



Two Distinct Group Reductions Leading to Integrable Coupled mKdV Models

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Received: 18 June 2025 / Accepted: 20 April 2026
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Abstract

We aim to explore pairs of distinct group reductions of the Ablowitz–Kaup–Newell–Segur matrix spectral problem that lead to integrable coupled modified Korteweg–de Vries (mKdV) models. Specifically, we formulate three representative group reduction schemes, each resulting in the derivation of new integrable coupled mKdV systems. This analysis not only introduces novel examples of integrable models but also provides fresh insights into the classification of third-order integrable equations.

Keywords Lax pair · Zero-curvature condition · Group reduction · Integrable hierarchy · Integrable mKdV models

Mathematics Subject Classification 37K10 · 35Q51 · 37K40

1 Introduction

The inverse scattering transform provides a nonlinear analogue of the Fourier transform for solving integrable models. Two well-known examples are the Korteweg–de Vries equation [1] and the sine-Gordon equation [2]. A fundamental question in soliton theory is how to construct and classify integrable models. Various scalar integrable models, such as the nonlinear Schrödinger (NLS) equations and the modified Korteweg–de Vries (mKdV) equations, both local [3, 4] and nonlocal [5, 6], have been extensively studied in the literature [7, 8]. However, there is a much wider diversity of multi-component integrable models (see, e.g., [9, 10]), and compara-

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tively fewer examples have been systematically documented in different contexts. One well-known example of a two-component integrable mKdV model is the standard Ablowitz–Kaup–Newell–Segur (AKNS) integrable mKdV system:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 6p_1 p_2 p_{1,x}, \\ p_{2,t} = p_{2,xxx} + 6p_1 p_2 p_{2,x}. \end{cases} \quad (1.1)$$

In this work, we show that the following two new systems are also integrable:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 3(4p_1^2 + p_2^2)p_{1,x} + 3p_1 p_2 p_{2,x}, \\ p_{2,t} = p_{2,xxx} + 6p_1 p_2 p_{1,x} + 6(p_1^2 + p_2^2)p_{2,x}, \end{cases} \quad (1.2)$$

and

$$\begin{cases} p_{1,t} = p_{1,xxx} + 6(p_1^2 + p_2^2)p_{1,x} + 12p_1 p_2 p_{2,x}, \\ p_{2,t} = p_{2,xxx} + 12p_1 p_2 p_{1,x} + 6(p_1^2 + p_2^2)p_{2,x}. \end{cases} \quad (1.3)$$

These systems are integrable in the sense that they admit reduced Lax pairs and possess infinitely many symmetries and conservation laws. They are derived from appropriately formulated matrix spectral problems through the application of dual group reductions or similarity transformations. Notably, these two systems do not appear in the comprehensive list of integrable systems compiled by Meshkov and Balakhnev using the symmetry approach [11]. Nevertheless, they should be derivable from the compatibility equations of the extended KP hierarchy [12] via Miura-type transformations.

Matrix spectral problems involving free potentials are fundamental and broadly applicable in the study of integrable models. In contrast, reduced matrix spectral problems are more restrictive and technically challenging to formulate. One effective approach for deriving such reduced problems is to apply group reductions, which have been used to generate integrable hierarchies (see, e.g., [13]). The main goal of using group reductions is to preserve the form of the associated zero-curvature equations, thereby facilitating the construction of integrable models. Two classical examples resulting from this approach are the NLS equation and the mKdV equation, both of which arise from the AKNS matrix spectral problem through a single group reduction.

Applying a pair of group reductions can lead to a broader class of integrable models (see, e.g., [14–17]). This dual-reduction framework introduces additional complexity, as the corresponding reductions on the potentials must be carefully balanced to preserve the compatibility of the associated zero-curvature equations. More recently, this dual-reduction approach has been extended to the construction of nonlocal integrable models (see, e.g., [6, 18, 19]). In particular, three types of reduced integrable NLS equations and two types of reduced integrable mKdV equations have been proposed and classified [20]. The inverse scattering transform has also been successfully developed to solve several nonlocal integrable models (see, e.g., [21, 22]). In addition to inverse scattering, several other powerful techniques have proven effective in constructing soliton solutions for both local and nonlocal integrable models. These include the Hirota bilinear method, Darboux transformations, Bäcklund transformations, and the Riemann–Hilbert approach. Substantial theoretical developments have been achieved

using these methods across a wide range of reduced integrable models (see, e.g., [13, 23–27]).

