A COMBINED DERIVATIVE NONLINEAR SCHRÖDINGER SOLITON HIERARCHY

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This paper aims to study a Kaup-Newell type matrix eigenvalue problem with four potentials, based on a specific matrix Lie algebra, and construct an associated soliton hierarchy of combined derivative nonlinear Schrödinger (NLS) equations, within the zero curvature formulation. The Liouville integrability of the resulting soliton hierarchy is shown by exploring its hereditary recursion operator and bi-Hamiltonian formulation. The first nonlinear example provides an integrable model consisting of combined derivative NLS equations with two arbitrary constants.

Keywords: matrix eigenvalue problem, zero curvature equation, integrable hierarchy, derivative nonlinear Schrödinger equations.

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1. Introduction

Integrable models are associated with matrix eigenvalue problems [1] and the underlying matrix Lie algebra is the basis [2, 3]. Matrix eigenvalue problems can also be used to establish inverse scattering transforms solving Cauchy problems, and to explore integrable properties, such as infinitely many symmetries and conservation laws [2]. Hamiltonian structures connecting symmetries with conservation laws can be furnished by the so-called trace identity [4]. Integrable models have various applications in physical sciences and engineering, including nonlinear optics, fluid dynamics and quantum mechanics [3].

There are abundant examples of integrable hierarchies, which include the Ablowitz-Kaup-Newell-Segur hierarchy [5] and its diverse hierarchies of integrable couplings [7]. Matrix Lie algebras provide a solid basis for constructing integrable models through the zero curvature formulation [4, 6, 7]. The key step is to find a spectral matrix which can successfully yield an integrable hierarchy. In this paper, we would like to propose a novel 4×4 Kaup-Newell type spectral matrix involving nonzero antidiagonal entries and minus signs, and compute an associated integrable hierarchy, within the zero curvature formulation.

The zero curvature formulation can be formulated as follows (see [4, 8] for details). As usual, let us denote a column potential vector by $u = (u_1, \cdots, u_q)^T$ and the spectral parameter by λ. We take a pseudo-regular element F_0 in a given loop matrix algebra $\tilde{g}$ with the loop parameter λ. The pseudo-regular property here is

$$\mathrm{Im}\,\mathrm{ad}_{F_0} \oplus \mathrm{Ker}\,\mathrm{ad}_{F_0} = \tilde{g}, \quad [\mathrm{Ker}\,\mathrm{ad}_{F_0}, \mathrm{Ker}\,\mathrm{ad}_{F_0}] = 0, \tag{1.1}$$

where ad_{F_0} denotes the adjoint action of F_0 on $\tilde{g}$. With q linear independent matrices $F_1, \cdots, F_q$ in $\tilde{g}$, we formulate a spatial spectral matrix,

$$\mathcal{M} = \mathcal{M}(u, \lambda) = F_0(\lambda) + u_1 F_1(\lambda) + \cdots + u_q F_q(\lambda), \tag{1.2}$$

and determine a Laurent series solution $Y = \sum_{n\geq 0} \lambda^{-n} Y^{[n]}$ to the stationary zero curvature equation

$$Y_x = [\mathcal{M}, Y] \tag{1.3}$$

in the underlying loop algebra $\tilde{g}$.

Next, we introduce an infinite sequence of temporal spectral matrices

$$\mathcal{N}^{[m]} = (\lambda^m Y)_+ + \Delta_m = \sum_{n=0}^{m} \lambda^{m-n} Y^{[n]} + \Delta_m, \quad m \geq 0, \tag{1.4}$$

where $\Delta_m \in \tilde{g}, \quad m \geq 0$, which provide the other parts of Lax pairs, such that the zero curvature equations:

$$\mathcal{M}_{t_m} - \mathcal{N}^{[m]}_x + [\mathcal{M}, \mathcal{N}^{[m]}] = 0, \quad m \geq 0, \tag{1.5}$$

generate an integrable hierarchy:

$$u_{t_m} = X^{[m]} = X^{[m]}(u), \quad m \geq 0, \tag{1.6}$$

which commutes pairwise. The equations in (1.5) are the solvability conditions of the spatial and temporal matrix eigenvalue problems:

$$\varphi_x = \mathcal{M}\varphi, \quad \varphi_{t_m} = \mathcal{N}^{[m]}\varphi, \quad m \geq 0. \tag{1.7}$$

One often needs the trial and error strategy while transforming the zero curvature equations into an integrable hierarchy.

