INTEGRABLE MATRIX MODIFIED KORTEWEG-DE VRIES EQUATIONS DERIVED FROM REDUCING AKNS LAX PAIRS

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Abstract. This paper aims to reduce Lax pairs of AKNS matrix spectral problems using pairs of group reductions or similarity transformations. The corresponding modified Korteweg-de Vries matrix integrable hierarchies are obtained from the reduced Lax pairs, amending the standard AKNS integrable hierarchies. A few exemplary cases are analyzed and computed to demonstrate the diversity of modified Korteweg-de Vries matrix integrable equations.

Key words: AKNS Lax pair, Matrix spectral problem, Zero-curvature equation, Soliton hierarchy.

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1. INTRODUCTION

Integrable hierarchies arise from Lax pairs of matrix spectral problems [1], with the key being the selection of a matrix spatial spectral problem. The inverse scattering transform offers a powerful method for solving initial value problems of integrable models [2, 3]. The nonlinear Schrödinger (NLS) equation and the modified Korteweg-de Vries (mKdV) equation are reduced from the Ablowitz-Kaup-Newell-Segur (AKNS) matrix integrable hierarchies through a single group reduction or similarity transformation [4]. Furthermore, a pair of group reductions or similarity transformations can generate a variety of integrable models [5]. The main challenge lies in achieving a balance between the two reductions of the potentials generated by these transformations, which often introduces additional requirements to maintain the invariance of the corresponding zero-curvature equations [6].

Recently, group reductions or similarity transformations have been employed to construct nonlocal integrable models involving reflection points [7]. All lower-order integrable models associated with the AKNS matrix spectral problems have been classified into three types of nonlocal nonlinear Schrödinger equations and two types of nonlocal modified Korteweg-de Vries equations [8]. Furthermore, the inverse scattering transform has been successfully applied to initial value problems of nonlocal integrable models (see, *e.g.* [9, 10]). Various other powerful approaches have been developed for reduced integrable models, particularly in formulating soliton solutions. Methods such as the Hirota bilinear approach, Darboux transformation, Bäcklund transforms, and the Riemann-Hilbert technique have proven to be efficient tools, and numerous mathematical theories have been developed for nonlocal reduced integrable models (see, *e.g.* [8, 11–14]).

In this paper, we introduce a pair of group reductions or similarity transformations for the AKNS matrix spectral problems, generating the corresponding reduced integrable models. The novel contribution lies in the formulation of two distinct types of similarity transformations: one involving diagonal block matrices and the other involving off-diagonal block matrices. In Section 2, we revisit the matrix AKNS spectral problems and their associated integrable models to set the stage for the subsequent analysis. We also propose two group reductions or similarity transformations for the AKNS matrix spectral problems, which lead to reduced matrix mKdV integrable hierarchies. In Section 3, we illustrate the theory with four concrete examples, demonstrating the diversity of reduced AKNS matrix spectral problems and the corresponding integrable models. The final section summarizes our results.

2. REDUCED MATRIX MKDV INTEGRABLE HIERARCHIES

2.1. THE AKNS INTEGRABLE HIERARCHIES REVISITED

Let m, n be two arbitrarily given natural numbers. We introduce two submatrix potentials p and q:

$$p = p(x,t) = (p_{jk})_{m\times n},\ q = q(x,t) = (q_{kj})_{n\times m}, \tag{1}$$

and denote the dependent variable consisting of p and q by $u = u(p,q)$. Then, for all $r \geq 0$, the standard matrix AKNS spectral problems are given as follows:

$$-i\phi_x = U\phi,\ -i\phi_t = V^{[r]}\phi, \tag{2}$$

where the Lax pairs are defined by

$$\begin{cases} U = U(u,\lambda) = \lambda\Lambda + P,\ \Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix},\ P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \\ V^{[r]} = V^{[r]}(u,\lambda) = \lambda^r\Omega + Q^{[r]},\ \Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix}, \\ Q^{[r]} = \sum_{s=0}^{r-1}\lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}. \end{cases} \tag{3}$$

