrable models. One can now work out that

$$\begin{cases} a^{[1]} = -\dfrac{1}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2], \\ f^{[1]} = -\dfrac{1}{\alpha}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2], \end{cases}$$

$$\begin{cases} b^{[1]} = \dfrac{1}{\alpha}\Big\{\beta u_{1,x} + \gamma u_{2,x} - \dfrac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_1 \\ \qquad -\dfrac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_2\Big\}, \\ c^{[1]} = \dfrac{1}{\alpha}\Big\{-\beta u_{3,x} - \gamma u_{4,x} - \dfrac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_3 \\ \qquad -\dfrac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_4\Big\}, \end{cases}$$

$$\begin{cases} e^{[1]} = \dfrac{1}{\alpha}\Big\{\gamma u_{1,x} + \beta u_{2,x} - \dfrac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_1 \\ \qquad -\dfrac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_2\Big\}, \\ g^{[1]} = \dfrac{1}{\alpha}\Big\{-\gamma u_{3,x} - \beta u_{4,x} - \dfrac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_3 \\ \qquad -\dfrac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_4\Big\}. \end{cases}$$

Upon observing the above recursion relations, one can introduce the temporal matrix eigenvalue problems:

$$\varphi_{t_m} = \mathcal{N}^{[m]}\varphi = \mathcal{N}^{[m]}(u,\lambda)\varphi, \quad \mathcal{N}^{[m]} = \lambda(\lambda^{2m+1}Y)_+, \quad m \geq 0, \tag{2.17}$$

where the subscript $+$ denotes the polynomial part of λ. The solvability conditions of the spatial and temporal matrix eigenvalue problems in (2.5) and (2.17) are the zero curvature equations in (1.5). All those equations engender a hierarchy of integrable models with four potentials:

$$u_{t_m} = X^{[m]} = X^{[m]}(u) = (b_x^{[m]}, e_x^{[m]}, c_x^{[m]}, g_x^{[m]})^T, \quad m \geq 0, \tag{2.18}$$

or more concretely,

$$u_{1,t_m} = b_x^{[m]}, \quad u_{2,t_m} = e_x^{[m]}, \quad u_{3,t_m} = c_x^{[m]}, \quad u_{4,t_m} = g_x^{[m]}, \quad m \geq 0. \tag{2.19}$$

The first nonlinear example in this hierarchy is the model of combined integrable nonlinear Schrödinger equations:

$$\begin{cases} u_{1,t_1} = \dfrac{1}{\alpha}(\beta u_{1,xx} + \gamma u_{2,xx}) - \dfrac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_1\}_x \\ \qquad - \dfrac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_2\}_x, \\ u_{2,t_1} = \dfrac{1}{\alpha}(\gamma u_{1,xx} + \beta u_{2,xx}) - \dfrac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_1\}_x \\ \qquad - \dfrac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_2\}_x, \\ u_{3,t_1} = -\dfrac{1}{\alpha}(\beta u_{3,xx} + \gamma u_{4,xx}) - \dfrac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_3\}_x \\ \qquad - \dfrac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_4\}_x, \\ u_{4,t_2} = -\dfrac{1}{\alpha}(\gamma u_{3,xx} + \beta u_{4,xx}) - \dfrac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_3\}_x \\ \qquad - \dfrac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_4\}_x. \end{cases} \tag{2.20}$$

This system provides a combined coupled integrable model with four components, which enlarges the category of coupled integrable models of nonlinear Schrödinger equations (see, e.g., [24, 27, 28]). One character is that each equation contains a linear combination of two derivative terms of the second order, and thus, we call them combined models.

Two special subcases, $\beta = 0$ or $\gamma = 0$, in the resulting hierarchy are interesting. They produce reduced hierarchies of uncombined integrable models.