To show the Liouville integrability, we furnish a bi-Hamiltonian formulation for the resulting hierarchy (1.6), via computing a hereditary recursion operator and applying the trace identity

$$\frac{\delta}{\delta u} \int \mathrm{tr}\left(Y \frac{\partial \mathcal{M}}{\partial \lambda}\right) dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa} \mathrm{tr}\left(Y \frac{\partial \mathcal{M}}{\partial u}\right), \tag{1.8}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to u, and κ is a constant, determined by

$$\kappa = -\frac{\lambda}{2} \frac{\partial}{\partial \lambda} \ln |\mathrm{tr}(Y^2)|. \tag{1.9}$$

The bi-Hamiltonian formulation implies the Liouville integrability of the resulting hierarchy (see, e.g., [4, 9]).

There exist various applications of the zero curvature formulation to hierarchies of Liouville integrable models in the literature [5–19]. One-component integrable hierarchies contain the Korteweg-de Vries hierarchy, the nonlinear Schrödinger hierarchy and the modified Korteweg-de Vries hierarchy [2, 3]. The well-known examples with two components include the Ablowitz-Kaup-Newell-Segur integrable hierarchy [5], the Heisenberg integrable hierarchy [20], the Kaup-Newell integrable hierarchy [21] and the Wadati-Konno-Ichikawa integrable hierarchy [22]. All those hierarchies are generated from 2×2 spectral matrices. The case of higher-order spectral matrices involves a high degree of difficulty.

The aim of this paper is to propose a specific 4×4 spectral matrix involving nonzero anti-diagonal entries and minus signs, and construct an associated hierarchy of four-component Liouville integrable models through the zero curvature formulation. A hereditary recursion operator and a bi-Hamiltonian formulation are determined to show the Liouville integrability for the resulting hierarchy. An illustrative example, consisting of generalized combined integrable derivative nonlinear Schrödinger equations, is explicitly presented. The last section are a conclusion and some concluding remarks.

2. An integrable hierarchy with four potentials

The starting point is a special matrix Lie algebra. Let δ be an arbitrary number and T, a square matrix of order $r \in \mathbb{N}$ such that

$$T^{-1} = -T. \tag{2.1}$$

We introduce a set $\tilde{g}$ of block matrices by

$$\tilde{g} = \left\{ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}_{2r \times 2r} \,\middle|\, A_4 = TA_1T^{-1}, \quad A_3 = \delta T A_2 T^{-1} \right\}. \tag{2.2}$$

Obviously, this forms a matrix Lie algebra under the matrix commutator $[A, B] = AB - BA$. We point out that the inclusion of an arbitrary constant in the first condition does not work, i.e. the reduction $A_4 = \sigma T A_1 T^{-1}$, where σ is an arbitrary constant, does not guarantee a matrix Lie algebra. We will use this Lie algebra with $r = 2$, $\delta = 1$ and

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.3}$$

to formulate a specific spectral matrix to generate an integrable hierarchy.