In the Lax pairs defined above, λ is the spectral parameter, α_1, α_2 and β_1, β_2 are two pairs of arbitrarily given distinct constants, I_k is the identity matrix of size k, $Q^{[0]}$ is the $(m+n)$-th-order zero matrix, and

$$W=\sum_{s\geq 0}\lambda^{-s}W^{[s]}=\sum_{s\geq 0}\lambda^{-s}\begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{4}$$

solves the stationary zero-curvature equation

$$W_x=i[U,W], \tag{5}$$

with the initial data $W^{[0]}=\Omega$. This formal series solution plays a key role in generating integrable hierarchies (see, *e.g.* [15, 16]).

It is now clear that the compatibility conditions of the two matrix spectral problems in (2) lead to the following zero-curvature equations:

$$U_t-V_x^{[r]}+i[U,V^{[r]}]=0,\ r\geq 0, \tag{6}$$

which describe a matrix AKNS integrable hierarchy:

$$p_t=i\alpha b^{[r+1]},\ q_t=-i\alpha c^{[r+1]},\ r\geq 0, \tag{7}$$

where $\alpha=\alpha_1-\alpha_2$. This generalizes the AKNS integrable hierarchy with scalar potentials [17]. Each system within the matrix AKNS integrable hierarchy possesses a bi-Hamiltonian structure, along with infinitely many symmetries and conserved quantities (see, *e.g.* [18–20]).

When $r=2s+1$, with $s\geq 0$, the matrix AKNS integrable hierarchy (7) reduces to the matrix mKdV integrable hierarchy. The first nonlinear (when $s=1$) integrable model in the resulted matrix mKdV integrable hierarchy gives the matrix mKdV equations:

$$\begin{cases} p_t=-\dfrac{\beta}{\alpha^3}(p_{xxx}+3pqp_x+3p_xqp), \\ q_t=-\dfrac{\beta}{\alpha^3}(q_{xxx}+3q_xpq+3qpq_x), \end{cases} \tag{8}$$

where $\beta=\beta_1-\beta_2$. The corresponding Lax matrix $V^{[3]}$ is given by

$$V^{[3]}=\lambda^3\Omega+\frac{\beta}{\alpha}\lambda^2P-\frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2+iP_x)-\frac{\beta}{\alpha^3}(i[P,P_x]+P_{xx}+2P^3), \tag{9}$$

where $I_{m,n}=\mathrm{diag}(I_m,-I_n)$. Other interesting examples of higher-order matrix AKNS integrable models can also be computed (see, *e.g.* [21]).

In what follows, we focus on a specific class of AKNS matrix spectral problems with

$$\alpha_1=-\alpha_2=1,\ \beta_1=-\beta_2=-4,\ m=n, \tag{10}$$

where n is an arbitrary natural number. In other words, we will consider integrable reductions of the matrix mKdV hierarchies associated with this particular class of AKNS spectral problems involving two square matrix potentials.

2.2. REDUCTIONS OF THE AKNS MATRIX SPECTRAL PROBLEMS

By taking two constant invertible symmetric square matrices of order n, Σ_1, Σ_2, along with two constant invertible square matrices of order n, Δ_1, Δ_2, we define two invertible constant square matrices of order $2n$ as follows:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},\ \Delta = \begin{bmatrix} 0 & \Delta_1 \\ \Delta_2 & 0 \end{bmatrix}. \tag{11}$$

Here, Σ is a block diagonal matrix, while Δ is a block off-diagonal matrix. It is clear that both Λ and Δ satisfy the important similarity properties

$$\Sigma\Lambda\Sigma^{-1} = -\Delta\Lambda\Delta^{-1} = \Lambda,\ \Sigma\Omega\Sigma^{-1} = -\Delta\Omega\Delta^{-1} = \Omega. \tag{12}$$