In this paper, we aim to formulate dual group reductions for the AKNS matrix spectral problem in order to construct integrable models with reduced Lax pairs. In Sect. 2, we revisit the AKNS framework of matrix spectral problems and integrable hierarchies, and recall a general approach to reduced integrable hierarchies via dual local group reductions, thereby laying the groundwork for subsequent analysis (see, e.g., [14, 28]). In Sect. 3, we present three specific dual group reductions and derive the corresponding reduced integrable hierarchies along with their associated AKNS-type reduced Lax pairs. Two concrete examples, mentioned earlier in this introduction, demonstrate the practical effectiveness of the theory. These results yield novel third-order integrable models and offer valuable insights into the classification of integrable model through the zero-curvature formulation. Finally, in the concluding section, we summarize our findings and provide some closing remarks.

2 Revisiting the AKNS Matrix Integrable Hierarchies and Dual Group Reductions

2.1 The AKNS Matrix Integrable Hierarchies

Within the AKNS framework for integrable models, we denote the dependent variable by $u = u(p, q)$, a column vector formed by two matrix-valued potentials:

$$p = p(x, t) = (p_{jk})_{m \times n}, \quad q = q(x, t) = (q_{kj})_{n \times m}, \quad (2.1)$$

where m and n are two positive integers. For each $r \geq 0$, the associated matrix AKNS spectral problems are given by

$$-i\phi_x = U\phi, \quad -i\phi_t = V^{[r]}\phi, \quad (2.2)$$

with the Lax pair defined as

$$U = U(u, \lambda) = \lambda\Lambda + P, \quad V^{[r]} = V^{[r]}(u, \lambda) = \lambda^r\Omega + Q^{[r]}, \quad (2.3)$$

where the matrices are

$$\Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix}, \quad P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \quad (2.4)$$

and

$$\Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix}, \quad Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}. \quad (2.5)$$

Here, I_k denotes the identity matrix of size k , λ is the spectral parameter, and $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$ are constant parameters. By convention, $Q^{[0]}$ is the zero matrix of order

$m + n$. The stationary zero-curvature equation,

$$W_x = i[U, W], \tag{2.6}$$

admits a unique solution in the form of a Laurent series:

$$W = \sum_{s \geq 0} \lambda^{-s} W^{[s]} = \sum_{s \geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}, \tag{2.7}$$

with given initial data $W^{[0]} = \Omega$. Taking a solution W to the stationary zero-curvature equation is a standard step in the Tu generating scheme [29], and this series expansion generates hierarchies of commuting integrable models (see, e.g., [30–33]).

Observing that (2.6) and (2.7) imply that

$$W_x^{[s]} = i[\Lambda, W^{[s+1]}] + i[P, W^{[s]}], \quad s \geq 0,$$

we can compute

$$\begin{aligned} V_x^{[r]} - i[U, V^{[r]}] &= \left(\sum_{s=0}^r \lambda^{r-s} W^{[s]} \right)_x - i[\lambda\Lambda + P, \sum_{s=0}^r \lambda^{r-s} W^{[s]}] \\ &= W_x^{[r]} - i[P, W^{[r]}] + \sum_{s=0}^{r-1} \lambda^{r-s} (W_x^{[s]} - i[\Lambda, W^{[s+1]}] - i[P, W^{[s]}]) - i\lambda^{r+1}[\Lambda, \Omega] \\ &= W_x^{[r]} - i[P, W^{[r]}] = i[\Lambda, W^{[r+1]}] = i \begin{bmatrix} 0 & \alpha b^{[r+1]} \\ -\alpha c^{[r+1]} & 0 \end{bmatrix}, \quad \alpha = \alpha_1 - \alpha_2. \end{aligned}$$

Based on this relation, the compatibility condition of the Lax pair (2.2),

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \tag{2.8}$$

yields the matrix AKNS hierarchy of integrable models:

$$p_t = i\alpha b^{[r+1]}, \quad q_t = -i\alpha c^{[r+1]}. \tag{2.9}$$

The scalar AKNS system with $m = n = 1$ is the classical example [4], which admits a reduction under a potential constraint [3]. Each system in the corresponding matrix integrable hierarchy admits a bi-Hamiltonian structure and possesses infinitely many symmetries and conserved quantities (see, e.g., [34–36]).