Let $u = u(x, t) = (u_1, u_2, u_3, u_4)^T$ be a column vector with four potentials, and α_1 and α_2, two arbitrary real numbers satisfying

$$\alpha = \alpha_1 + \alpha_2 \neq 0. \tag{2.4}$$

Motivated by recent studies on matrix eigenvalue problems involving four potentials (see, e.g., [23–25] and [26, 27] for examples of matrix eigenvalue problems of arbitrary-order and fourth-order, respectively), we would like to consider a matrix eigenvalue problem of the form

$$\varphi_x = \mathcal{M}\varphi = \mathcal{M}(u,\lambda)\varphi, \quad \mathcal{M} = \begin{bmatrix} 0 & \lambda u_1 & \lambda u_2 & \alpha_1\lambda^2 \\ \lambda u_3 & 0 & \alpha_2\lambda^2 & \lambda u_4 \\ \lambda u_4 & -\alpha_2\lambda^2 & 0 & -\lambda u_3 \\ -\alpha_1\lambda^2 & \lambda u_2 & -\lambda u_1 & 0 \end{bmatrix}, \tag{2.5}$$

where, as usual, λ stands for the spectral parameter. This spectral matrix $\mathcal{M}$ belongs to the above matrix Lie algebra $\tilde{g}$, and it is a kind of 4×4 matrix generalization of the Kaup-Newell eigenvalue problem [21]. In this spectral matrix, the constant terms appear in the antidiagonal, all diagonal entries are zero, and there are four minus signs. Interestingly, beginning with this eigenvalue problem, an associated hierarchy of bi-Hamiltonian integrable models can be generated. All models in the hierarchy possess a particular combined structure.

To generate an associated integrable hierarchy, we first solve the corresponding stationary zero curvature equation (1.3). Let us take

$$Y = \begin{bmatrix} a & b & e & f \\ c & -a & f & g \\ g & -f & -a & -c \\ -f & e & -b & a \end{bmatrix} = \sum_{n\ge 0}\lambda^{-n}Y^{[n]}. \tag{2.6}$$

The reason to take this form is that with the spectral matrix $\mathcal{M}$ in (2.5), an arbitrary matrix in $\tilde{g}$ will yield a commutator matrix of the above form in (2.6). In doing so, the corresponding stationary zero curvature equation (1.3) equivalently engenders

$$\begin{cases} a_x = \lambda c u_1 + \lambda g u_2 - \lambda b u_3 - \lambda e u_4, \\ b_x = \alpha\lambda^2 e - 2\lambda a u_1 - 2\lambda f u_2, \\ c_x = \alpha\lambda^2 g + 2\lambda a u_3 - 2\lambda f u_4, \end{cases} \tag{2.7}$$

$$\begin{cases} e_x = -\alpha\lambda^2 b + 2\lambda f u_1 - 2\lambda a u_2, \\ g_x = -\alpha\lambda^2 c + 2\lambda f u_3 + 2\lambda a u_4, \\ f_x = \lambda g u_1 - \lambda c u_2 + \lambda e u_3 - \lambda b u_4. \end{cases} \tag{2.8}$$

In order to get a solution Y recursively, we assume that the basic objects of Y are taken as follows:

$$\begin{cases} a = \sum_{n\ge0}\lambda^{-2n}a^{[n]}, \quad b = \sum_{n\ge0}\lambda^{-2n-1}b^{[n]}, \quad c = \sum_{n\ge0}\lambda^{-2n-1}c^{[n]}, \\ e = \sum_{n\ge0}\lambda^{-2n-1}e^{[n]}, \quad f = \sum_{n\ge0}\lambda^{-2n}f^{[n]}, \quad g = \sum_{n\ge0}\lambda^{-2n-1}g^{[n]}. \end{cases} \tag{2.9}$$

