AKNS TYPE REDUCED INTEGRABLE HIERARCHIES WITH HAMILTONIAN FORMULATIONS

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Abstract. The aim of this paper is to generate a kind of integrable hierarchies of four-component evolution equations with Hamiltonian structures, from a kind of reduced Ablowitz-Kaup-Newell-Segur (AKNS) matrix spectral problems. The zero curvature formulation is the basic tool and the trace identity is the key to establishing Hamiltonian structures. Two examples of Hamiltonian equations in the resulting integrable hierarchies are added to the category of coupled integrable nonlinear Schrödinger equations and coupled integable modified Korteweg-de Vries equations.

Key words: Integrable hierarchy, Lax pair, Zero curvature equation, NLS equations, mKdV equations.

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1. INTRODUCTION

The zero curvature formulation is a powerful approach for constructing integrable equations in soliton theory [1, 2]. It allows us to express a system of partial differential equations (PDEs) in terms of a compatibility condition between two matrices, known as the zero curvature condition. By imposing this condition, we can obtain various integrable properties for the equations under consideration.

To construct integrable equations using the zero curvature formulation, we proceed as follows: The starting point is to formulate an appropriate matrix spatial spectral problem, whose spectral matrix reads

$$\mathcal{M} = \mathcal{M}(u,\lambda) = u_1 e_1(\lambda) + \cdots + u_q e_q(\lambda) + e_0(\lambda), \tag{1}$$

where λ is the spectral parameter, $u = (u_1, \cdots, u_q)^T$ is the dependent variable, and $e_1, \cdots, e_q$ are linear independent elements and e_0 is a pseudo-regular element in a loop algebra $\tilde{g}$. The pseudo-regular conditions

$$\mathrm{Ker}\,\mathrm{ad}_{e_0} \oplus \mathrm{Im}\,\mathrm{ad}_{e_0} = \tilde{g}, \text{ and } \mathrm{Ker}\,\mathrm{ad}_{e_0} \text{ is commutative}$$

guarantee that there exists a Laurent series solution $\mathcal{Y} = \sum_{s\geq 0} \lambda^{-s}\mathcal{Y}^{[s]}$ to the stationary zero curvature equation:

$$\mathcal{Y}_x = i[\mathcal{M},\mathcal{Y}]. \tag{2}$$

Hamiltonian structures of associated integrable equations can be established through the trace identity [3]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}\big(\mathcal{Y}\frac{\partial \mathcal{M}}{\partial \lambda}\big)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\big(\mathcal{Y}\frac{\partial \mathcal{M}}{\partial u}\big), \tag{3}$$

where γ is a constant, independent of λ, and $\frac{\delta}{\delta u}$ is the variational derivative with respect to u.

An integrable hierarchy can then be presented through zero curvature equations:

$$\mathcal{M}_t - \mathcal{N}^{[r]}_x + i[\mathcal{M},\mathcal{N}^{r]}] = 0,\ r \geq 0, \tag{4}$$

where $\mathcal{N}^{[r]}$, $r \geq 0$, are generated from the solution Z. These equations are the compatibility conditions between the spatial and temporal matrix spectral problems:

$$-i\phi_x = \mathcal{M}\phi,\ -i\phi_t = \mathcal{N}^{[r]}\phi,\ r \geq 0. \tag{5}$$

Many integrable hierarchies are computed in this way, associated with the special linear algebras (see, *e.g.*, [4–8]), and the special orthogonal algebras (see, *e.g.*, [9–11]). Bi-Hamiltonian structures can be often furnished, which immediately exhibit the Liouville integrability of the associated zero curvature equations [3, 12]. There are many integrable hierarchies with two components, p and q. Such famous integrable hierarchies contain the Ablowitz-Kaup-Newell-Segur hierarchy [4], the Heisenberg hierarchy [13], the Kaup-Newell hierarchy [14] and the Wadati-Konno-Ichikawa hierarchy [15], which are associated with the following spectral matrices:

$$\mathcal{M}(u,\lambda) = \begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix}, \begin{bmatrix} \lambda v & \lambda p \\ \lambda q & -\lambda v \end{bmatrix}, \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix}, \begin{bmatrix} \lambda & \lambda p \\ \lambda q & -\lambda \end{bmatrix},$$

where $pq + v^2 = 1$, respectively. In theoretical physics, one often uses $u^2 + v^2 + w^2 = 1$ upon setting $p = u + iw$ and $q = u - iw$.

