INTEGRABLE NONLOCAL NONLINEAR SCHRÖDINGER HIERARCHIES OF TYPE $(-\lambda^*, \lambda)$ AND SOLITON SOLUTIONS

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Two simultaneous nonlocal group constraints of the Ablowitz–Kaup–Newell–Segur matrix eigenvalue problems are discussed, of which one constraint changes the eigenvalue parameter into its negative of the complex conjugate and the other constraint does not change the eigenvalue parameter. Under those two constraints, mixed-type nonlocal integrable nonlinear Schrödinger hierarchies are generated. Further, based on specific distributions of eigenvalues and adjoint eigenvalues, a formulation of soliton solutions is established via the corresponding reflectionless generalized Riemann–Hilbert problems, where eigenvalues and adjoint eigenvalues could be equal.

Keywords: matrix eigenvalue problem, integrable hierarchy, nonlocal integrable equation, nonlinear Schrödinger equations, Riemann–Hilbert technique, soliton solution.

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1. Introduction

Zero curvature equations are the staring point to study nonlinear integrable equations. Based on matrix eigenvalue problems, with which zero curvature equations are associated, the inverse scattering transform serves an efficient method to solve Cauchy problems of integrable equations. Conducting group constraints for matrix eigenvalue problems, which keep the zero curvature equations invariant, one can obtain reduced integrable equations, either local or nonlocal. There is only one kind of local group constraints resulting in reduced hierarchies of integrable nonlinear Schrödinger (NLS) equations, but there are two kinds of local group constraints resulting in reduced hierarchies of integrable modified Korteweg–de Vries (mKdV) equations, based on the Ablowitz–Kaup–Newell–Segur (AKNS) matrix eigenvalue problems. Nonlocal groups constraints have recently attracted much attention from soliton researchers and nonlocal integrable equations represent an area of research that has seen a significant growth acceleration in recent publications, supplementing the classical theory of partial differential equations. Three types of nonlocal integrable NLS equations and two types of nonlocal integrable mKdV equations can be constructed via the AKNS matrix eigenvalue problems by taking one nonlocal group constraint [1, 2].

The traditional inverse scattering transform has been successfully applied to nonlocal integrable equations (see, e.g. [3–6]). There are some other efficient approaches in the nonlocal case, including the Hirota bilinear method, Darboux transformation, Bäcklund transforms and the Riemann–Hilbert technique, in constructing soliton solutions (see, for example, [7–10]). Particularly, the Riemann–Hilbert problems are systematically used to deal with nonlocal integrable NLS and mKdV equations [2, 11–14]. In this paper, we would like to propose novel mixed-type reduced nonlocal integrable NLS equations of even order by introducing two simultaneous nonlocal group constraints, and construct their soliton solutions by solving the corresponding reflectionless generalized Riemann–Hilbert problems.

The other sections of the paper are arranged as follows. In the next section, we first recall the AKNS hierarchies of matrix integrable equations and their matrix eigenvalue problems to make the subsequent discussion smoothly, and then we consider two simultaneous nonlocal group constraints for the AKNS matrix eigenvalue problems and generate reduced nonlocal integrable NLS hierarchies of type $(-\lambda^*, \lambda)$, in which λ is the eigenvalue parameter and $*$ denotes the complex conjugate. Two scalar paradigmatic examples of the presented nonlocal integrable equations read

$$r_{1,t} = -\frac{\beta}{\alpha^2} i [r_{1,xx} - 2\sigma(r_1 r_1^*(-x,t) + r_1(-x,-t) r_1^*(x,-t)) r_1],$$

and

$$r_{1,t} = -\frac{\beta}{\alpha^2} i [r_{1,xx} + 2\delta(r_1 r_1(-x,-t) + r_1^*(-x,t) r_1^*(x,-t)) r_1],$$

where i stands for the imaginary unit, $\alpha, \beta \in \mathbb{R}$ are arbitrary nonzero constants, $\sigma = \pm 1$ and $\delta = \pm 1$. In Section 3, on the basis of the scrutinized distribution of eigenvalues and adjoint eigenvalues, we manage the associated reflectionless generalized Riemann–Hilbert problems, where eigenvalues and adjoint eigenvalues could be equal, and work out soliton solutions to the presented novel reduced nonlocal integrable NLS equations of all even orders. In the last section, we give rise to a conclusion and some concluding remarks.

2. Reduced integrable nonlocal NLS hierarchies of type $(-\lambda^*, \lambda)$

2.1. The AKNS matrix integrable equations revisited

In order to facilitate the subsequent exposition, we first state the matrix hierarchies of AKNS integrable equations and their matrix eigenvalue problems.

First, let us denote two matrix potentials by r and s:

$$r = r(x,t) = (r_{jk})_{p\times q}, \quad s = s(x,t) = (s_{kj})_{q\times p}, \tag{1}$$

where p and q are two arbitrarily given natural numbers. It is known that the matrix AKNS eigenvalue problems are given by

$$\begin{cases} -i\phi_x = U\phi = U(u,\lambda)\phi = (\lambda\Lambda + R)\phi, \\ -i\phi_t = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi = (\lambda^m\Omega + S^{[m]})\phi, \quad m \geq 0, \end{cases} \tag{2}$$

where λ is an eigenvalue parameter and ϕ is a column eigenfunction. The pair of square matrices of order $(p+q)$, Λ and Ω, is determined by

$$\Lambda = \mathrm{diag}(\alpha_1 I_p, \alpha_2 I_q), \quad \Omega = \mathrm{diag}(\beta_1 I_p, \beta_2 I_q), \tag{3}$$

where α_1, α_2 and β_1, β_2 are two pairs of arbitrarily given real distinct constants, and I_n stands for the identity matrix of order n. The other pair of square matrices of order $(p+q)$, R and $S^{[r]}$, is given by

$$R = R(u) = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix}, \tag{4}$$

called the potential matrix, and

$$S^{[m]} = \sum_{n=0}^{m-1} \lambda^n \begin{bmatrix} a^{[m-n]} & b^{[m-n]} \\ c^{[m-n]} & d^{[m-n]} \end{bmatrix}, \tag{5}$$

in which the entries $a^{[n]}, b^{[n]}, c^{[n]}$ and $d^{[n]}$ are defined recursively by

$$b^{[0]} = 0, \quad c^{[0]} = 0, \quad a^{[0]} = \beta_1 I_p, \quad d^{[0]} = \beta_2 I_q, \tag{6a}$$