For $r = 2s + 1, s \geq 1$, the hierarchy (2.9) gives rise to a matrix mKdV integrable hierarchy. In particular, when $s = 1$, the Lax matrix $V^{[3]}$ is

$$V^{[3]} = \lambda^3 \Omega + \frac{\beta}{\alpha} \lambda^2 P - \frac{\beta}{\alpha^2} \lambda I_{m,n} (P^2 + iP_x) - \frac{\beta}{\alpha^3} (i[P, P_x] + P_{xx} + 2P^3), \tag{2.10}$$

where $I_{m,n} = \text{diag}(I_m, -I_n)$ and $\beta = \beta_1 - \beta_2$. This yields the AKNS matrix mKdV integrable model:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pq p_x + 3p_x q p), \quad q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_x p q + 3q p q_x). \quad (2.11)$$

These serve as foundational examples for studying matrix mKdV integrable models, and higher-order generalizations can be systematically constructed (see, e.g., [37]).

2.2 The Formulation of Dual Group Reductions

2.2.1 Reducing AKNS Matrix Spectral Problems

Let Σ_1 and Σ_2 be two constant, invertible, symmetric matrices of orders m and n , respectively, and let Δ_1 and Δ_2 be two other constant, invertible matrices of the same respective orders. We then construct two bigger constant invertible matrices of order $m + n$ as follows:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \quad (2.12)$$

To construct reduced AKNS matrix spectral problems, we consider two types of group reductions applied to a given AKNS spectral matrix U as defined in (2.3):

$$\Sigma U(\lambda) \Sigma^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T, \quad \Delta U(\lambda) \Delta^{-1} = U(\lambda), \quad (2.13)$$

where A^{-1} and A^T denote the inverse and transpose of a matrix A , respectively. As demonstrated in [38], the first group reduction preserves the invariance of the associated zero-curvature equations, while the second does so naturally. Given the specific block structure of U , these group reductions imply the following conditions on the potential matrix P :

$$\Sigma P \Sigma^{-1} = -P^T, \quad \Delta P \Delta^{-1} = P. \quad (2.14)$$

These in turn impose the following constraints on the two matrix potentials p and q :

$$p = -\Sigma_1^{-1} q^T \Sigma_2 \quad \text{or} \quad q = -\Sigma_2^{-1} p^T \Sigma_1, \quad (2.15)$$

and

$$p = \Delta_1 p \Delta_2^{-1}, \quad q = \Delta_2 q \Delta_1^{-1}. \quad (2.16)$$

Consequently, from (2.15) and (2.16), the matrix potential p must satisfy

$$\Delta_1 p = p \Delta_2, \quad \Sigma_1^{-1} \Delta_1^T \Sigma_1 p = p \Sigma_2^{-1} \Delta_2^T \Sigma_2, \quad (2.17)$$

or alternatively, the matrix potential q must satisfy

$$q \Delta_1 = \Delta_2 q, \quad q \Sigma_1^{-1} \Delta_1^T \Sigma_1 = \Sigma_2^{-1} \Delta_2^T \Sigma_2 q. \quad (2.18)$$

Thus, the AKNS spectral problem can be consistently reduced to the following form using either p or q :

$$-i\phi_x = U\phi, \quad U = \begin{bmatrix} \alpha_1\lambda I_m & p \\ -\Sigma_2^{-1}p^T \Sigma_1 & \alpha_2\lambda I_n \end{bmatrix}, \tag{2.19}$$

where p satisfies the constraints in (2.17), or alternatively,

$$-i\phi_x = U\phi, \quad U = \begin{bmatrix} \alpha_1\lambda I_m & -\Sigma_1^{-1}q^T \Sigma_2 \\ q & \alpha_2\lambda I_n \end{bmatrix}, \tag{2.20}$$

where q satisfies the constraints in (2.18) (see, e.g., [39, 40]).