This paper aims to present integrable hierarchies of Hamiltonian evolution equations with four components. The zero curvature formulation is the tool to generate integrable hierarchies and the trace identity is the key to establishing Hamiltonian structures for the resulting integrable hierarchies. Two illustrative examples are a sort of coupled integrable nonlinear Schrödinger equations and coupled integrable modified Korteweg-de Vries equations. The last section is devoted to a conclusion and some concluding remarks.

2. LAX PAIRS AND AN INTEGRABLE HIERARCHY

We begin with a 4×4 matrix spectral problem of the form:

$$-i\phi_x = \mathcal{M}\phi = \mathcal{M}(u,\lambda)\phi,\ \mathcal{M} = \begin{bmatrix} \alpha_1\lambda & p_1 & p_2 & p_1 \\ q_1 & \alpha_2\lambda & 0 & 0 \\ q_2 & 0 & \alpha_2\lambda & 0 \\ q_1 & 0 & 0 & \alpha_2\lambda \end{bmatrix}, \tag{6}$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ are two distinct constants, λ is the spectral parameter and u is the four-dimensional potential

$$u = u(x,t) = (p_1, p_2, q_1, q_2)^T. \tag{7}$$

This spectral problem is a specific reduction of the Ablowitz-Kaup-Newell-Segur (AKNS) spectral problem with two vector potentials (see, *e.g.*, [4, 16, 17] for details). We would like to show that such a reduced matrix spectral problem can be added to the category of matrix spectral problems that yield integrable hierarchies.

To derive an associated integrable hierarchy, let us first solve the stationary zero curvature equation (2) by assuming that a solution takes a Laurent series form:

$$\mathcal{Y} = \begin{bmatrix} a & b_1 & b_2 & b_1 \\ c_1 & d_{1,1} & d_{1,2} & d_{1,1} \\ c_2 & d_{2,1} & d_{2,2} & d_{2,1} \\ c_1 & d_{1,1} & d_{1,2} & d_{1,1} \end{bmatrix} = \sum_{s\geq 0} \lambda^{-s}\mathcal{Y}^{[s]}, \tag{8}$$

with $\mathcal{Y}^{[s]]}$ being determined by

$$\mathcal{Y}^{[s]} = \begin{bmatrix} a^{[s]} & b_1^{[s]} & b_2^{[s]} & b_1^{[s]} \\ c_1^{[s]} & d_{1,1}^{[s]} & d_{1,2}^{[s]} & d_{1,1}^{[s]} \\ c_2^{[s]} & d_{2,1}^{[s]} & d_{2,2}^{[s]} & d_{2,1}^{[s]} \\ c_1^{[s]} & d_{1,1}^{[s]} & d_{1,2}^{[s]} & d_{1,1}^{[s]} \end{bmatrix},\ s\geq 0. \tag{9}$$

It is direct to see that the corresponding stationary zero curvature equation requires the initial conditions on $\mathcal{Y}^{[0]}$:

$$a_x^{[0]} = 0,\ b_j^{[0]} = c_j^{[0]} = 0,\ (d_{k,l}^{[0]})_x = 0,\ 1\leq j,k,l\leq 2, \tag{10}$$

and yields the recursion relations for defining $\mathcal{Y}^{[s]},\ s\geq 1$:

$$b_j^{[s+1]} = \frac{1}{\alpha}(-ib_{j,x}^{[s]} + p_j a^{[s]} - 2p_1 d_{1,j}^{[s]} - p_2 d_{2,j}^{[s]}),\ 1\leq j\leq 2, \tag{11}$$

$$c_j^{[s+1]} = \frac{1}{\alpha}(ic_{j,x}^{[s]} + q_j a^{[s]} - 2q_1 d_{j,1}^{[s]} - q_2 d_{j,2}^{[s]}),\ 1\leq j\leq 2, \tag{12}$$

$$(d_{k,l}^{[s+1]})_x = i(q_k b_l^{[s+1]} - p_l c_k^{[s+1]}),\ 1\leq k,l\leq 2, \tag{13}$$