$$b^{[n+1]} = \frac{1}{\alpha}(-ib_x^{[n]} - rd^{[n]} + a^{[n]}r), \quad n \geq 0, \tag{6b}$$

$$c^{[n+1]} = \frac{1}{\alpha}(ic_x^{[n]} + sa^{[n]} - d^{[n]}s), \quad n \geq 0, \tag{6c}$$

$$a_x^{[n]} = i(rc^{[n]} - b^{[n]}s), \quad d_x^{[n]} = i(sb^{[n]} - c^{[n]}r), \quad n \geq 1, \tag{6d}$$

where zero constants of integration are selected. To illustrate, one can compute that

$$S^{[1]} = \frac{\beta}{\alpha}R, \quad S^{[2]} = \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2}I_{p,q}(R^2 + iR_x),$$

and

$$S^{[3]} = \frac{\beta}{\alpha}\lambda^2 R - \frac{\beta}{\alpha^2}\lambda I_{p,q}(R^2 + iR_x) - \frac{\beta}{\alpha^3}(i[R, R_x] + R_{xx} + 2R^3),$$

where $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ and $I_{p,q} = \mathrm{diag}(I_p, -I_q)$. From the recursive relations in (6), we also see that

$$W = \sum_{n\geq 0} \lambda^{-n} \begin{bmatrix} a^{[n]} & b^{[n]} \\ c^{[n]} & d^{[n]} \end{bmatrix}, \tag{7}$$

in which $a^{[n]}, b^{[n]}, c^{[n]}$ and $d^{[n]}$ are formulated by (6), determines a Laurent series solution to the corresponding stationary zero curvature equation

$$W_x = i[U, W]. \tag{8}$$

It now follows that all zero curvature equations

$$U_t - V_x^{[m]} + i[U, V^{[m]}] = 0, \quad m \geq 0, \tag{9}$$

generate one AKNS matrix integrable hierarchy (see, e.g. [15, 16] for details),

$$r_t = i\alpha b^{[m+1]}, \quad s_t = -i\alpha c^{[m+1]}, \quad m \geq 0, \tag{10}$$

in which r and s are the previous matrix potentials given by (1). The zero curvature equations in (9) are exactly the compatibility conditions of the two matrix eigenvalue problems in (2).

Moreover, through the trace identity [17] and a Lax operator algebra theory [18, 19] we can directly verify that the hierarchy (10) gives infinitely many commuting iso-spectral flows, each of which has a bi-Hamiltonian formulation and thus a hierarchy of commuting conserved quantities as well. It is easy to see that the first nonlinear (i.e. $m = 2$) member in the hierarchy presents the AKNS matrix NLS equations:

$$r_t = -\frac{\beta}{\alpha^2} i(r_{xx} + 2rsr), \quad s_t = \frac{\beta}{\alpha^2} i(s_{xx} + 2srs), \tag{11}$$

which includes the standard NLS equation and the multi-component NLS equations.

2.2. New type reduced nonlocal NLS hierarchies

Let us now assume that Σ_1 and Σ_2 are two constant invertible Hermitian matrices of orders p and q, respectively, Δ_1 and Δ_2 are other two constant invertible symmetric matrices of orders p and q, respectively, and form the two bigger invertible constant matrices of order $p + q$:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{12}$$

For the spectral matrix U, we propose a pair of nonlocal group constraints

$$U^\dagger(-x, t, -\lambda^*) = (U(-x, t, -\lambda^*))^\dagger = -\Sigma U(x, t, \lambda)\Sigma^{-1}, \tag{13}$$

and

$$U^T(-x, -t, \lambda) = (U(-x, -t, \lambda))^T = \Delta U(x, t, \lambda)\Delta^{-1}, \tag{14}$$

in which $\dagger$ and T means the Hermitian transpose and the matrix transpose, respectively.

Equivalently, we see that these two group constraints lead to

$$R^\dagger(-x,t) = (R(-x,t))^\dagger = -\Sigma R(x,t)\Sigma^{-1}, \tag{15}$$

and

$$R^T(-x,-t) = (R(-x,-t))^T = \Delta R(x,t)\Delta^{-1}, \tag{16}$$

respectively. These actually require the following corresponding constraints for the two matrix potentials r and s:

$$s(x,t) = -\Sigma_2^{-1} r^\dagger(-x,t)\Sigma_1, \tag{17}$$

and

$$s(x,t) = \Delta_2^{-1} r^T(-x,-t)\Delta_1. \tag{18}$$

Consequently, the first matrix potential r needs to satisfy

$$-\Sigma_2^{-1} r^\dagger(-x,t)\Sigma_1 = \Delta_2^{-1} r^T(-x,-t)\Delta_1, \tag{19}$$

or the second matrix potential s needs to satisfy

$$-\Sigma_1^{-1} s^\dagger(-x,t)\Sigma_2 = \Delta_1^{-1} s^T(-x,-t)\Delta_2, \tag{20}$$

to ensure that both nonlocal group constraints in (13) and (14) hold.

Moreover, following the two group constraints in (13) and (14), one can obtain

$$\begin{cases} W^\dagger(-x,t,-\lambda^*) = (W(-x,t,-\lambda^*))^\dagger = \Sigma W(x,t,\lambda)\Sigma^{-1}, \\ W^T(-x,-t,\lambda) = (W(-x,-t,\lambda))^T = \Delta W(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{21}$$

where W is defined by (7). These relations imply that

$$\begin{cases} V^{[2n]\dagger}(-x,t,-\lambda^*) = (V^{[2n]}(-x,t,-\lambda^*))^\dagger = \Sigma V^{[2n]}(x,t,\lambda)\Sigma^{-1}, \\ V^{[2n]T}(-x,-t,\lambda) = (V^{[2n]}(-x,-t,\lambda))^T = \Delta V^{[2n]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{22}$$

and

$$\begin{cases} Q^{[2n]\dagger}(-x,t,-\lambda^*) = (Q^{[2n]}(-x,t,-\lambda^*))^\dagger = \Sigma Q^{[2n]}(x,t,\lambda)\Sigma^{-1}, \\ Q^{[2n]T}(-x,-t,\lambda) = (Q^{[2n]}(-x,-t,\lambda))^T = \Delta Q^{[2n]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{23}$$

where $n \geq 0$.