If one takes $\alpha = \beta = 1$ and $\gamma = 0$ in the model (2.20), one gets a coupled integrable nonlinear Schrödinger model:

$$\begin{cases} u_{1,t_1} = u_{1,xx} - 2[(u_1u_3 + u_2u_4)u_1 + (u_1u_4 + u_2u_3)u_2]_x, \\ u_{2,t_1} = u_{2,xx} - 2[(u_1u_4 + u_2u_3)u_1 + (u_1u_3 + u_2u_4)u_2]_x, \\ u_{3,t_1} = -u_{3,xx} - 2[(u_1u_3 + u_2u_4)u_3 + (u_1u_4 + u_2u_3)u_4]_x, \\ u_{4,t_1} = -u_{4,xx} - 2[(u_1u_4 + u_2u_3)u_3 + (u_1u_3 + u_2u_4)u_4]_x. \end{cases} \tag{2.21}$$

If one takes $\alpha = \gamma = 1$ and $\beta = 0$ in the model (2.20), one obtains another coupled integrable nonlinear Schrödinger model:

$$\begin{cases} u_{1,t_1} = u_{2,xx} - 2[(u_1u_4 + u_2u_3)u_1 + (u_1u_3 + u_2u_4)u_2]_x, \\ u_{2,t_1} = u_{1,xx} - 2[(u_1u_3 + u_2u_4)u_1 + (u_1u_4 + u_2u_3)u_2]_x, \\ u_{3,t_1} = -u_{4,xx} - 2[(u_1u_4 + u_2u_3)u_3 + (u_1u_3 + u_2u_4)u_4]_x, \\ u_{4,t_1} = -u_{3,xx} - 2[(u_1u_3 + u_2u_4)u_3 + (u_1u_4 + u_2u_3)u_4]_x. \end{cases} \tag{2.22}$$

There is an interesting character that the resulting two models just exchange the first component with the second component and the third component with the fourth component in the vector fields on the right-hand sides.

3. Recursion Operator and Bi-Hamiltonian Formulation

To propose a Hamiltonian formulation to exhibit the Liouville integrability for the soliton hierarchy (2.19), we can make use of the trace identity (1.8) in the case of the spatial matrix eigenvalue problem (2.5). Noting the expression of the Laurent series solution Y by (2.6), we can readily work out

$$\begin{cases} \mathrm{tr}\left(Y\dfrac{\partial \mathcal{M}}{\partial \lambda}\right) = 2(2\alpha\lambda a + bu_3 + cu_1 + eu_4 + gu_2), \\ \mathrm{tr}\left(Y\dfrac{\partial \mathcal{M}}{\partial u}\right) = 2(\lambda c, \lambda g, \lambda b, \lambda e)^T \end{cases} \tag{3.1}$$

and consequently, an application of the trace identity yields

$$\begin{aligned} &\frac{\delta}{\delta u}\int \lambda^{-2n-1}(2\alpha a^{[n+1]} + u_3 b^{[n]} + u_4 e^{[n]} + u_1 c^{[n]} + u_2 g^{[n]})dx \\ &\quad = \lambda^{-\kappa}\frac{\partial}{\partial \lambda}\lambda^{\kappa-2n}(c^{[n]}, g^{[n]}, b^{[n]}, e^{[n]})^T, \quad n \geq 0. \end{aligned} \tag{3.2}$$

Checking with $n = 1$ tells $\kappa = 0$, and accordingly, one obtains

$$\frac{\delta}{\delta u}\mathcal{H}^{[n]} = (c^{[n]}, g^{[n]}, b^{[n]}, e^{[n]})^T, \quad n \geq 0, \tag{3.3}$$

where the Hamiltonian functionals are determined by

$$\begin{cases} \mathcal{H}^{[0]} = \displaystyle\int \frac{1}{2}[u_1(\beta u_3 + \gamma u_4) + u_2(\beta u_4 + \gamma u_3) + u_3(\beta u_1 + \gamma u_2) \\ \qquad\quad +\ u_4(\beta u_2 + \gamma u_1)]dx, \\ \mathcal{H}^{[n]} = -\displaystyle\int \frac{1}{2n}(2\alpha a^{[n+1]} + u_3 b^{[n]} + u_1 c^{[n]} + u_4 e^{[n]} + u_2 g^{[n]})dx, \quad n \geq 1. \end{cases} \tag{3.4}$$