and

$$a_x^{[s+1]} = i(-2q_1 b_1^{[s+1]} - q_2 b_2^{[s+1]} + 2p_1 c_1^{[s+1]} + p_2 c_2^{[s+1]}), \tag{14}$$

where $s \geq 0$. Based on (10), let us further take the initial values,

$$a^{[0]} = \beta,\ d_{k,l}^{[0]} = 0,\ 1 \leq k,l \leq 2, \tag{15}$$

where $\beta \in \mathbb{C}$ is an arbitrary constant, and choose the constant of integration as zero,

$$a^{[s]}|_{u=0} = 0,\ d_{k,l}^{[s]}|_{u=0} = 0,\ 1 \leq k,l \leq 2,\ s \geq 1, \tag{16}$$

so that we can determine all required differential polynomials $a^{[s]}, b_j^{[s]}, c_j^{[s]}, d_{k,l}^{[s]},\ 1 \leq j,k,l \leq 2,\ s \geq 1$, uniquely. In this way, we can work out that

$$b_j^{[1]} = \frac{\beta}{\alpha} p_j,\ c_j^{[1]} = \frac{\beta}{\alpha} q_j,\ a^{[1]} = 0,\ d_{k,l}^{[1]} = 0,\ 1 \leq j,k,l \leq 2;$$

$$\begin{cases} b_j^{[2]} = -\frac{\beta}{\alpha^2} i p_{j,x},\ c_j^{[2]} = \frac{\beta}{\alpha^2} i q_{j,x},\ 1 \leq j \leq 2, \\ a^{[2]} = -\frac{\beta}{\alpha^2}(2p_1q_1 + p_2q_2),\ d_{k,l}^{[2]} = \frac{\beta}{\alpha^2} p_l q_k,\ 1 \leq k,l \leq 2; \end{cases}$$

$$\begin{cases} b_1^{[3]} = -\frac{\beta}{\alpha^3}(p_{1,xx} + 2p_1p_2q_2 + 4p_1^2q_1), \\ b_2^{[3]} = -\frac{\beta}{\alpha^3}(p_{2,xx} + 2p_2^2q_2 + 4p_1p_2q_1), \end{cases}$$

$$\begin{cases} c_1^{[3]} = -\frac{\beta}{\alpha^3}(q_{1,xx} + 2p_2q_1q_2 + 4p_1q_1^2), \\ c_2^{[3]} = -\frac{\beta}{\alpha^3}(q_{2,xx} + 2p_2q_2^2 + 4p_1q_1q_2), \end{cases}$$

$$\begin{cases} a^{[3]} = \frac{\beta}{\alpha^3} i(2p_{1,x}q_1 + p_{2,x}q_2 - 2p_1q_{1,x} - p_2q_{2,x}), \\ d_{k,l}^{[3]} = \frac{\beta}{\alpha^3} i(p_l q_{k,x} - q_k p_{l,x}),\ 1 \leq k,l \leq 2; \end{cases}$$

and

$$\begin{cases} b_1^{[4]} = \frac{\beta}{\alpha^4} i(p_{1,xxx} + 3p_1p_{2,x}q_2 + 12p_1p_{1,x}q_1 + 3p_{1,x}p_2q_2), \\ b_2^{[4]} = \frac{\beta}{\alpha^4} i(p_{2,xxx} + 6p_1p_{2,x}q_1 + 6p_{1,x}p_2q_1 + 6p_2p_{2,x}q_2), \end{cases}$$

$$\begin{cases} c_1^{[4]} = -\frac{\beta}{\alpha^4} i(q_{1,xxx} + 12p_1q_1q_{1,x} + 3p_2q_1q_{2,x} + 3p_2q_{1,x}q_2), \\ c_2^{[4]} = -\frac{\beta}{\alpha^4} i(q_{2,xxx} + 6p_1q_{1,x}q_2 + 6p_1q_1q_{2,x} + 6p_2q_2q_{2,x}), \end{cases}$$

$$\begin{cases} a^{[4]} = \frac{\beta}{\alpha^4}\big[2p_{1,xx}q_1 + 2p_1q_{1,xx} + p_{2,xx}q_2 + p_2q_{2,xx} \\ \qquad -2p_{1,x}q_{1,x} - p_{2,x}q_{2,x} + 3(2p_1q_1 + p_2q_2)^2\big], \\ d_{k,l}^{[4]} = -\frac{\beta}{\alpha^4}(6p_1q_1p_lq_k + 3p_2q_2p_lq_k + p_{l,xx}q_k + p_lq_{k,xx} - p_{l,x}q_{k,x}),\ 1 \leq k,l \leq 2; \end{cases}$$

which will be used to present examples of integrable Hamiltonian equations below.