Now, as a consequence of the potential constraints (17) and (18), we see that the matrix AKNS integrable equations in (10) with $m = 2n,\quad n \geq 0$, become a hierarchy of reduced nonlocal integrable NLS type equations

$$r_t = i\alpha b^{[2n+1]}|_{s=-\Sigma_2^{-1}r^\dagger(-x,t)\Sigma_1=\Delta_2^{-1}r^T(-x,-t)\Delta_1}, \quad n \geq 0, \tag{24}$$

in which r is a reduced $p\times q$ matrix potential satisfying (19), Σ_1 and Σ_2 are two arbitrary invertible constant Hermitian matrices of orders p and q, respectively, and Δ_1 and Δ_2 are the other two arbitrary invertible constant symmetric matrices of orders p and q, respectively. Moreover, every member in the reduced hierarchy (24) has a Lax pair consisting of the reduced matrix eigenvalue problems in (2) with $m = 2n, \quad n \geq 0$, and possesses a hierarchy of commuting symmetries and conserved densities, which are reduced from those for the matrix integrable AKNS equations in (10) with $m = 2n, \quad n \geq 0$.

2.3. Illustrative examples in the nonlocal NLS case

If we take $n = 1$, namely $m = 2$, then the resulting reduced nonlocal integrable NLS type equations defined in (24) with $n = 1$ present a type of reduced integrable nonlocal NLS equations

$$\begin{aligned} r_t &= -\frac{\beta}{\alpha^2} i(r_{xx} - 2r\Sigma_2^{-1} r^\dagger(-x,t)\Sigma_1 r) \\ &= -\frac{\beta}{\alpha^2} i(r_{xx} + 2r\Delta_2^{-1} r^T(-x,-t)\Delta_1 r), \end{aligned} \tag{25}$$

in which the $p \times q$ matrix potential r needs to satisfy the constraint (19).

Let us below illustrate the above reduced integrable nonlocal NLS equations, with a few concrete examples associated with selected choices for Σ, Δ and small integer values for p, q.

First, we would like to consider the case of $p = 1$ and $q = 2$. We choose

$$\Sigma_1 = 1, \quad \Sigma_2^{-1} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \quad \Delta_1 = 1, \quad \Delta_2^{-1} = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix},$$

in which σ and δ are the plus or minus sign, i.e. two real constants satisfying $\sigma^2 = \delta^2 = 1$. Obviously, the potential constraint (19) just requires

$$r_2 = -\sigma\delta r_1^*(x,-t),$$

where $r = (r_1, r_2)$, and consequently, the corresponding reduced potential matrix R becomes

$$R = \begin{bmatrix} 0 & r_1 & -\sigma\delta r_1^*(x,-t) \\ -\sigma r_1^*(-x,t) & 0 & 0 \\ \delta r_1(-x,-t) & 0 & 0 \end{bmatrix}. \tag{26}$$

Further, the corresponding resulting integrable nonlocal NLS equations read

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} - 2\sigma(r_1 r_1^*(-x,t) + r_1(-x,-t)r_1^*(x,-t))r_1], \tag{27}$$

in which r_1^* is the complex conjugate of r_1 and $\sigma = \pm 1$. We point out that three pairs of nonlocal points are involved simultaneously.

Similarly, we choose

$$\Sigma_1 = 1, \quad \Sigma_2^{-1} = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \quad \Delta_1 = 1, \quad \Delta_2^{-1} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix},$$

in which σ and δ are the plus or minus sign, i.e. two real constants which meet $\sigma^2 = \delta^2 = 1$ so that there are four choices. Evidently, the potential constraint (19) also just requires

$$r_2 = -\sigma\delta r_1^*(x, -t),$$

where $r = (r_1, r_2)$, and accordingly, the corresponding potential matrix R is reduced to

$$R = \begin{bmatrix} 0 & r_1 & -\sigma\delta r_1^*(x, -t) \\ \delta r_1(-x, -t) & 0 & 0 \\ -\sigma r_1^*(-x, t) & 0 & 0 \end{bmatrix}. \tag{28}$$

Further, the corresponding resulting new reduced integrable nonlocal NLS equations become

$$r_{1,t} = -\frac{\beta}{\alpha^2} i[r_{1,xx} + 2\delta(r_1 r_1(-x, -t) + r_1^*(-x, t) r_1^*(x, -t)) r_1], \tag{29}$$

in which r_1^* denotes the complex conjugate of r_1 again and $\delta = \pm 1$. This pair of equations involves a different nonlocality pattern from the one in (27).

Second, we consider the case of $p = 1$ and $q = 4$. Using analogous arguments as above with

$$\Sigma_1 = 1, \quad \Sigma_2^{-1} = \mathrm{diag}\left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}, \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}\right),$$
$$\Delta_1 = 1, \quad \Delta_2^{-1} = \mathrm{diag}\left(\begin{bmatrix} 0 & \delta_1 \\ \delta_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \delta_2 \\ \delta_2 & 0 \end{bmatrix}\right),$$

and

$$\Sigma_1 = 1, \quad \Sigma_2^{-1} = \mathrm{diag}\left(\begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}\right),$$
$$\Delta_1 = 1, \quad \Delta_2^{-1} = \mathrm{diag}\left(\begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix}, \begin{bmatrix} \delta_2 & 0 \\ 0 & \delta_2 \end{bmatrix}\right),$$

in which σ_j and δ_j, $j = 1, 2$, are real constants which satisfy $\sigma_j^2 = \delta_j^2 = 1$, $j = 1, 2$, we obtain two classes of two-component mixed-type integrable nonlocal NLS equations:

$$\begin{cases} r_{1,t} = -\dfrac{\beta}{\alpha^2} i[r_{1,xx} - 2\sigma_1(r_1 r_1^*(-x, t) + r_1(-x, -t) r_1^*(x, -t)) r_1 \\ \qquad -\sigma_2(r_3 r_3^*(-x, t) + r_3(-x, -t) r_3^*(x, -t)) r_1, \\ r_{3,t} = -\dfrac{\beta}{\alpha^2} i[r_{3,xx} - 2\sigma_1(r_1 r_1^*(-x, t) + r_1(-x, -t) r_1^*(x, -t)) r_3 \\ \qquad -\sigma_2(r_3 r_3^*(-x, t) + r_3(-x, -t) r_3^*(x, -t)) r_3, \end{cases} \tag{30}$$

and

$$\begin{cases} r_{1,t} = -\dfrac{\beta}{\alpha^2} i[r_{1,xx} + 2\delta_1(r_1 r_1(-x,-t) + r_1^*(-x,t) r_1^*(x,-t)) r_1 \\ \qquad +2\delta_2(r_3 r_3(-x,-t) + r_3^*(-x,t) r_3^*(x,-t)) r_1, \\ r_{3,t} = -\dfrac{\beta}{\alpha^2} i[r_{3,xx} + 2\delta_1(r_1 r_1(-x,-t) + r_1^*(-x,t) r_1^*(x,-t)) r_3 \\ \qquad +2\delta_2(r_3 r_3(-x,-t) + r_3^*(-x,t) r_3^*(x,-t)) r_3, \end{cases} \tag{31}$$

respectively, where we have adopted $r = (r_1, r_2, r_3, r_4)$.