This enables us to produce a Hamiltonian formulation for the hierarchy (2.19):

$$u_{t_m} = X^{[m]} = J_1\frac{\delta \mathcal{H}^{[m]}}{\delta u}, \quad m \geq 0, \tag{3.5}$$

where the Hamiltonian operator J_1 is given by

$$J_1 = \left[\begin{array}{cc|cc} & & \partial & 0 \\ \multicolumn{2}{c|}{0} & 0 & \partial \\ \hline \partial & 0 & & \\ 0 & \partial & \multicolumn{2}{c}{0} \end{array}\right] \tag{3.6}$$

and the functionals $\mathcal{H}^{[m]}$ are defined by (3.4). As a consequence, we have an inter-relation $S = J_1 \frac{\delta \mathcal{H}}{\delta u}$ between a symmetry S and a conserved functional $\mathcal{H}$ of each model in the hierarchy.

The characteristic commutative property for the vector fields $X^{[n]}$

$$[\![X^{[n_1]}, X^{[n_2]}]\!] = X^{[n_1]\prime}(u)[X^{[n_2]}] - X^{[n_2]\prime}(u)[X^{[n_1]}] = 0, \quad n_1, n_2 \geq 0, \tag{3.7}$$

follows from an algebra of Lax operators:

$$\begin{aligned} [\![\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}]\!] &= \mathcal{N}^{[n_1]\prime}(u)[X^{[n_2]}] - \mathcal{N}^{[n_2]\prime}(u)[X^{[n_1]}] \\ &\quad + [\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}] = 0, \quad n_1, n_2 \geq 0. \end{aligned} \tag{3.8}$$

This can directly be verified by analyzing the relation between the isospectral zero curvature equations (see [31] for details).

On the other hand, from the recursion relation $X^{[m+1]} = \Phi X^{[m]}$, we can compute a hereditary recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$ [29] for the hierarchy (2.19), which reads as follows:

$$\begin{cases} \Phi_{11} = \dfrac{1}{\alpha}\partial_x - \dfrac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_3 + \partial u_2 \partial^{-1} u_4), \\ \Phi_{12} = -\dfrac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_4 + \partial u_2 \partial^{-1} u_3), \\ \Phi_{13} = -\dfrac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_1 + \partial u_2 \partial^{-1} u_2), \\ \Phi_{14} = -\dfrac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_2 + \partial u_2 \partial^{-1} u_1); \end{cases} \tag{3.9}$$

$$\begin{cases} \Phi_{21} = -\dfrac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_4 + \partial u_2 \partial^{-1} u_3), \\ \Phi_{22} = \dfrac{1}{\alpha}\partial_x - \dfrac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_3 + \partial u_2 \partial^{-1} u_4), \\ \Phi_{23} = -\dfrac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_2 + \partial u_2 \partial^{-1} u_1), \\ \Phi_{24} = -\dfrac{2}{\alpha^2}(\partial u_1 \partial^{-1} u_1 + \partial u_2 \partial^{-1} u_2); \end{cases} \tag{3.10}$$

$$\begin{cases} \Phi_{31} = -\dfrac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_3 + \partial u_4 \partial^{-1} u_4), \\ \Phi_{32} = -\dfrac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_4 + \partial u_4 \partial^{-1} u_3), \\ \Phi_{33} = -\dfrac{1}{\alpha}\partial_x - \dfrac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_1 + \partial u_4 \partial^{-1} u_2), \\ \Phi_{34} = -\dfrac{2}{\alpha^2}(\partial u_3 \partial^{-1} u_2 + \partial u_4 \partial^{-1} u_1); \end{cases} \tag{3.11}$$

$$\begin{cases} \Phi_{41} = -\dfrac{2}{\alpha^2}(\partial u_3\partial^{-1}u_4 + \partial u_4\partial^{-1}u_3), \\ \Phi_{42} = -\dfrac{2}{\alpha^2}(\partial u_3\partial^{-1}u_3 + \partial u_4\partial^{-1}u_4), \\ \Phi_{43} = -\dfrac{2}{\alpha^2}(\partial u_3\partial^{-1}u_2 + \partial u_4\partial^{-1}u_1), \\ \Phi_{44} = -\dfrac{1}{\alpha}\partial_x - \dfrac{2}{\alpha^2}(\partial u_3\partial^{-1}u_1 + \partial u_4\partial^{-1}u_2). \end{cases} \tag{3.12}$$