Based on the previous analyses, we introduce the following two group reductions or similarity transformations:

$$\Sigma U(\lambda)\Sigma^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T,\ \Delta U(\lambda)\Delta^{-1} = -U^T(\lambda) = -(U(\lambda))^T, \tag{13}$$

where A^T represents the matrix transpose of a matrix A. The first group reduction has been used to present reduced local integrable models (see, *e.g.* [4]). We will demonstrate that these group reductions or similarity transformations will preserve the invariance of the original zero-curvature equations. Following the specific form of the spectral matrix U, we can observe that these two group reductions or similarity transformations lead to the following relations for the potential matrix P:

$$\Sigma P\Sigma^{-1} = -P^T,\ \Delta P\Delta^{-1} = -P^T. \tag{14}$$

These reductions give rise to the following pairs of constraints for the submatrix potentials p and q:

$$p^T = -\Sigma_2 q\Sigma_1^{-1},\ q^T = -\Sigma_1 p\Sigma_2^{-1}, \tag{15}$$

and

$$p^T = -\Delta_2 p\Delta_1^{-1},\ q^T = -\Delta_1 q\Delta_2^{-1}. \tag{16}$$

Clearly, the two constraints in (15) are compatible because Σ is Hermitian. To ensure the compatibility of the two constraints in (16), we introduce a sufficient condition

$$\Sigma_2\Delta_1^{-1}\Sigma_1 = \eta\Delta_2, \tag{17}$$

where $\eta \in \mathbb{C}$ and $\eta \neq 0$.

To summarize, with the condition (17) on the constant matrices Σ and Δ, the two group reductions or similarity transformations in (13) generate a class of reduced AKNS matrix spectral problems:

$$-i\phi_x = U\phi,\ U = \begin{bmatrix} \lambda I_n & p \\ -\Sigma_2^{-1}p^T\Sigma_1 & -\lambda I_n \end{bmatrix}, \tag{18}$$

where p must satisfy $p^T = -\Delta_2 p \Delta_1^{-1}$. Alternatively, we can express the class of reduced AKNS matrix spectral problems as

$$-i\phi_x = U\phi,\ U = \begin{bmatrix} \lambda I_n & -\Sigma_1^{-1} q^T \Sigma_2 \\ q & -\lambda I_n \end{bmatrix}, \tag{19}$$

where q must satisfy $q^T = -\Delta_1 q \Delta_2^{-1}$.

2.3. REDUCED MATRIX MKDV INTEGRABLE HIERARCHIES

Let us consider the effects of the solution W determined by (4), with the initial data $W^{[0]} = \Omega$, under the two group reductions or similarity transformations in (13). First, we can easily check

$$\Sigma W(\lambda)\Sigma^{-1}|_{\lambda=\infty} = (W(-\lambda))^T|_{\lambda=\infty},\ \Delta W(\lambda)\Delta^{-1}|_{\lambda=\infty} = -(W(\lambda))^T|_{\lambda=\infty}. \tag{20}$$

Then, it follows from the uniqueness of solutions to the stationary zero-curvature equation that

$$\Sigma W(\lambda)\Sigma^{-1} = (W(-\lambda))^T,\ \Delta W(\lambda)\Delta^{-1} = -(W(\lambda))^T. \tag{21}$$

Furthermore, for all $r, s \geq 0$, we can show that

$$\Sigma V^{[2s+1]}(\lambda)\Sigma^{-1} = -(V^{[2s+1]}(-\lambda))^T,\ \Delta V^{[r]}(\lambda)\Delta^{-1} = -(V^{[r]}(\lambda))^T.$$