The two obvious relations

$$\begin{cases} -\alpha\lambda a_x = u_4 b_x - u_2 c_x - u_3 e_x + u_1 g_x, \\ \alpha\lambda f_x = u_3 b_x + u_1 c_x + u_4 e_x + u_2 g_x, \end{cases} \tag{2.10}$$

help us get the required recursion relations. Now by a careful check, we can see that the above equations in (2.7) and (2.8) generate the two initial equations

$$\begin{cases} a_x^{[0]} = u_1 c^{[0]} + u_2 g^{[0]} - u_3 b^{[0]} - u_4 e^{[0]}, \\ f_x^{[0]} = u_1 g^{[0]} - u_2 c^{[0]} + u_3 e^{[0]} - u_4 b^{[0]}, \end{cases} \tag{2.11}$$

and the recursion relations which determine a Laurent series solution:

$$\begin{cases} a_x^{[n+1]} = -\frac{1}{\alpha}(u_4 b_x^{[n]} - u_2 c_x^{[n]} - u_3 e_x^{[n]} + u_1 g_x^{[n]}), \\ f_x^{[n+1]} = \frac{1}{\alpha}(u_3 b_x^{[n]} + u_1 c_x^{[n]} + u_4 e_x^{[n]} + u_2 g_x^{[n]}), \end{cases}$$

$$\begin{cases} b^{[n+1]} = \frac{1}{\alpha}(-e_x^{[n]} + 2u_1 f^{[n+1]} - 2u_2 a^{[n+1]}), \\ c^{[n+1]} = \frac{1}{\alpha}(-g_x^{[n]} + 2u_3 f^{[n+1]} + 2u_4 a^{[n+1]}), \end{cases}$$

$$\begin{cases} e^{[n+1]} = \frac{1}{\alpha}(b_x^{[n]} + 2u_1 a^{[n+1]} + 2u_2 f^{[n+1]}), \\ g^{[n+1]} = \frac{1}{\alpha}(c_x^{[n]} - 2u_3 a^{[n+1]} + 2u_4 f^{[n+1]}), \end{cases}$$

where $n \geq 0$. A simple solution to the initial equations in (2.11) is given by

$$\begin{cases} b^{[0]} = \beta u_1 + \gamma u_2, \quad c^{[0]} = \beta u_3 - \gamma u_4, \\ e^{[0]} = \beta u_2 - \gamma u_1, \quad g^{[0]} = \beta u_4 + \gamma u_3, \\ a^{[0]} = \text{const}, \quad f^{[0]} = \text{const}, \end{cases} \tag{2.12}$$

where β and γ are two arbitrary constants. For brevity, we choose the zero constants of integration,

$$a^{[n]}|_{u=0} = 0, \quad f^{[n]}|_{u=0} = 0, \quad n \geq 1. \tag{2.13}$$

The initial values for $a^{[0]}$ and $f^{[0]}$ do not create any effect on all other coefficients in the Laurent series solution, but the two constants β and γ bring the diversity of associated integrable models, particularly a combined structure. Now, based on (2.12) and (2.13), one can work out that

$$\begin{cases} a^{[1]} = -\frac{1}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2], \\ f^{[1]} = \frac{1}{\alpha}[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2], \end{cases}$$

$$\begin{cases} b^{[1]} = \frac{1}{\alpha}\{\gamma u_{1,x} - \beta u_{2,x} + \frac{2}{\alpha}[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_1 \\ \qquad + \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_2\}, \\ c^{[1]} = \frac{1}{\alpha}\{-\gamma u_{3,x} - \beta u_{4,x} + \frac{2}{\alpha}[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_3 \\ \qquad - \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_4\}, \end{cases}$$

$$\begin{cases} e^{[1]} = \frac{1}{\alpha}\{\beta u_{1,x} + \gamma u_{2,x} - \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_1 \\ \qquad + \frac{2}{\alpha}[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_2\}, \\ g^{[1]} = \frac{1}{\alpha}\{\beta u_{3,x} - \gamma u_{4,x} + \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_3 \\ \qquad + \frac{2}{\alpha}[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_4\}. \end{cases}$$