Third, we consider the more general case of $p = 2$ and $q = 2$. Analogous arguments with

$$\Sigma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \quad \Sigma_2^{-1} = \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} 0 & \delta_1 \\ \delta_1 & 0 \end{bmatrix}, \quad \Delta_2^{-1} = \begin{bmatrix} 0 & \delta_2 \\ \delta_2 & 0 \end{bmatrix},$$

$$\Sigma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \quad \Sigma_2^{-1} = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix}, \quad \Delta_2^{-1} = \begin{bmatrix} 0 & \delta_2 \\ \delta_2 & 0 \end{bmatrix},$$

$$\Sigma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \quad \Sigma_2^{-1} = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} 0 & \delta_1 \\ \delta_1 & 0 \end{bmatrix}, \quad \Delta_2^{-1} = \begin{bmatrix} \delta_2 & 0 \\ 0 & \delta_2 \end{bmatrix},$$

and

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}, \quad \Sigma_2^{-1} = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} 0 & \delta_1 \\ \delta_1 & 0 \end{bmatrix}, \quad \Delta_2^{-1} = \begin{bmatrix} 0 & \delta_2 \\ \delta_2 & 0 \end{bmatrix},$$

in which σ_j and δ_j, $j = 1, 2$, are real constants which satisfy $\sigma_j^2 = \delta_j^2 = 1$, $j = 1, 2$, lead respectively to the corresponding reduced matrix potentials:

$$r = \begin{bmatrix} r_{11} & -\sigma\delta r_{11}^*(x,-t) \\ r_{21} & -\sigma\delta r_{21}^*(x,-t) \end{bmatrix}, \qquad s = \begin{bmatrix} -\sigma r_{21}^*(-x,t) & -\sigma r_{11}^*(-x,t) \\ \delta r_{21}(-x,-t) & \delta r_{11}(-x,-t) \end{bmatrix}, \tag{32}$$

$$r = \begin{bmatrix} r_{11} & r_{12} \\ -\sigma\delta r_{11}^*(x,-t) & -\sigma\delta r_{12}^*(x,-t) \end{bmatrix}, \quad s = \begin{bmatrix} \delta r_{12}(-x,-t) & -\sigma r_{12}^*(-x,t) \\ \delta r_{11}(-x,-t) & -\sigma r_{11}^*(-x,t) \end{bmatrix}, \tag{33}$$

$$r = \begin{bmatrix} r_{11} & -\sigma\delta r_{11}^*(x,-t) \\ r_{21} & -\sigma\delta r_{21}^*(x,-t) \end{bmatrix}, \qquad s = \begin{bmatrix} \delta r_{21}(-x,-t) & \delta r_{11}(-x,-t) \\ -\sigma r_{21}^*(-x,t) & -\sigma r_{11}^*(-x,t) \end{bmatrix}, \tag{34}$$

and

$$r = \begin{bmatrix} r_{11} & r_{12} \\ -\sigma\delta r_{11}^*(x,-t) & -\sigma\delta r_{12}^*(x,-t) \end{bmatrix}, \quad s = \begin{bmatrix} -\sigma r_{12}^*(-x,t) & \delta r_{12}(-x,-t) \\ -\sigma r_{11}^*(-x,t) & \delta r_{11}(-x,-t) \end{bmatrix}, \tag{35}$$

where $\sigma = \sigma_1\sigma_2$ and $\delta = \delta_1\delta_2$. These enable us to present four classes of two-component mixed-type integrable nonlocal NLS equations:

$$\begin{cases} r_{11,t} = -\dfrac{\beta}{\alpha^2} i[r_{11,xx} - 2\sigma(r_{11}^*(x,-t)r_{21}(-x,-t) + r_{11}r_{21}^*(-x,t))\, r_{11} \\ \qquad -2\sigma(r_{11}^*(x,-t)r_{11}(-x,-t) + r_{11}r_{11}^*(-x,t))\, r_{21}], \\ r_{21,t} = -\dfrac{\beta}{\alpha^2} i[r_{21,xx} - 2\sigma(r_{21}^*(x,-t)r_{21}(-x,-t) + r_{21}r_{21}^*(-x,t))\, r_{11} \\ \qquad -2\sigma(r_{21}^*(x,-t)r_{11}(-x,-t) + r_{21}r_{11}^*(-x,t))\, r_{21}], \end{cases} \tag{36}$$

$$\begin{cases} r_{11,t} = -\dfrac{\beta}{\alpha^2} i[r_{11,xx} + 2\delta(r_{11}r_{12}(-x,-t) + r_{12}r_{11}(-x,-t))\, r_{11} \\ \qquad +2\delta(r_{11}r_{12}^*(-x,t) + r_{12}r_{11}^*(-x,t))\, r_{11}^*(x,-t)], \\ r_{12,t} = -\dfrac{\beta}{\alpha^2} i[r_{12,xx} + 2\delta(r_{11}r_{12}(-x,-t) + r_{12}r_{11}(-x,-t))\, r_{12} \\ \qquad +2\delta(r_{11}r_{12}^*(-x,t) + r_{12}r_{11}^*(-x,t))\, r_{12}^*(x,-t)], \end{cases} \tag{37}$$

$$\begin{cases} r_{11,t} = -\dfrac{\beta}{\alpha^2} i[r_{11,xx} + 2\delta(r_{11}r_{21}(-x,-t) + r_{11}^*(x,-t)r_{21}^*(-x,t))\, r_{11} \\ \qquad +2\delta(r_{11}r_{11}(-x,-t) + r_{11}^*(x,-t)r_{11}^*(-x,t))\, r_{21}], \\ r_{21,t} = -\dfrac{\beta}{\alpha^2} i[r_{21,xx} + 2\delta(r_{21}r_{21}(-x,-t) + r_{21}^*(x,-t)r_{21}^*(-x,t))\, r_{11} \\ \qquad +2\delta(r_{21}r_{11}(-x,-t) + r_{21}^*(x,-t)r_{11}^*(-x,t))\, r_{21}], \end{cases} \tag{38}$$

and

$$\begin{cases} r_{11,t} = -\dfrac{\beta}{\alpha^2} i[r_{11,xx} - 2\sigma(r_{11}r_{12}^*(-x,t) + r_{12}r_{11}^*(-x,t))\, r_{11} \\ \qquad -2\sigma(r_{11}r_{12}(-x,-t) + r_{12}r_{11}(-x,-t))\, r_{11}^*(x,-t)], \\ r_{12,t} = -\dfrac{\beta}{\alpha^2} i[r_{12,xx} - 2\sigma(r_{11}r_{12}^*(-x,t) + r_{12}r_{11}^*(-x,t))\, r_{12} \\ \qquad -2\sigma(r_{11}r_{12}(-x,-t) + r_{12}r_{11}(-x,-t))\, r_{12}^*(x,-t)], \end{cases} \tag{39}$$

respectively, where $\sigma = \sigma_1\sigma_2 = \pm 1$ and $\delta = \delta_1\delta_2 = \pm 1$.