With some direct analysis, we can see that J_1 and $J_2 = \Phi J_1$ constitute a Hamiltonian pair. Namely, an arbitrary linear combination of J_1 and J_2 is again Hamiltonian. Accordingly, the hierarchy (2.19) possesses a bi-Hamiltonian structure [30]:

$$u_{t_m} = X^{[m]} = J_1\frac{\delta\mathcal{H}^{[m]}}{\delta u} = J_2\frac{\delta\mathcal{H}^{[m-1]}}{\delta u}, \quad m \geq 1. \tag{3.13}$$

It then follows that the associated Hamiltonian functionals commute with each other under the corresponding two Poisson brackets [6]:

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_1} = \int \left(\frac{\delta\mathcal{H}^{[n_1]}}{\delta u}\right)^T J_1\frac{\delta\mathcal{H}^{[n_2]}}{\delta u}\,dx = 0, \quad n_1, n_2 \geq 0 \tag{3.14}$$

and

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_2} = \int \left(\frac{\delta\mathcal{H}^{[n_1]}}{\delta p}\right)^T J_2\frac{\delta\mathcal{H}^{[n_2]}}{\delta u}\,dx = 0, \quad n_1, n_2 \geq 0. \tag{3.15}$$

The bi-Hamiltonian formulation also implies the hereditary property of the recursion operator Φ and thus Φ is a common recursion operator for the integrable hierarchy (2.19).

To conclude, each model in the hierarchy (2.19) is Liouville integrable and possesses infinitely many commuting symmetries $\{X^{[n]}\}_{n=0}^{\infty}$ and conserved functionals $\{\mathcal{H}^{[n]}\}_{n=0}^{\infty}$. One particular illustrative integrable model is the system in (2.20), which adds to the existing category of nonlinear combined Liouville integrable Hamiltonian models with four components.

4. Concluding Remarks

From a specific special 4×4 matrix eigenvalue problem, a hierarchy of four-component Liouville integrable models has been generated through the zero curvature formulation. The crucial step is to determine a particular Laurent series solution of the corresponding stationary zero curvature equation. The resulting integrable hierarchy has been shown to be bi-Hamiltonian by determining a hereditary recursion operator and applying the trace identity in the case the underlying matrix eigenvalue problem.

We are curious to know what kind of mathematical structures of soliton solutions there exist for the obtained integrable models. Various powerful and effective approaches are available for use, which include the Riemann–Hilbert technique [32], the Zakharov–Shabat dressing method [33], the Darboux transformation [34–36], and the determinant approach [37]. In addition to solitons, lump, kink, breather and rogue wave solutions, particularly their interaction solutions (see, e.g., [38–45]), are also interesting, and it is possible to compute them from soliton solutions by taking wave number reductions. Another important aspect of the study of integrable models is to get nonlocal reduced integrable models by conducting nonlocal group reductions or similarity transformations of matrix eigenvalue problems, and to explore their solitons, which are significant in mathematics as well as physics (see, e.g., [46–48]). Nonlocality reveals a diverse array of novel phenomena and solutions (see, e.g., [49, 50]).

Integrable models are of great interest, and they are built around connections of all kinds that bridge mathematics with the real world problems. The study of integrable models offers insights into universal behaviors across different physical system scenarios, and underpins the fundamental understanding of complex nonlinear mathematical and physical phenomena.

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References

[1] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, 1981).
[2] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968) 467–490.
[3] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform-Fourier Analysis for nonlinear problems, *Stud. Appl. Math.* **53** (1974) 249–315.
[4] V. Drinfeld and V. V. Sokolov, Lie algebras and equations of Kortewegde Vries type, *Sov. J. Math.* **30** (1985) 1975–2036.
[5] W. X. Ma, Integrable couplings and matrix loop algebras, in *Nonlinear and Modern Mathematical Physics*, eds. W. X. Ma and D. Kaup, AIP Conference Proceedings, Vol. 1562 (American Institute of Physics, Melville, New York, 2013), pp. 105–122.
[6] G. Z. Tu, On Liouville integrability of zero-curvature equations and the Yang hierarchy, *J. Phys. A: Math. Gen.* **22** (1989) 2375–2392.
[7] W. X. Ma, A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction, *Chin. Ann. Math. Ser. A* **13** (1992) 115–123.