By a further inspection on the above recursion relations, one can introduce the temporal matrix eigenvalue problems:

$$\varphi_{t_m} = \mathcal{N}^{[m]}\varphi = \mathcal{N}^{[m]}(u,\lambda)\varphi, \quad \mathcal{N}^{[m]} = \lambda(\lambda^{2m+1}Y)_+, \quad m \geq 0, \tag{2.14}$$

where the subscript + stands for the polynomial part of λ. The solvability conditions of the spatial and temporal matrix eigenvalue problems in (2.5) and (2.14), i.e. the zero curvature equations in (1.5), engender a hierarchy of integrable models with four potentials:

$$u_{t_m} = X^{[m]} = X^{[m]}(u) = \left(b_x^{[m]}, e_x^{[m]}, c_x^{[m]}, g_x^{[m]}\right)^T, \quad m \geq 0, \tag{2.15}$$

or more concretely,

$$u_{1,t_m} = b_x^{[m]}, \quad u_{2,t_m} = e_x^{[m]}, \quad u_{3,t_m} = c_x^{[m]}, \quad u_{4,t_m} = g_x^{[m]}, \quad m \geq 0. \tag{2.16}$$

The first nonlinear example in this hierarchy is the model of combined integrable derivative nonlinear Schrödinger equations:

$$\begin{cases} u_{1,t_1} = \frac{1}{\alpha}(\gamma u_{1,xx} - \beta u_{2,xx}) + \frac{2}{\alpha^2}\{[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_1\}_x \\ \qquad + \frac{2}{\alpha^2}\{(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_2\}_x, \\ u_{2,t_1} = \frac{1}{\alpha}(\beta u_{1,xx} + \gamma u_{2,xx}) - \frac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_1\}_x \\ \qquad + \frac{2}{\alpha^2}\{(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_2\}_x, \\ u_{3,t_1} = -\frac{1}{\alpha}(\gamma u_{3,xx} + \beta u_{4,xx}) + \frac{2}{\alpha^2}\{[(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_3\}_x \\ \qquad - \frac{2}{\alpha^2}\{(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_4\}_x, \\ u_{4,t_2} = \frac{1}{\alpha}(\beta u_{3,xx} - \gamma u_{4,xx}) + \frac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \gamma u_4)u_2]u_3\}_x \\ \qquad + \frac{2}{\alpha^2}\{(\beta u_3 - \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_4\}_x. \end{cases} \tag{2.17}$$

This system provides a coupled integrable model with four components, enlarging the category of coupled integrable models of nonlinear Schrödinger type equations (see, e.g., [24, 28, 29]). One characteristic property is that each equation contains a linear combination of two derivative terms of the second order, and thus, we call it a combined model.

Two special cases with $\beta = 0$ and $\gamma = 0$ in the resulting hierarchy are particularly interesting. They produce two reduced hierarchies of uncombined integrable models.

If one first takes $\alpha = \beta = 1$ and $\gamma = 0$ in the model (2.17), one obtains a coupled integrable nonlinear Schrödinger model:

$$\begin{cases} u_{1,t_1} = -u_{2,xx} + 2[(u_1u_3 + u_2u_4)u_1 + (u_1u_4 - u_2u_3)u_2]_x, \\ u_{2,t_1} = u_{1,xx} - 2[(u_1u_4 - u_2u_3)u_1 - (u_1u_3 + u_2u_4)u_2]_x, \\ u_{3,t_1} = -u_{4,xx} + 2[(u_1u_3 + u_2u_4)u_3 - (u_1u_4 - u_2u_3)u_4]_x, \\ u_{4,t_1} = u_{3,xx} + 2[(u_1u_4 - u_2u_3)u_3 + (u_1u_3 + u_2u_4)u_4]_x. \end{cases} \tag{2.18}$$