3. Formulating soliton solutions

3.1. Properties of eigenvalues and associated eigenfunctions

On one hand, taking the group constraint in (13) (or (14)) into consideration, one can easily find a relation between eigenvalues and adjoint eigenvalues. This is, one can guarantee that λ presents an eigenvalue of the matrix eigenvalue problems defined in (2) provided that $\hat{\lambda} = -\lambda^*$ (or $\hat{\lambda} = \lambda$) presents an adjoint eigenvalue, that is to say, the adjoint matrix eigenvalue problems hold:

$$i\tilde{\phi}_x = \tilde{\phi}U = \tilde{\phi}U(u,\hat{\lambda}), \quad i\tilde{\phi}_t = \tilde{\phi}V^{[2n]} = \tilde{\phi}V^{[2n]}(u,\hat{\lambda}), \tag{40}$$

where $n \geq 0$. Consequently, we can make the assumption that there are eigenvalues $\lambda:\ \mu,\ -\mu^*$, and the corresponding adjoint eigenvalues $\hat{\lambda}:\ -\mu^*,\ \mu$, in which $\mu \in \mathbb{C}$.

On the other hand, taking the group constraints in (13) and (14) into account, one can show that both of

$$\phi^\dagger(-x,t,-\lambda^*)\Sigma \quad \text{and} \quad \phi^T(-x,-t,\lambda)\Delta, \tag{41}$$

solve the above adjoint matrix eigenvalue problems with the originally given eigenvalue λ, once $\phi(\lambda)$ solves the matrix eigenvalue problems defined in (2) with an arbitrarily given eigenvalue λ.

3.2. Soliton solutions via the Riemann–Hilbert technique

This subsection aims to establish a skeleton frame of soliton solutions for the relevant mixed-type nonlocal integrable NLS type equations by considering the associated reflectionless generalized Riemann–Hilbert problems. Let us fix two integers $N_1, N_2 \geq 0$ such that $N := 2N_1 + N_2 \geq 1$.

First, we begin with the following N eigenvalues λ_k and N adjoint eigenvalues $\hat{\lambda}_k$:

$$\{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_N\} = \{\mu_1,\ \ldots,\ \mu_{N_1}, -\mu_1^*,\ \ldots,\ -\mu_{N_1}^*,\ \nu_1,\ \ldots,\ \nu_{N_2}\}, \tag{42}$$

and

$$\{\hat{\lambda}_1,\ \hat{\lambda}_2,\ \ldots,\ \hat{\lambda}_N\} = \{-\mu_1^*,\ \ldots,\ -\mu_{N_1}^*, \mu_1,\ \ldots,\ \mu_{N_1},\ -\nu_1,\ \ldots,\ -\nu_{N_2}\}, \tag{43}$$

where $\mu_k \notin i\mathbb{R}$, $\quad 1 \leq k \leq N_1$, and $\nu_k \in \mathbb{R}$, $\quad 1 \leq k \leq N_2$, and suppose that associated with those eigenvalues and adjoint eigenvalues, the relevant eigenfunctions and adjoint eigenfunctions are denoted by

$$\{v_1,\ v_2,\ \ldots, v_N\} \quad \text{and} \quad \{\hat{v}_1,\ \hat{v}_2, \ldots,\ \hat{v}_N\}, \tag{44}$$

respectively. Obviously, in this nonlocal case, one does not keep the following property

$$\{\lambda_1,\ \lambda_2,\ \ldots, \lambda_N\} \cap \{\hat{\lambda}_1,\ \hat{\lambda}_2, \ldots,\ \hat{\lambda}_N\} = \emptyset, \tag{45}$$

as in the traditional local case.

Let us next introduce a pair of square matrices of order $p+q$,

$$G^+(\lambda) = I_{p+q} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l}, \quad (G^-)^{-1}(\lambda) = I_{p+q} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}, \tag{46}$$

in which M is a square matrix of order N, whose entries are defined by

$$m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, & \text{if } \lambda_l \neq \hat{\lambda}_k, \\ 0, & \text{if } \lambda_l = \hat{\lambda}_k, \end{cases} \quad \text{where} \quad 1 \leq k, l \leq N. \tag{47}$$

This pair of square matrices of order $p+q$, $G^+(\lambda)$ and $G^-(\lambda)$, solves the associated reflectionless generalized Riemann–Hilbert problem [12], namely, they satisfy

$$(G^-)^{-1}(\lambda)G^+(\lambda) = I_{p+q}, \quad \text{where} \quad \lambda \in \mathbb{R}, \tag{48}$$

as long as an orthogonal condition between eigenfunctions and adjoint eigenfunctions

$$\hat{v}_k v_l = 0 \quad \text{if} \quad \lambda_l = \hat{\lambda}_k, \tag{49}$$

is satisfied for all pairs of $1 \leq k,\ l \leq N$.

To use the original matrix spectral problems to determine soliton solutions, let us now take account of an asymptotic expansion

$$G^+(\lambda) = I_{p+q} + \frac{1}{\lambda}G_1^+ + \mathrm{O}\left(\frac{1}{\lambda^2}\right), \tag{50}$$

as $\lambda \to \infty$, and then, we acquire

$$G_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl}\hat{v}_l. \tag{51}$$

Obviously, a substitution of this expression into the spatial eigenvalue problems in (2) engenders

$$R = -[\Lambda, G_1^+] = \lim_{\lambda\to\infty} \lambda[G^+(\lambda), \Lambda]. \tag{52}$$

Noting (4), we see that this formula allows us to arrive at soliton solutions for the AKNS matrix integrable equations (10):

$$r = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl}\hat{v}_l^2, \quad s = -\alpha \sum_{k,l=1}^{N} v_k^2 (M^{-1})_{kl}\hat{v}_l^1, \tag{53}$$

where for each $1 \leq k \leq N$, we have made the splitting $v_k = ((v_k^1)^T, (v_k^2)^T)^T$ and $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2)$, in which v_k^1 and $\hat{v}_k^1$ are column and row vectors of dimension p, respectively, and v_k^2 and $\hat{v}_k^2$ are column and row vectors of dimension q, respectively.