As a result, under the two group reductions or similarity transformations in (13), we observe that

$$\begin{cases} \Sigma(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Sigma^{-1} = -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(-\lambda))^T, \\ \Delta(U_t - V_x^{[r]} + i[U, V^{[r]}])(\lambda)\Delta^{-1} \qquad = -((U_t - V_x^{[r]} + i[U, V^{[r]}])(\lambda))^T, \end{cases}$$

and thus, the matrix mKdV integrable models in (7) become a reduced hierarchy of integrable models:

$$p_t = 2ib^{[2s+2]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1},\ s \geq 0, \tag{22}$$

where the matrix potential p satisfies $p^T = -\Delta_2 p \Delta_1^{-1}$, or equivalently, a reduced hierarchy of integrable models:

$$q_t = -2ic^{[2s+2]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2},\ s \geq 0, \tag{23}$$

where the matrix potential q satisfies $q^T = -\Delta_1 q \Delta_2^{-1}$. The matrix spectral problems, consisting of (18) and

$$-i\phi_t = V^{[2s+1]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi,\ s \geq 0, \tag{24}$$

present a Lax pair for every member in the reduced integrable hierarchy (22), or equivalently, the matrix spectral problems, consisting of (19) and

$$-i\phi_t = V^{[2s+1]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}\phi,\ s \geq 0, \tag{25}$$

present a Lax pair for each member in the reduced integrable hierarchy (23).

Note that Σ_1,Σ_2 are arbitrary invertible symmetric matrices, and Δ_1,Δ_2 are arbitrary invertible square matrices. There is a single condition, given by (17), that we need to check for Σ and Δ. Once the matrices Σ_1,Σ_2 and Δ_1,Δ_2 are appropriately chosen, we can generate a variety of integrable hierarchies corresponding to reduced matrix mKdV models.

3. APPLICATIONS TO FOUR SPECIAL CASES

In this section, we apply the general framework to four specific cases, offering illustrative examples of reduced matrix AKNS spectral problems and integrable mKdV equations. We consider four distinct combinations of the parameters by assuming that:

$$\sigma=\pm 1,\ \delta=\pm 1, \tag{26}$$

which leads to four possible scenarios to explore.

Example 3.1: Let us begin by considering the case where $n=2$. We choose the following specific values for the matrices:

$$\Sigma_1=I_2,\ \Sigma_2=-\sigma I_2,\ \Delta_1=\begin{bmatrix} 0 & 1\\ \delta & 0\end{bmatrix},\ \Delta_2=-\delta\Delta_1. \tag{27}$$

Then, the two group reductions or similarity transformations in (13) lead to

$$U=U(u,\lambda)=\begin{bmatrix} \lambda I_2 & p\\ \sigma p^T & -\lambda I_2\end{bmatrix},\ p=\begin{bmatrix} p_2 & p_1\\ p_3 & \delta p_2\end{bmatrix}, \tag{28}$$

where $u=(p_1,p_2,p_3)^T$, and the corresponding reduced matrix integrable mKdV equations are expressed as

$$\begin{cases} p_{1,t}=p_{1,xxx}+6\sigma[(p_1^2+p_2^2)p_{1,x}+p_2(p_1+\delta p_3)p_{2,x}],\\ p_{2,t}=p_{2,xxx}+3\sigma[p_2(p_1+\delta p_3)p_{1,x}+(p_1^2+2p_2^2+p_3^2)p_{2,x}+p_2(\delta p_1+p_3)p_{3,x}]\\ p_{3,t}=p_{3,xxx}+6\sigma[p_2(\delta p_1+p_3)p_{2,x}+(p_2^2+p_3^2)p_{3,x}].\end{cases} \tag{29}$$

Example 3.2: Next, let us consider $n=2$ and take the following specific values for the matrices:

$$\Sigma_1=I_2,\ \Sigma_2=-\sigma I_2,\ \Delta_1=\begin{bmatrix} -\delta & 1\\ 1 & 0\end{bmatrix},\ \Delta_2=-\Delta_1. \tag{30}$$