If one second takes $\alpha = \gamma = 1$ and $\beta = 0$ in the model (2.17), one gets another coupled integrable derivative nonlinear Schrödinger model:

$$\begin{cases} u_{1,t_1} = u_{1,xx} - 2[(u_1u_4 - u_2u_3)u_1 - (u_1u_3 + u_2u_4)u_2]_x, \\ u_{2,t_1} = u_{2,xx} - 2[(u_1u_3 + u_2u_4)u_1 + (u_1u_4 - u_2u_3)u_2]_x, \\ u_{3,t_1} = -u_{3,xx} - 2[(u_1u_4 - u_2u_3)u_3 + (u_1u_3 + u_2u_4)u_4]_x, \\ u_{4,t_1} = -u_{4,xx} + 2[(u_1u_3 + u_2u_4)u_3 - (u_1u_4 - u_2u_3)u_4]_x. \end{cases} \tag{2.19}$$

Checking the vector fields on the right-hand sides, we see an interesting phenomenon that the resulting two reduced models just exchange the first component with the second component and the third component with the fourth component. Surprisingly, those two reduced models still commute with each other.

3. Recursion operator and bi-Hamiltonian formulation

To explore the Liouville integrability of the resulting hierarchy (2.16), we furnish a Hamiltonian formulation by using the trace identity (1.8) in the case of the spatial matrix eigenvalue problem (2.5).

By the expression of the Laurent series solution Y by (2.6), we can readily work out

$$\begin{cases} \operatorname{tr}\left(Y\dfrac{\partial \mathcal{M}}{\partial \lambda}\right) = 2(-2\alpha\lambda f + bu_3 + cu_1 + eu_4 + gu_2), \\ \operatorname{tr}\left(Y\dfrac{\partial \mathcal{M}}{\partial u}\right) = 2(\lambda c, \lambda g, \lambda b, \lambda e)^T, \end{cases} \tag{3.1}$$

and then, an application of the trace identity leads to

$$\frac{\delta}{\delta u}\int \lambda^{-2n-1}(-2\alpha f^{[n+1]}+u_3b^{[n]}+u_4e^{[n]}+u_1c^{[n]}+u_2g^{[n]})\,dx$$
$$=\lambda^{-\kappa}\frac{\partial}{\partial\lambda}\lambda^{\kappa-2n}(c^{[n]},g^{[n]},b^{[n]},e^{[n]})^T,\quad n\geq 0. \tag{3.2}$$

Checking with $n=1$ yields $\kappa=0$, and consequently, one arrives at

$$\frac{\delta}{\delta u}\mathcal{H}^{[n]}=(c^{[n]},g^{[n]},b^{[n]},e^{[n]})^T,\quad n\geq 0, \tag{3.3}$$

where the Hamiltonian functionals are determined by

$$\begin{cases}\mathcal{H}^{[0]}=\int \frac{1}{2}[u_1(\beta u_3-\gamma u_4)+u_2(\beta u_4+\gamma u_3)+u_3(\beta u_1+\gamma u_2)+u_4(\beta u_2-\gamma u_1)]\,dx,\\ \mathcal{H}^{[n]}=\int \frac{1}{2n}(2\alpha f^{[n+1]}-u_3b^{[n]}-u_1c^{[n]}-u_4e^{[n]}-u_2g^{[n]})\,dx,\quad n\geq 1.\end{cases} \tag{3.4}$$

The first Hamiltonian functional above was computed directly. This enables us to establish a Hamiltonian formulation for the hierarchy (2.16),

$$u_{t_m}=X^{[m]}=J_1\frac{\delta\mathcal{H}^{[m]}}{\delta u},\quad m\geq 0, \tag{3.5}$$

where the Hamiltonian operator J_1 is given by

$$J_1=\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \partial & 0\\ & & 0 & \partial\\ \hline \partial & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}}\\ 0 & \partial & & \end{array}\right], \tag{3.6}$$

and the functionals $\mathcal{H}^{[m]}$ are defined by (3.4). As a consequence of this Hamiltonian formulation, we have an interrelation $Z=J_1\frac{\delta\mathcal{H}}{\delta u}$ between a symmetry Z and a conserved functional $\mathcal{H}$ of each model in the hierarchy.