If we take zero matrix potentials, namely, set $r = 0$ and $s = 0$, then the associated matrix eigenvalue problems defined in (2) tell the corresponding N eigenfunctions:

$$v_k = v_k(x,t,\lambda_k) = e^{i\lambda_k\Lambda x + i\lambda_k^{2s}\Omega t} w_k, \quad 1 \leq k \leq N, \tag{54}$$

where the constant column vectors w_k, $1 \leq k \leq N$, are arbitrary. Taking the preceding analysis in Subsection 3.1 into account, one can assume that the relevant adjoint eigenfunctions are taken as

$$\hat{v}_k = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(-x,t,\lambda_k)\Sigma = \hat{w}_k e^{-i\hat{\lambda}_k\Lambda x - i\hat{\lambda}_k^{2s}\Omega t}, \quad 1 \leq k \leq N, \tag{55}$$

in which

$$\hat{w}_k = w_k^\dagger \Sigma, \quad 1 \leq k \leq N. \tag{56}$$

In this way, the orthogonal condition (49) becomes

$$w_k^\dagger \Sigma w_l = 0 \quad \text{when} \quad \lambda_l = \hat{\lambda}_k, \tag{57}$$

for pairs of $1 \leq k, l \leq N$.

Lastly, to guarantee that the above formulation gives soliton solutions to the resulting nonlocal matrix integrable even-order NLS equations (24), the matrix G_1^+ determined by (51) needs to satisfy the two corresponding involution properties:

$$(G_1^+)^\dagger(-x,t) = \Sigma G_1^+(x,t)\Sigma^{-1}, \quad (G_1^+)^T(-x,-t) = -\Delta G_1^+(x,t)\Delta^{-1}. \tag{58}$$

When those requirements are met, the resulting potential matrix R defined by (52) satisfies both of the two nonlocal group constraint conditions in (15) and (16). Consequently, we arrive at the following soliton solutions,

$$q = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2, \tag{59}$$

for the resulting mixed-type nonlocal matrix integrable even-order NLS equations (24). These solutions are, obviously, constraints of the soliton solutions, determined by (53), to the matrix AKNS integrable equations (10).

3.3. Realization of the involution properties

In what follows, we would like to establish a theoretical formulation for checking the involution properties in (58).

First, taking the previous analyses into account, one can compute the adjoint eigenfunctions $\hat{v}_k$, $1 \le k \le N$, by

$$\hat{v}_k = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(-x,t,\lambda_k)\Sigma = v_{N_1+k}^T(-x,-t,\lambda_{N_1+k})\Delta, \quad 1 \le k \le N_1, \tag{60}$$

$$\hat{v}_{N_1+k} = \hat{v}_{N_1+k}(x,t,\hat{\lambda}_{N_1+k}) = v_{N_1+k}^\dagger(-x,t,\lambda_{N_1+k})\Sigma = v_k^T(-x,-t,\lambda_k)\Delta, \quad 1 \le k \le N_1, \tag{61}$$

and

$$\hat{v}_k = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(-x,t,\lambda_k)\Sigma = v_k^T(-x,-t,\lambda_k)\Delta, \quad 2N_1+1 \le k \le N. \tag{62}$$

To achieve these selections in (60), (61) and (62), we need the following conditions for w_k, $1 \le k \le N$:

$$\begin{cases} w_k^T(\Sigma^*\Delta^{*-1} - \Delta\Sigma^{-1}) = 0, & 1 \le k \le N_1, \\ w_k = \Delta^{-1}\Sigma^T w_{k-N_1}^*, & N_1+1 \le k \le 2N_1, \\ w_k^\dagger \Sigma = w_k^T \Delta, & 2N_1+1 \le k \le N, \end{cases} \tag{63}$$

in which A^* stands for the complex matrix of a matrix A. The aim of making these conditions is to ensure that the constraint conditions in (15) and (16) will be satisfied.

Next, observe that once we determine solutions of the reflectionless generalized Riemann–Hilbert problems through (46) and (47), and meet the following involution requirements

$$(G^+)^\dagger(-\lambda^*) = \Sigma(G^-)^{-1}(\lambda)\Sigma^{-1}, \quad (G^+)^T(\lambda) = \Delta(G^-)^{-1}(\lambda)\Delta^{-1}, \tag{64}$$

the associated resulting matrix G_1^+ possesses the previous involution properties in (58). Those two conditions in (58) come from the nonlocal group constraints in (13) and (14). In this way, as long as the preceding constraints in (63) and the required orthogonal property in (57) on w_k, $1 \le k \le N$, are satisfied, the formula (59), along with (46), (47), (54) and (55), provides a kind of soliton solutions to the newly presented mixed-type reduced integrable nonlocal matrix NLS equations of even order in (24).

3.4. Illustrative examples of one-soliton solutions

In the remainder of this section, we present a few examples of one-soliton solutions to some resultant integrable nonlocal NLS equations, following the basic solution procedure above.

First, in the case where $p = 1$ and $q = 2$, we can work out two classes of one-soliton ($N = 1$) solutions to the mixed-type integrable nonlocal NLS equations in (27) and (29). Let us choose $\lambda_1 = \nu_1$, $\hat{\lambda}_1 = -\nu_1$, $\nu_1 \in \mathbb{R}$, and set $w_1 = (w_{1,1}, w_{1,2}, w_{1,3})^T$, where $w_{1,1}, w_{1,2}, w_{1,3} \in \mathbb{R}$. This choice yields a class of one-breather solutions

$$r_1 = \frac{2\sigma(\alpha_1 - \alpha_2)\nu_1 w_{1,1} w_{1,2} e^{i(\alpha_1-\alpha_2)\nu_1 x + i(\beta_1-\beta_2)\nu_1^2 t}}{w_{1,1}^2 e^{2i(\alpha_1-\alpha_2)\nu_1 x} + \sigma(w_{1,2}^2 + w_{1,3}^2)}. \tag{65}$$

It solves the mixed-type nonlocal integrable NLS equation (27) when

$$w_{1,2}^2 = w_{1,3}^2, \quad w_{1,1}^2 = 2w_{1,3}^2, \tag{66}$$

and the mixed-type nonlocal integrable NLS equation (29) when

$$w_{1,2}^2 = w_{1,3}^2, \quad w_{1,1}^2 = -2\delta w_{1,3}^2. \tag{67}$$

The two required conditions are generated from the involution properties in (58). The class of solutions has singularity when $e^{2i(\alpha_1-\alpha_2)\nu_1 x} + \sigma = 0$ for (27) and when $e^{2i(\alpha_1-\alpha_2)\nu_1 x} - \sigma\delta = 0$ for (29).