We take advantage of the zero curvature formulation, and a direct computation shows that the following temporal matrix spectral problems

$$-i\phi_t = \mathcal{N}^{[r]}\phi = \mathcal{N}^{[r]}(u,\lambda)\phi,\ \mathcal{N}^{[r]} = (\lambda^r \mathcal{Y})_+ = \sum_{s=0}^{r} \lambda^s \mathcal{Y}^{[r-s]},\ r \geq 0, \tag{17}$$

are appropriate other parts of Lax pairs so that the compatibility conditions of the resulting Lax pairs, *i.e.*, the zero curvature equations in (4), present a four-component integrable hierarchy:

$$u_{t_r} = K^{[r]} = (\alpha i b_1^{[r+1]}, \alpha i b_2^{[r+1]}, -\alpha i c_1^{[r+1]}, -\alpha i c_2^{[r+1]})^T,\ r \geq 0, \tag{18}$$

or more specifically,

$$p_{1,t_r} = \alpha i b_1^{[r+1]},\ p_{2,t_r} = \alpha i b_2^{[r+1]},\ q_{1,t_r} = -\alpha i c_1^{[r+1]},\ q_{2,t_r} = -\alpha i c_2^{[r+1]},\ r \geq 0. \tag{19}$$

Based on the previous expressions of $b_1^{[s]}, b_2^{[s]}, c_1^{[s]}$ and $c_2^{[s]}$, $s \geq 1$, we immediately obtain the first two examples of integrable nonlinear evolution equations. The first one is the integrable coupled nonlinear Schrödinger equations:

$$\begin{cases} ip_{1,t_2} = \frac{\beta}{\alpha^2}(p_{1,xx} + 2p_1p_2q_2 + 4p_1^2q_1), \\ ip_{1,t_2} = \frac{\beta}{\alpha^2}(p_{2,xx} + 2p_2^2q_2 + 4p_1p_2q_1), \\ iq_{1,t_2} = -\frac{\beta}{\alpha^2}(q_{1,xx} + 2p_2q_1q_2 + 4p_1q_1^2), \\ iq_{2,t_2} = -\frac{\beta}{\alpha^2}(q_{2,xx} + 4p_1q_1q_2 + 2p_2q_2^2), \end{cases} \tag{20}$$

and the second is the integrable coupled modified Korteweg-de Vries equations:

$$\begin{cases} p_{1,t_3} = -\frac{\beta}{\alpha^3}(p_{1,xxx} + 3p_1p_{2,x}q_2 + 12p_1p_{1,x}q_1 + 3p_{1,x}p_2q_2), \\ p_{2,t_3} = -\frac{\beta}{\alpha^3}(p_{2,xxx} + 6p_1p_{2,x}q_1 + 6p_{1,x}p_2q_1 + 6p_2p_{2,x}q_2), \\ q_{1,t_3} = -\frac{\beta}{\alpha^3}(q_{1,xxx} + 12p_1q_1q_{1,x} + 3p_2q_{1,x}q_2 + 3p_2q_1q_{2,x}), \\ q_{2,t_3} = -\frac{\beta}{\alpha^3}(q_{2,xxx} + 6p_1q_{1,x}q_2 + 6p_1q_1q_{2,x} + 6p_2q_2q_{2,x}). \end{cases} \tag{21}$$

Those two examples enrich the category of integrable multi-component nonlinear Schrödinger equations and integrable multi-component modified Korteweg-de Vries equations (see, *e.g.*, [18–20]).