Then, the two group reductions or similarity transformations in (13) yield

$$U=U(u,\lambda)=\begin{bmatrix} \lambda I_2 & p\\ \sigma p^T & -\lambda I_2\end{bmatrix},\ p=\begin{bmatrix} p_2 & p_1\\ p_3 & \delta p_1+p_2\end{bmatrix}, \tag{31}$$

where $u=(p_1,p_2,p_3)^T$, and the corresponding reduced matrix integrable mKdV equations are given by

$$\begin{cases} p_{1,t}=p_{1,xxx}+3\sigma \\ \qquad [(4p_1^2+2p_2^2+3\delta p_1p_2+\delta p_2p_3)p_{1,x}(\delta p_1^2+2p_1p_2+\delta p_1p_3+2p_2p_3)p_{2,x}], \\ p_{2,t}=p_{2,xxx}+3\sigma[(p_1p_2+\delta p_1p_3+p_2p_3)p_{1,x}+(p_1^2+2p_2^2+p_3^2)p_{2,x} \\ \qquad +(\delta p_1^2+p_1p_2+p_2p_3)p_{3,x}], \\ p_{3,t}=p_{3,xxx}+3\sigma[(\delta p_1p_2+p_1p_3+\delta p_2p_3)p_{1,x}+(\delta p_1+2p_2)(p_1+p_3)p_{2,x} \\ \qquad +(p_1^2+2\delta p_1p_2+2p_2^2+2p_3^2)p_{3,x}]. \end{cases} \tag{32}$$

Example 3.3: For $n=3$, we take the following set of matrix choices

$$\Sigma_1=I_3,\ \Sigma_2=-\sigma I_3,\ \Delta_1=\Delta_2=\begin{bmatrix} 1 & 0 & \delta \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}. \tag{33}$$

Now, the two group reductions or similarity transformations in (13) generate

$$U=U(u,\lambda)=\begin{bmatrix} \lambda I_3 & p \\ \sigma p^T & -\lambda I_3 \end{bmatrix},\ p=\begin{bmatrix} p_2 & p_1 & 0 \\ p_3 & 0 & -p_1 \\ -\delta p_2 & -\delta p_1-p_3 & -p_2 \end{bmatrix}, \tag{34}$$

where $u=(p_1,p_2,p_3)^T$. The resulting system of equations for $n=3$ involves complex interactions between the three components of the potential matrix, exhibiting nonlinear interactions and differential terms. These interactions reflect the structure of the reduced AKNS matrix spectral problems and the associated integrable mKdV equations. In this case, the associated reduced matrix integrable mKdV equations are given as follows:

$$\begin{cases} p_{1,t}=p_{1,xxx}+3\sigma[(4p_1^2+3\delta p_1p_3+2p_2^2+p_3^2)p_{1,x} \\ \qquad +p_2(2p_1+\delta p_3)p_{2,x}+(\delta p_1^2+p_1p_3+\delta p_2^2)p_{3,x}], \\ p_{2,t}=p_{2,xxx}+3\sigma[p_2(2p_1+\delta p_3)p_{1,x} \\ \qquad +(2p_1^2+\delta p_1p_3+4p_2^2+p_3^2)p_{2,x}+p_2p_3p_{3,x}], \\ p_{3,t}=p_{3,xxx}+3\sigma[(p_1p_3-\delta p_2^2)p_{1,x} \\ \qquad -p_2(\delta p_1-2p_3)p_{2,x}+(p_1^2+2p_2^2+2p_3^2)p_{3,x}]. \end{cases} \tag{35}$$

Example 3.4: For $n=3$, let us consider another set of matrix choices:

$$\Sigma_1=I_3,\ \Sigma_2=-\sigma I_3,\ \Delta_1=\Delta_2=\begin{bmatrix} -\delta & 0 & \delta \\ 0 & -1 & 0 \\ \delta & 0 & \delta \end{bmatrix}. \tag{36}$$