The characteristic commutative property for the vector fields $X^{[n]}$,

$$[\![X^{[n_1]},X^{[n_2]}]\!]=X^{[n_1]\prime}(u)[X^{[n_2]}]-X^{[n_2]\prime}(u)[X^{[n_1]}]=0,\quad n_1,n_2\geq 0, \tag{3.7}$$

follows from an algebra of Lax operators:

$$[\![\mathcal{N}^{[n_1]},\mathcal{N}^{[n_2]}]\!]=\mathcal{N}^{[n_1]\prime}(u)[X^{[n_2]}]-\mathcal{N}^{[n_2]\prime}(u)[X^{[n_1]}]+[\mathcal{N}^{[n_1]},\mathcal{N}^{[n_2]}]$$
$$=0,\quad n_1,n_2\geq 0. \tag{3.8}$$

This can directly be derived from the relation between the isospectral zero curvature equations (see [34, 35] for details).

Furthermore, based on the recursion relation $X^{[m+1]}=\Phi X^{[m]}$, we can compute a hereditary recursion operator $\Phi=(\Phi_{jk})_{4\times 4}$ [30] for the hierarchy (2.16), and the recursion operator Φ is determined by

$$\begin{cases}\Phi_{11}=\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_3+\partial u_2\partial^{-1}u_4), & \Phi_{12}=-\frac{1}{\alpha}\partial_x+\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_4-\partial u_2\partial^{-1}u_3),\\ \Phi_{13}=\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_1-\partial u_2\partial^{-1}u_2), & \Phi_{14}=\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_2+\partial u_2\partial^{-1}u_1);\end{cases} \tag{3.9}$$

$$\begin{cases}\Phi_{21}=\frac{1}{\alpha}\partial_x-\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_4-\partial u_2\partial^{-1}u_3), & \Phi_{22}=\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_3+\partial u_2\partial^{-1}u_4),\\ \Phi_{23}=\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_2+\partial u_2\partial^{-1}u_1), & \Phi_{24}=-\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_1-\partial u_2\partial^{-1}u_2);\end{cases} \tag{3.10}$$

$$\begin{cases}\Phi_{31}=\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_3-\partial u_4\partial^{-1}u_4), & \Phi_{32}=\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_4+\partial u_4\partial^{-1}u_3),\\ \Phi_{33}=\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_1+\partial u_4\partial^{-1}u_2), & \Phi_{34}=-\frac{1}{\alpha}\partial_x+\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_2-\partial u_4\partial^{-1}u_1);\end{cases} \tag{3.11}$$

$$\begin{cases}\Phi_{41}=\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_4+\partial u_4\partial^{-1}u_3), & \Phi_{42}=-\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_3-\partial u_4\partial^{-1}u_4),\\ \Phi_{43}=\frac{1}{\alpha}\partial_x-\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_2-\partial u_4\partial^{-1}u_1), & \Phi_{44}=\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_1+\partial u_4\partial^{-1}u_2).\end{cases} \tag{3.12}$$

The hereditary property of Φ means [31] that it satisfies

$$L_{\Phi X}\Phi=\Phi L_X\Phi, \tag{3.13}$$

where X is an arbitrary vector field and the Lie derivative $L_X\Phi$ is defined via

$$(L_X\Phi)Z=\Phi[[X,Z]]-[[X,\Phi Z]], \tag{3.14}$$

in which Z is an arbitrary vector field. Observe that an operator $\Psi=\Psi(x,t,u,u_x,\cdots)$ is a recursion operator of an evolution equation $u_t=X(u)$ if and only if the operator Ψ needs to satisfy

$$\frac{\partial\Psi}{\partial t}+L_X\Psi=0. \tag{3.15}$$

We can readily prove that the autonomous operator Φ is a recursion operator of $u_{t_0}=X^{[0]}$, i.e. we have $L_{X[0]}\Phi=0$. In view of this, we can compute that

$$L_{X[m]}\Phi=L_{\Phi X[m-1]}\Phi=\Phi L_{X[m-1]}\Phi=\cdots=\Phi^m L_{X[0]}\Phi=0,\quad m\geq 1. \tag{3.16}$$

Consequently, we see that Φ is a common recursion operator for all models in the hierarchy (2.16). Symbolic algorithms are also available for computing recursion operators of nonlinear partial differential equations by computer algebra systems (see, e.g., [32]).