3. HAMILTONIAN STRUCTURES

In order to establish Hamiltonian structures for the presented integrable hierarchy (19), we apply the trace identity (3) associated with the matrix spectral problem (6). Using the solution $\mathcal{Y}$ determined by (8), we can derive

$$\mathrm{tr}\big(\mathcal{Y}\frac{\partial \mathcal{M}}{\partial \lambda}\big) = \alpha_1 a + \alpha_2(2d_{1,1} + d_{2,2}),\ \mathrm{tr}\big(\mathcal{Y}\frac{\partial \mathcal{M}}{\partial u}\big) = (2c_1, c_2, 2b_1, b_2)^T,$$

and thus, we arrive at

$$\frac{\delta}{\delta u}\int[\alpha_1 a^{[s+1]}+\alpha_2(2d_{1,1}^{[s]}+d_{2,2}^{[s]})]\lambda^{-s-1}\,dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma-s}(2c_1^{[s]},c_2^{[s]},2b_1^{[s]},b_2^{[s]})^T,$$

where $s\geq 0$. Checking the case with $s=2$, we see $\gamma=0$. Consequently, we obtain

$$\frac{\delta}{\delta u}\mathcal{H}^{[s]}=(2c_1^{[s+1]},c_2^{[s+1]},2b_1^{[s+1]},b_2^{[s+1]})^T,\ s\geq 0, \tag{22}$$

where the Hamiltonian functionals are defined by

$$\mathcal{H}^{[s]}=\int H^{[s]}\,dx,\ H^{[s]}=-\frac{\alpha_1 a^{[s+2]}+\alpha_2(2d_{1,1}^{[s+2]}+d_{2,2}^{[s+2]})}{s+1},\ s\geq 0, \tag{23}$$

of which the first three Hamiltonian functional are

$$\mathcal{H}^{[0]}=\int\frac{\beta}{\alpha}(2p_1q_1+p_2q_2)\,dx, \tag{24}$$

$$\mathcal{H}^{[1]}=\int\frac{\beta}{2\alpha^2}i\big[2(p_1q_{1,x}-p_{1,x}q_1)+(p_2q_{2,x}-p_{2,x}q_2)\big]\,dx, \tag{25}$$

and

$$\begin{aligned}\mathcal{H}^{[2]}=\int\frac{\beta}{3\alpha^3}\big[&-2p_{1,xx}q_1-p_{2,xx}q_2-2p_1q_{1,xx}-p_2q_{2,xx}\\&+2p_{1,x}q_{1,x}+p_{2,x}q_{2,x}-3(2p_1q_1+p_2q_2)^2\big]\,dx.\end{aligned} \tag{26}$$

Those identities allow us to present the Hamiltonian structures for the obtained integrable hierarchy (19):

$$u_{t_r}=K^{[r]}=J\frac{\delta\mathcal{H}^{[r]}}{\delta u},\ J=\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \frac{1}{2}\alpha i & 0\\ & & 0 & \alpha i\\ \hline -\frac{1}{2}\alpha i & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}}\\ 0 & -\alpha i & & \end{array}\right],\ r\geq 0, \tag{27}$$

where J is skew-symmetric and thus Hamiltonian, and the Hamiltonian functionals $\mathcal{H}^{[r]}$, $r\geq 0$, are determined by (23). It is known that the Hamiltonian structures exhibit a connection from a conserved functional $\mathcal{H}$ to a symmetry S by $S=J\frac{\delta\mathcal{H}}{\delta u}$.

A direct computation shows that we can have an isospectral Lax operator algebra (see [21, 22] for details):

$$\begin{aligned}&[\![\mathcal{N}^{[s_1]},\mathcal{N}^{[s_2]}]\!]=\mathcal{N}^{[s_1]\prime}(u)[K^{[s_2]}]-\mathcal{N}^{[s_2]\prime}(u)[K^{[s_1]}]+[\mathcal{N}^{[s_1]},\mathcal{N}^{[s_2]}]\\&=\frac{\partial}{\partial\epsilon}\big[\mathcal{N}^{[s_1]}(u+\epsilon K^{[s_2]})-\mathcal{N}^{[s_2]}(u+\epsilon K^{[s_1]})\big]\big|_{\epsilon=0}+[\mathcal{N}^{[s_1]},\mathcal{N}^{[s_2]}]=0,\ s_1,s_2\geq 0,\end{aligned} \tag{28}$$

which is a consequence of the isospectral zero curvature equations [22]. This Lax operator algebra guarantees the Abelain algebra of infinitely many symmetries $\{K^{[s]}\}_{s=0}^{\infty}$:

$$\begin{aligned}&[\![K^{[s_1]},K^{[s_2]}]\!] = K^{[s_1]\prime}(u)[K^{[s_2]}] - K^{[s_2]\prime}(u)[K^{[s_1]}]\\ &= \frac{\partial}{\partial\epsilon}\big[K^{[s_1]}(u+\epsilon K^{[s_2]}) - K^{[s_2]}(u+\epsilon K^{[s_1]})\big]\big|_{\epsilon=0} = 0,\ s_1,s_2 \geq 0.\end{aligned} \tag{29}$$