[8] W. X. Ma, Novel integrable Hamiltonian hierarchies with six potentials, *Acta. Math. Sci.* **44B** (2024) 2498–2508.
[9] M. Antonowicz and A. P. Fordy, Coupled KdV equations with multi-Hamiltonian structures, *Phys. D* **28** (1987) 345–357.
[10] T. C. Xia, F. J. Yu and Y. Zhang, The multi-component coupled Burgers hierarchy of soliton equations and its multi-component integrable couplings system with two arbitrary functions, *Phys. A* **343** (2004) 238–246.
[11] W. X. Ma, An extended AKNS eigenvalue problem and its affiliated integrable Hamiltonian hierarchies, *Chaos Solitons Fractals* **188** (2024) 115580.
[12] W. X. Ma, Reduced AKNS spectral problems and associated complex matrix integrable models, *Acta Appl. Math.* **187** (2023) 17.
[13] W. X. Ma, Integrable couplings and two-dimensional unital algebras, *Axioms* **13** (2024) 481.
[14] S. Manukure, Finite-dimensional Liouville integrable Hamiltonian systems generated from Lax pairs of a bi-Hamiltonian soliton hierarchy by symmetry constraints, *Commun. Nonlinear Sci. Numer. Simul.* **57** (2018) 125–135.
[15] H. F. Wang and Y. F. Zhang, Application of Riemann–Hilbert method to an extended coupled nonlinear Schrödinger equations, *J. Comput. Appl. Math.* **420** (2023) 114812.
[16] V. S. Gerdjikov, Nonlinear evolution equations related to Kac–Moody algebras $A_r^{(1)}$: spectral aspects, *Turkish J. Math.* **46** (2022) 1828–1844.
[17] W. X. Ma, AKNS type reduced integrable hierarchies with Hamiltonian formulations, *Rom. J. Phys.* **68** (2023) 116.
[18] W. X. Ma, A matrix integrable Hamiltonian hierarchy and its two integrable reductions, *Int. J. Geom. Methods Mod. Phys.* **21** (2024) 2450293.
[19] L. A. Takhtajan, Integration of the continuous Heisenberg spin chain through the inverse scattering method, *Phys. Lett. A* **64** (1977) 235–237.
[20] D. J. Kaup and A. C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, *J. Math. Phys.* **19** (1978) 798–801.
[21] M. Wadati, K. Konno and Y. H. Ichikawa, New integrable nonlinear evolution equations, *J. Phys. Soc. Japan* **47** (1979) 1698–1700.
[22] W. X. Ma, A combined derivative nonlinear Schrödinger soliton hierarchy, *Rep. Math. Phys.* **93** (2024) 313–325.
[23] W. X. Ma, Four-component integrable hierarchies and their Hamiltonian structures, *Commun. Nonlinear Sci. Numer. Simul.* **126** (2023) 107460.
[24] W. X. Ma, A combined Liouville integrable hierarchy associated with a fourth-order matrix spectral problem, *Commun. Theor. Phys.* **76** (2024) 107460.
[25] Y. F. Zhang, A few expanding integrable models, Hamiltonian structures and constrained flows, *Commun. Theor. Phys.* **55** (2011) 273–290.
[26] Zhaqilao, A generalized AKNS hierarchy, bi-Hamiltonian structure, and Darboux transformation, *Commun. Nonlinear Sci. Numer. Simul.* **17** (2012) 2319–2332.
[27] W. X. Ma, Novel Liouville integrable Hamiltonian models with six components and three signs, *Chin. J. Phys.* **86** (2023) 292–299.
[28] W. X. Ma, Four-component combined integrable equations possessing bi-Hamiltonian formulations, *Mod. Phys. Lett. B* **38** (2024) 2450319.
[29] B. Fuchssteiner and A. S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Phys. D* **4** (1981) 47–66.
[30] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.* **19** (1978) 1156–1162.
[31] W. X. Ma, The algebraic structure of zero curvature representation and application to coupled KdV systems, *J. Phys. A: Math. Gen.* **26** (1993) 2573–2582.