Second, in the case where $p = 1$ and $q = 4$, we can compute two classes of one-soliton ($N = 1$) solutions for the mixed-type integrable nonlocal NLS equations in (30) and (31). Let us similarly take $\lambda_1 = \nu_1$, $\hat{\lambda}_1 = -\nu_1$, $\nu_1 \in \mathbb{R}$, and set $w_1 = (w_{1,1}, w_{1,2}, w_{1,3}, w_{1,4}, w_{1,5})^T$, where $w_{1,k} \in \mathbb{R}$, $1 \le k \le 5$. Such a choice generates a class of candidate one-soliton solutions:

$$\begin{cases} r_1 = \dfrac{2\sigma_1(\alpha_1 - \alpha_2)\nu_1 w_{1,1} w_{1,2} e^{i(\alpha_1-\alpha_2)\nu_1 x + i(\beta_1-\beta_2)\nu_1^2 t}}{w_{1,1}^2 e^{2i(\alpha_1-\alpha_2)\nu_1 x} + \sigma_1(w_{1,2}^2 + w_{1,3}^2) + \sigma_2(w_{1,4}^2 + w_{1,5}^2)}, \\ r_3 = \dfrac{2\sigma_2(\alpha_1 - \alpha_2)\nu_1 w_{1,1} w_{1,4} e^{i(\alpha_1-\alpha_2)\nu_1 x + i(\beta_1-\beta_2)\nu_1^2 t}}{w_{1,1}^2 e^{2i(\alpha_1-\alpha_2)\nu_1 x} + \sigma_1(w_{1,2}^2 + w_{1,3}^2) + \sigma_2(w_{1,4}^2 + w_{1,5}^2)}, \end{cases} \tag{68}$$

for the mixed-type nonlocal integrable NLS equation (30), and a class of candidate one-soliton solutions:

$$\begin{cases} r_1 = \dfrac{2\delta_1(\alpha_1-\alpha_2)\nu_1 w_{1,1} w_{1,2} e^{i(\alpha_1-\alpha_2)\nu_1 x + i(\beta_1-\beta_2)\nu_1^2 t}}{w_{1,1}^2 e^{2i(\alpha_1-\alpha_2)\nu_1 x} + \delta_1(w_{1,2}^2 + w_{1,3}^2) + \delta_2(w_{1,4}^2 + w_{1,5}^2)}, \\ r_3 = \dfrac{2\delta_2(\alpha_1-\alpha_2)\nu_1 w_{1,1} w_{1,4} e^{i(\alpha_1-\alpha_2)\nu_1 x + i(\beta_1-\beta_2)\nu_1^2 t}}{w_{1,1}^2 e^{2i(\alpha_1-\alpha_2)\nu_1 x} + \delta_1(w_{1,2}^2 + w_{1,3}^2) + \delta_2(w_{1,4}^2 + w_{1,5}^2)}, \end{cases} \tag{69}$$

for the mixed-type nonlocal integrable NLS equation (31). The functions defined by (68) solve the mixed-type nonlocal integrable NLS equation (30), when we require

$$w_{1,2}^2 = w_{1,3}^2, \quad w_{1,4}^2 = w_{1,5}^2, \quad w_{1,1}^2 = 2\gamma_1 w_{1,3}^2, \quad w_{1,1}^2 = 2\gamma_2 w_{1,5}^2, \tag{70}$$

where the two real constants γ_1 and γ_2 need to satisfy

$$(\sigma_2\gamma_1 + \sigma_1\gamma_2)\gamma_1^2\gamma_2^2 - (\sigma_2\gamma_1^3 + \sigma_1\gamma_2^3) - 3(\sigma_1\gamma_1 + \sigma_2\gamma_2)\gamma_1\gamma_2 = 0. \tag{71}$$

Similarly, the functions determined by (69) present solutions to the mixed-type nonlocal integrable NLS equation (31), when we impose

$$w_{1,2}^2 = w_{1,3}^2, \quad w_{1,4}^2 = w_{1,5}^2, \quad w_{1,1}^2 = 2\kappa_1 w_{1,3}^2, \quad w_{1,1}^2 = 2\kappa_2 w_{1,5}^2, \tag{72}$$

where the two real constants κ_1 and κ_2 need to fulfill

$$\delta_1\delta_2(\kappa_1^2 + \kappa_2^2) + (\delta_1\kappa_1 + \delta_2\kappa_2 + 2)\kappa_1\kappa_2 = 0. \tag{73}$$

The above two sets of required conditions are generated from the involution properties in (58).

The condition (71) for γ_1 and γ_2 is satisfied if we select

$$\gamma_1 = \pm\sigma_1, \pm\sigma_2, \quad \gamma_2 = \mp\sigma_2, \mp\sigma_1, \tag{74}$$

respectively, and similarly the condition (73) for κ_1 and κ_2 is fulfilled if we take

$$\kappa_1 = \pm\delta_1, \pm\delta_2, \quad \kappa_2 = \mp\delta_2, \mp\delta_1, \tag{75}$$

respectively. The above classes of one-soliton solutions possess singularity, when

$$e^{2i(\alpha_1-\alpha_2)\nu_1 x} + \frac{\sigma_1}{\gamma_1} + \frac{\sigma_2}{\gamma_2} = 0 \tag{76}$$

for (30), and when

$$e^{2i(\alpha_1-\alpha_2)\nu_1 x} + \frac{\delta_1}{\kappa_1} + \frac{\delta_2}{\kappa_2} = 0 \tag{77}$$

for (31). Clearly, the singularity happens at the values of x determined by $e^{2i(\alpha_1-\alpha_2)\nu_1 x} = \pm 1$. However, for the above four simple selections in (74) (or (75)), we have $\frac{\sigma_1}{\gamma_1} + \frac{\sigma_2}{\gamma_2} = 0$ (or $\frac{\delta_1}{\kappa_1} + \frac{\delta_2}{\kappa_2} = 0$) and so the corresponding one-soliton solutions are analytical in the whole (x,t)-plane.