It further follows from the Hamiltonian structures that the conserved functionals $\{\mathcal{H}^{[s]}\}_{s=0}^{\infty}$ form an Abelian algebra:

$$\{\mathcal{H}^{[s_1]},\mathcal{H}^{[s_2]}\}_J = \int (\frac{\delta\mathcal{H}^{[s_1]}}{\delta u})^T J \frac{\delta\mathcal{H}^{[s_2]}}{\delta u}\,dx = 0,\ s_1,s_2 \geq 0. \tag{30}$$

This implies that each equation in the resulting hierarchy (19) is Liouville integrable, or more precisely, each possesses infinitely many commuting conserved densities $\{\mathcal{H}^{[s]}\}_{s=0}^{\infty}$ and symmetries $\{K^{[s]}\}_{s=0}^{\infty}$. Furthermore, a combination of J with a recursion operator Φ [23], generated from the recursion relation $K^{[s+1]} = \Phi K^{[s]}$, leads to bi-Hamiltonian structures [12] for the hierarchy.

4. HIGHER-ORDER LAX PAIRS AND INTEGRABLE HIERARCHIES

Let $m \geq 1$ be an arbitrarily given natural number. We can consider a generalization of the matrix spatial spectral problem (6):

$$-i\phi_x = \mathcal{M}\phi,\ \mathcal{M} = \begin{bmatrix} \alpha_1\lambda & \mathbf{p}_1 & p_2 & \mathbf{p}_1 \\ \mathbf{q}_1 & \alpha_2 I_m\lambda & 0 & 0 \\ q_2 & 0 & \alpha_2\lambda & 0 \\ \mathbf{q}_1 & 0 & 0 & I_m\alpha_2\lambda \end{bmatrix}_{(2m+2)\times(2m+2)}, \tag{31}$$

where I_m is the m-th order identity matrix, and

$$\mathbf{p}_1 = (\underbrace{p_1,\cdots,p_1}_{m}),\ \mathbf{q}_1 = (\underbrace{q_1,\cdots,q_1}_{m})^T. \tag{32}$$

Similarly, a Laurent series solution to the corresponding stationary zero curvature equation could be taken as

$$\mathcal{Y} = \begin{bmatrix} a & \mathbf{b}_1 & b_2 & \mathbf{b}_1 \\ \mathbf{c}_1 & d_{1,1}E_{m,m} & d_{1,2}E_{m,1} & d_{1,1}E_{m,m} \\ c_2 & d_{2,1}E_{1,m} & d_{2,2} & d_{2,1}E_{1,m} \\ \mathbf{c}_1 & d_{1,1}E_{m,m} & d_{1,2}E_{m,1} & d_{1,1}E_{m,m} \end{bmatrix}_{(2m+2)\times(2m+2)} = \sum_{s\geq 0}\lambda^{-s}\mathcal{Y}^{[s]}, \tag{33}$$

where $E_{k,l}$ is the $k\times l$ matrix of ones, $\mathbf{b}_1$ and $\mathbf{c}_1$ are given by

$$\mathbf{b}_1 = (\underbrace{b_1,\cdots,b_1}_{m}),\ \mathbf{c}_1 = (\underbrace{c_1,\cdots,c_1}_{m})^T, \tag{34}$$

and a, b_j, c_j and $d_{k,l}$ are assumed to be of Laurent series form

$$a = \sum_{s\geq 0} \lambda^{-s} a^{[s]},\ b_j = \sum_{s\geq 0} \lambda^{-s} b_j^{[s]},\ c_j = \sum_{s\geq 0} \lambda^{-s} c_j^{[s]},\ d_{k,l} = \sum_{s\geq 0} \lambda^{-s} d_{k,l}^{[s]}, \tag{35}$$

in which $1 \leq j,k,l \leq 2$.