With some direct analysis, we can further show that J_1 and $J_2=\Phi J_1$ constitute a Hamiltonian pair. Namely, an arbitrary linear combination J of J_1 and J_2 is again Hamiltonian, since it satisfies

$$\int (Z^{[1]})^T J'(u)[JZ^{[2]}]Z^{[3]}\,dx+\text{cycle}(Z^{[1]},Z^{[2]},Z^{[3]})=0, \tag{3.17}$$

where $Z^{[i]}$'s are arbitrary vector fields. Accordingly, the hierarchy (2.16) possesses a bi-Hamiltonian formulation [33],

$$u_{t_m} = X^{[m]} = J_1 \frac{\delta \mathcal{H}^{[m]}}{\delta u} = J_2 \frac{\delta \mathcal{H}^{[m-1]}}{\delta u}, \qquad m \geq 1. \tag{3.18}$$

It then follows that the associated Hamiltonian functionals commute with each other under the corresponding two Poisson brackets [4]:

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_1} = \int \left(\frac{\delta \mathcal{H}^{[n_1]}}{\delta u}\right)^T J_1 \frac{\delta \mathcal{H}^{[n_2]}}{\delta u}\, dx = 0, \quad n_1, n_2 \geq 0, \tag{3.19}$$

and

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_2} = \int \left(\frac{\delta \mathcal{H}^{[n_1]}}{\delta u}\right)^T J_2 \frac{\delta \mathcal{H}^{[n_2]}}{\delta u}\, dx = 0, \quad n_1, n_2 \geq 0. \tag{3.20}$$

The bi-Hamiltonian formulation also implies the hereditariness of the recursion operator Φ.

To conclude, each model in the hierarchy (2.16) is Liouville integrable and possesses infinitely many commuting symmetries $\{X^{[n]}\}_{n=0}^{\infty}$ and conserved functionals $\{\mathcal{H}^{[n]}\}_{n=0}^{\infty}$. One particular illustrative integrable model is the system (2.17) of derivative nonlinear Schrödinger equations, which adds to the existing category of coupled nonlinear Liouville integrable Hamiltonian models with four components.

4. Concluding remarks

From a special matrix Lie algebra, a specific 4×4 matrix eigenvalue problem was proposed and an associated hierarchy of four-component Liouville integrable models was generated through the zero curvature formulation. The key is to determine a particular Laurent series solution of the corresponding stationary zero curvature equation. The resulting integrable hierarchy has been shown to possess a hereditary recursion operator and a bi-Hamiltonian formulation, and thus, all members in the hierarchy are Liouville integrable.

We are curious to know about mathematical structures of soliton solutions to the obtained integrable models. Abundant powerful and effective approaches are available for use, which include the Riemann-Hilbert technique [36], the Darboux transformation [37–39], the Zakharov-Shabat dressing method [40] and the determinant approach [41]. In addition to solitons, lump, kink, breather and rogue wave solutions, particularly their interaction solutions (see, e.g., [42–49)], are also greatly significant, and many of them can often be generated from soliton solutions by conducting wave number reductions. Another important aspect of the study of integrable models is to look for nonlocal reduced integrable models, and nonlocal group reductions and similarity transformations of matrix eigenvalue problems are helpful. Solitons in the nonlocal case are significantly important in mathematics as well as physics (see, e.g., [50–53]).

In summary, integrable models are of great importance, due to their role in understanding nonlinear phenomena and their impact on diverse scientific disciplines. Their study continues to contribute valuable insights that shape our understanding of complex physical systems and advance our knowledge in various fields.

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