Fig. 1 and Fig. 2 show parts of the solutions given by (65) in the case of parameters:

$$\begin{cases} \nu_1 = 1, \quad \sigma = 1, \quad \alpha_1 = \beta_1 = 2, \quad \alpha_2 = \beta_2 = 1, \\ w_{1,1} = -2\sqrt{2}, \quad w_{1,2} = 2, \quad w_{1,3} = -2, \end{cases} \tag{78}$$

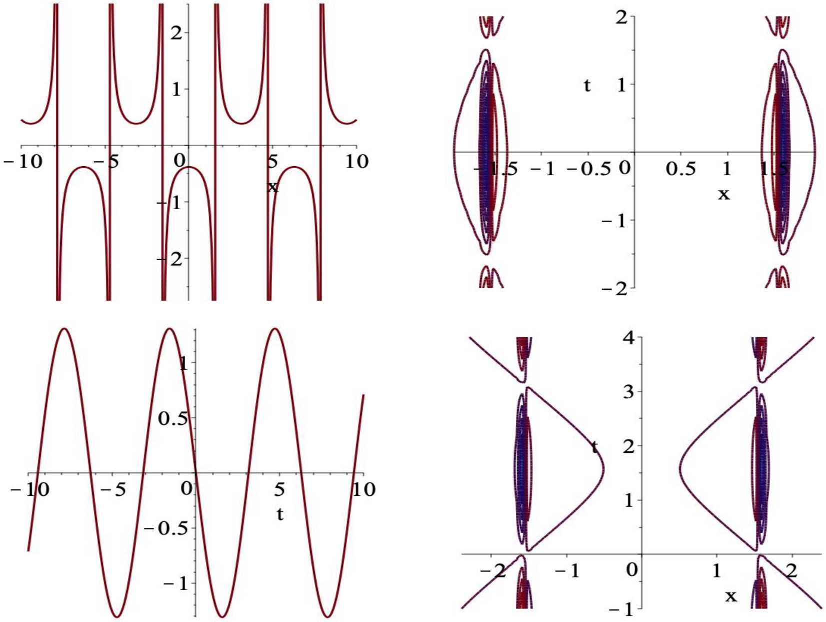

Fig. 1. An x-curve ($t = 1$) and a contour plot of the real part (top), and a t-curve ($x = 1$) and a contour plot of the imaginary part (bottom) of r_1 with (79).

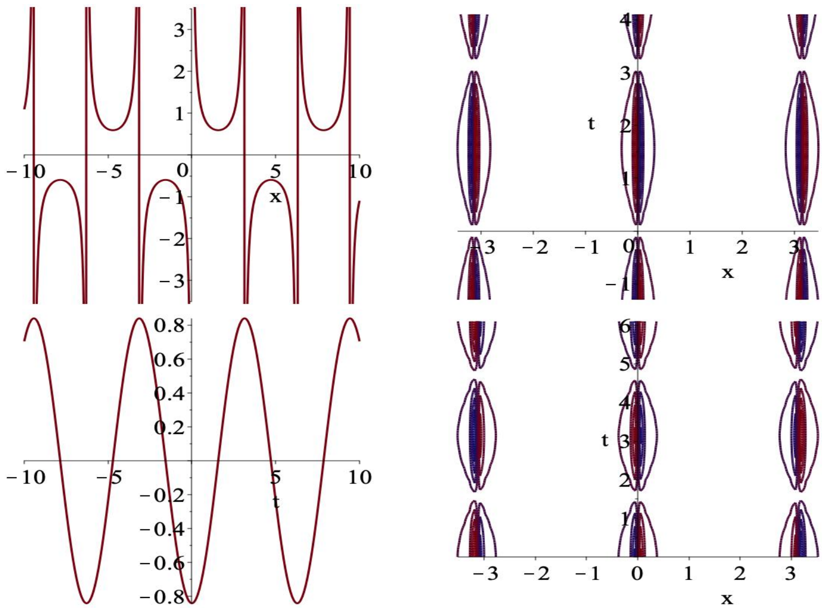

Fig. 2. An x-curve ($t = 1$) and a contour plot of the real part (top), and a t-curve ($x = 1$) and a contour plot of the imaginary part (bottom) of r_1 with (79).

for Eq. (27), and in the case of parameters:

$$\begin{cases} \nu_1 = 1, \quad \delta = \sigma = -1, \quad \alpha_1 = \beta_1 = 2, \quad \alpha_2 = \beta_2 = 1, \\ w_{1,1} = 2\sqrt{2}, \quad w_{1,2} = -2, \quad w_{1,3} = -2, \end{cases} \tag{79}$$

for Eq. (29), respectively. Each figure displays an x-curve and a contour plot of the real part and a t-curve and a contour plot of the imaginary part of the corrresponding solution. The other two presented one-soliton solutions in (68) and (69) possess similar properties.

4. Conclusion and remarks

Reduced integrable nonlocal NLS equations of type $(-\lambda^*, \lambda)$ have been presented and their soliton solutions have been formulated via the associated reflectionless generalized Riemann–Hilbert problems, in which eigenvalues and adjoint eigenvalues could be equal. The starting point is the AKNS matrix eigenvalue problems and the key is to take into account two nonlocal group constraints for the original AKNS matrix eigenvalue problems at the same time. The obtained integrable nonlocal NLS equations of even order are mixed-type, involving reverse-space, reverse-time and reverse-spacetime nonlocalities.

It will be of particular importance to look for soliton type solutions through different approaches, for example, the Darboux transformation, the Hirota bilinear tool, Bäcklund transforms and the Wronskian determinant technique. It will also be very important to investigate dynamical properties of different kinds of exact and explicit solutions in the nonlocal case, such as soliton, lump wave and breather wave solutions [20–23], algebro-geometric solutions [24, 25], optical solitons [26, 27], and solitonless solutions [28], based on Riemann–Hilbert's approach. Moreover, the basic idea of similarity transformations could be applied to nonlocal deformations of Lax reprentations for ordinary differential systems [29, 30]. The nonlocalities involved create big challenges even for establishing global existence of solutions, but infinitely many explicit symmetries and conserved densities may help. Another interesting problem is to explore more reduced nonlocal integrable equations from matrix eigenvalue problems formulated with other matrix loop algebras, including the special orthogonal Lie algebras (see, e.g. [31, 32] for examples).

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