In this general case, we have

$$\frac{\delta}{\delta u} \int \left[\alpha_1 a + \alpha_2(2md_{1,1]} + d_{2,2})\right] dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} (2mc_1, c_2, 2mb_1, b_2)^T.$$

Consequently, the associated integrable equations and their Hamiltonian structures read

$$u_{t_r} = K^{[r]} = \Phi^r K^{[0]} = (\alpha i b_1^{[r+1]}, \alpha i b_2^{[r+1]}, -\alpha i c_1^{[r+1]}, -\alpha i c_2^{[r+1]})^T = J \frac{\delta \mathcal{H}^{[r]}}{\delta u},\ r \geq 0, \tag{36}$$

where the Hamiltonian operator J is defined by

$$J = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \frac{1}{2m}\alpha i & 0 \\ & & 0 & \alpha i \\ \hline -\frac{1}{2m}\alpha i & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -\alpha i & & \end{array}\right], \tag{37}$$

and the Hamiltonian functionals are determined by

$$\mathcal{H}^{[r]} = -\int \frac{\alpha_1 a^{[r+2]} + \alpha_2(2md_{1,1}^{[s+2]} + d_{2,2}^{[s+2]})}{r+1},\ r \geq 0. \tag{38}$$

Each equation in every hierarchy in (36) is Liouville integrable, and actually possesses infinitely many commuting conserved densities and symmetries, as shown in (29) and (30).

When taking the initial values in (15), and the Lax operators, $\mathcal{N}^{[s]},\ s \geq 0$, as in (17), we can have the first two integrable nonlinear equations in the hierarchy (36):

$$\begin{cases} ip_{1,t_2} = \frac{\beta}{\alpha^2}(p_{1,xx} + 4mp_1^2 q_1 + 2p_1 p_2 q_2), \\ ip_{1,t_2} = \frac{\beta}{\alpha^2}(p_{2,xx} + 2p_2^2 q_2 + 4mp_1 p_2 q_1), \\ iq_{1,t_2} = -\frac{\beta}{\alpha^2}(q_{1,xx} + 4mp_1 q_1^2) + 2p_2 q_1 q_2, \\ iq_{2,t_2} = -\frac{\beta}{\alpha^2}(q_{2,xx} + 2p_2 q_2^2 + 4mp_1 q_1 q_2), \end{cases} \tag{39}$$

and

$$\begin{cases} p_{1,t_3} = -\frac{\beta}{\alpha^3}(p_{1,xxx} + 12mp_1 p_{1,x} q_1 + 3(p_1 p_2)_x q_2), \\ p_{2,t_3} = -\frac{\beta}{\alpha^3}(p_{2,xxx} + 6m(p_1 p_2)_x q_1 + 6p_2 p_{2,x} q_2), \\ q_{1,t_3} = -\frac{\beta}{\alpha^3}(q_{1,xxx} + 12mp_1 q_1 q_{1,x} + 3p_2(q_1 q_2)_x), \\ q_{2,t_3} = -\frac{\beta}{\alpha^3}(q_{2,xxx} + 6mp_1(q_1 q_2)_X + 6p_2 q_2 q_{2,x}). \end{cases} \tag{40}$$

5. CONCLUDING REMARKS

A set of integrable Hamiltonian hierarchies with four components has been constructed, based on a class of special matrix spectral problems, through the zero curvature formulation. The resulting integrable equations possess Hamiltonian structures, furnished *via* applications of the trace identity to the underlying matrix spatial spectral problems.

Other generalizations could be formulated by taking more copies of p_1 as did for p_2. Also, we can naturally have more components in matrix spatial spectral problems to generate integrable Hamiltonian equations with six or more components.

It would be interesting to find soliton solutions to the obtained integrable Hamiltonian equations. The Darboux transformation [24], the Riemann-Hilbert technique [25] and the Zakharov-Shabat dressing method [26] should be helpful. It is worth pointing out that if the underlying algebra is taken to be $\mathrm{gl}(\infty)$, then we can have a τ-function theory, which generates soliton type solutions in a natural way. Other interesting solutions (see, *e.g.*, [27–29]) can be generated by taking group reductions. Nonlocal integrable counterparts could also be formulated under similarity transformations of spectral matrices (see, *e.g.*, [31, 32] for details). Any theories of soliton solutions in nonlocal cases (see, *e.g.*, [33–35] for novel nonlocal nonlinear Schrödinger equations) are very helpful in recognizing characteristics of nonlinear waves.

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