In this case, the two group reductions or similarity transformations in (13) engender

$$U = U(u,\lambda) = \begin{bmatrix} \lambda I_3 & p \\ \sigma p^T & -\lambda I_3 \end{bmatrix}, \; p = \begin{bmatrix} p_2 & p_1 & p_2 \\ p_3 & 0 & 2\delta p_1 + p_3 \\ p_2 & p_1 + \delta p_3 & -p_2 \end{bmatrix}, \tag{37}$$

where $u = (p_1, p_2, p_3)^T$. This shows that by choosing a different set of matrices, we obtain a distinct set of interactions in the corresponding reduced AKNS spectral matrix and mKdV equations. The structure of the equations remains similar to Example 3.3, but with important differences in the nonlinear interaction terms due to the variations in the matrices $\Sigma_1, \Sigma_2, \Delta_1, \Delta_2$. The corresponding associated reduced matrix integrable mKdV equations are expressed as follows:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 3\sigma[(4p_1^2 + 3\delta p_1 p_3 + 2p_2^2 + p_3^2)p_{1,x} \\ \qquad +2p_1 p_2 p_{2,x} + p_1(\delta p_1 + p_3)p_{3,x}], \\ p_{2,t} = p_{2,xxx} + 3\sigma[p_2(2p_1 + \delta p_3)p_{1,x} \\ \qquad +(2p_1^2 + 3\delta p_1 p_3 + 4p_2^2 + 2p_3^2)p_{2,x} + 2p_2(\delta p_1 + p_3)p_{3,x}], \\ p_{3,t} = p_{3,xxx} + 6\sigma[(2p_1 + \delta p_3)p_3 p_{1,x} \\ \qquad +p_2 p_3 p_{2,x} + (2p_1^2 + 3\delta p_1 p_3 + p_2^2 + 2p_3^2)p_{3,x}]. \end{cases} \tag{38}$$

In these examples, the integration of the spectral matrix into the system of equations reveals deep nonlinear interactions that form the structure of the integrable mKdV equations. The varying parameters σ and δ play a crucial role in shaping the dynamics of the system, influencing the nature of the interactions between the components.

These examples highlight the flexibility and richness of the Lax pair formulation for constructing integrable models. The ability to perform different similarity transformations on the zero-curvature equations provides a pathway to generate a wide range of specific integrable reductions (see, *e.g.* [22–25]). Such transformations allow for the exploration of various types of nonlinear wave behaviors, with potential applications across different fields of study. These examples further supplement integrable models associated with the 4×4 matrix spectral problems, as explored in [26, 27].

4. CONCLUDING REMARKS

This paper introduces and analyzes a pair of local similarity transformations applied to a specific class of AKNS matrix spectral problems, resulting in reduced matrix mKdV integrable hierarchies. Several concrete examples of reduced AKNS matrix spectral problems and corresponding integrable mKdV hierarchies have been presented. A novel aspect of this work is the proposal of a pair of group reductions or similarity transformations, one involving a diagonal block matrix and the other an off-diagonal block matrix. These transformations offer a distinct approach compared to previous studies [6, 28], where the similarity matrices were exclusively of the diagonal block matrix type.

Our examples highlight the versatility of the Lax pair formulation in constructing integrable models, demonstrating how various reductions and transformations can produce a wide range of integrable mKdV equations with different nonlinear interactions. The selection of parameters such as σ and δ plays a crucial role in shaping the structure of these systems. The inherent flexibility of the Lax pair approach enables the creation of customized models, making it a valuable tool for capturing and analyzing diverse phenomena in both theoretical research and practical applications.

By delving deeper into this approach and examining various forms of similarity transformations for the zero-curvature equations, we can uncover more intricate structures and special cases of integrable models, revealing fascinating phenomena such as rogue waves, lump waves, and soliton waves (see, *e.g.* [29–32]). This process paves the way for a broader class of integrable models, which could be applied to diverse fields such as water waves, fluid dynamics, and nonlinear optics.

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