[32] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons: the Inverse Scattering Method* (Consultantn Bureau, 1984).

[33] E. V. Doktorov and S. B. Leble, *A Dressing Method in Mathematical Physics* (Springer, 2007).

[34] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer-Verlag, 1991).

[35] X. G. Geng, R. M. Li and B. Xue, A vector general nonlinear Schrödinger equation with $(m+n)$ components, *J. Nonlinear Sci.* **30** (2020) 991–1013.

[36] R. S. Ye and Y. Zhang, A vectorial Darboux transformation for the Fokas–Lenells system, *Chaos Solitons Fractals* **169** (2023) 113233.

[37] T. Aktosun, T. Busse, F. Demontis and C. van der Mee, Symmetries for exact solutions to the nonlinear Schrödinger equation, *J. Phys. A: Math. Theor.* **43** (2010) 025202.

[38] L. Cheng, Y. Zhang and M. J. Lin, Lax pair and lump solutions for the $(2+1)$-dimensional DJKM equation associated with bilinear Bäcklund transformations, *Anal. Math. Phys.* **9** (2019) 1741–1752.

[39] T. A. Sulaiman, A. Yusuf, A. Abdeljabbar and M. Alquran, Dynamics of lump collision phenomena to the $(3+1)$-dimensional nonlinear evolution equation, *J. Geom. Phys.* **169** (2021) 104347.

[40] A. Yusuf, T. A. Sulaiman, A. Abdeljabbam and M. Alquran, Breathem waves, analytical solutions and conservation lawn using LieBäcklund symmetries to the $(2+1)$-dimensional Chaffee-Infante equation, *J. Ocean Eng. Sci.* **8** (2023) 145–151.

[41] S. Manukure, A. Chowdhury and Y. Zhou, Complexiton solutions to the asymmetric Nizhnik–Novikov–Veselov equation, *Int. J. Mod. Phys. B* **33** (2019) 1950098.

[42] Y. Zhou, S. Manukure and M. McAnally, Lump and rogue wave solutions to a $(2+1)$-dimensional Boussinesq type equation, *J. Geom. Phys.* **167** (2021) 104275.

[43] W. X. Ma, Lump waves in a spatial symmetric nonlinear dispersive wave model in $(2+1)$-dimensions, *Mathematics* **11** (2023) 4664.

[44] S. Manukure and Y. Zhou, A study of lump and line rogue wave solutions to a $(2+1)$-dimensional nonlinear equation, *J. Geom. Phys.* **167** (2021) 104274.

[45] S. X. Yang, Y. F. Wang and X. Zhang, Conservation laws, Darboux transformation and localized waves for the N-coupled nonautonomous Gross-Pitaevskii equations in the Bose-Einstein condensates, *Chaos Solitons Fractals* **169** (2023) 113272.

[46] W. X. Ma, Type (λ^*, λ) reduced nonlocal integrable AKNS equations and their soliton solutions, *Appl. Numer. Math.*, **199** (2024) 105–113.

[47] W. X. Ma, Darboux transformation of vector nonlocal reverse-space nonlinear Schrödinger equations, *Int. J. Geom. Methods Mod. Phys.* **21** (2024) 2450182.

[48] W. X. Ma, Nonlocal integrable equations in soliton theory, in *Nonlinear and Modern Mathematical Physics*, eds. S. Manukure and W. X. Ma (Springer, 2024), pp. 151–266.

[49] W. X. Ma, General solution to a nonlocal linear differential equation of first-order, *Qual. Theory Dyn. Syst.* **23** (2024) 177.

[50] W. X. Ma, Solving a non-local linear differential equation model of the Newtonian-type, *Pramana — J. Phys.* **98** (2024) 68.