2.3 Reducing Matrix mKdV Integrable Hierarchies

Due to the uniqueness of the Laurent series solution to the stationary zero-curvature equation (2.6), we can derive the following invariant properties:

$$\Sigma W(\lambda)\Sigma^{-1} = W^T(-\lambda) = (W(-\lambda))^T, \quad \Delta W(\lambda)\Delta^{-1} = W(\lambda), \tag{2.21}$$

where W is defined by (2.7). The structures of the solution formulas correspond to the two group reductions imposed on the AKNS spectral problem. For example, the first one can be derived as follows. From the stationary zero-curvature (2.6), we obtain

$$\begin{aligned} (\Sigma W(\lambda)\Sigma^{-1})_x &= i[\Sigma U(\lambda)\Sigma^{-1}, \Sigma W(\lambda)\Sigma^{-1}], \\ (W^T(-\lambda))_x &= i[W^T(-\lambda), U^T(-\lambda)] = i[-U^T(-\lambda), W^T(-\lambda)]. \end{aligned}$$

Using the first group reduction in (2.13), we observe that $\Sigma W(\lambda)\Sigma^{-1}$ and $W^T(-\lambda)$ satisfy the same stationary zero-curvature equation with the same initial data

$$\Sigma W^{[0]}\Sigma^{-1} = W^{[0]T} = \Omega.$$

By uniqueness, this establishes the first invariant property in (2.21). The second property follows by an entirely analogous argument. These invariance relations imply that for each $s \geq 0$, the Lax matrices $V^{[2s+1]} = (\lambda^{2s+1}W)_+$, as defined in (2.3), satisfy

$$\Sigma V^{[2s+1]}(\lambda)\Sigma^{-1} = -V^{[2s+1]T}(-\lambda) = -(V^{[2s+1]}(-\lambda))^T, \quad \Delta V^{[2s+1]}(\lambda)\Delta^{-1} = V^{[2s+1]}(\lambda). \tag{2.22}$$

Consequently, the associated zero-curvature equation

$$U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}] = 0 \tag{2.23}$$

retains its form under both group reductions:

$$\begin{cases} \Sigma(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}](\lambda)\Sigma^{-1} = -(U_t^T + V_x^{[2s+1]T} + i[U^T, V^{[2s+1]T}])(-\lambda), \\ \Delta(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}](\lambda)\Delta^{-1} = (U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}](\lambda)). \end{cases} \tag{2.24}$$

It follows that the matrix AKNS integrable models given in (2.11) for $r = 2s + 1, s \geq 0$, reduces to the following simplified matrix mKdV integrable hierarchy:

$$p_t = i\alpha b^{[2s+2]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}, \quad s \geq 0, \tag{2.25}$$

in term of p , satisfying the constraints in (2.17), or alternatively,

$$q_t = -i\alpha c^{[2s+2]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}, \quad s \geq 0, \tag{2.26}$$

in term of q , satisfying the constraints in (2.18).

Every member in the reduced hierarchy (2.25) or (2.26) is associated with the reduced spatial matrix spectral problems in (2.19) or (2.20), respectively. Moreover, each system admits an infinite hierarchy of commuting symmetries and conserved densities, inherited from for the original matrix AKNS integrable hierarchy. The corresponding temporal parts of the Lax pairs for the reduced models are given by

$$-i\phi_t = V^{[2s+1]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi, \quad s \geq 0, \tag{2.27}$$

or, alternatively,

$$-i\phi_t = V^{[2s+1]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}\phi, \quad s \geq 0. \tag{2.28}$$

Together with the respective spatial spectral problems, these spectral problems constitute the Lax pairs for the reduced p -based hierarchy (2.25) and the q -based hierarchy (2.26), respectively.

We summarize the above results in the following theorem:

Theorem 2.1 *Let α_1, α_2 and β_1, β_2 be two pairs of distinct constants. Define*

$$U = \begin{bmatrix} \alpha_1 I_m & p \\ q & \alpha_2 I_n \end{bmatrix},$$

where p and q are $m \times n$ and $n \times m$ matrix functions, respectively. Assume that

$$W = \sum_{s \geq 0} \lambda^{-s} W^{[s]} = \sum_{s \geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}$$

satisfies the stationary zero-curvature equation

$$W_x = i[U, W],$$

with initial condition $W^{[0]} = \text{diag}(\beta_1 I_m, \beta_2 I_n)$. For each $r \geq 0$, define

$$V^{[r]} = (\lambda^r W)_+ = \sum_{j=1}^r \lambda^j W^{[r-j]},$$

i.e., the polynomial part of $\lambda^r W$ in λ .

Let Σ_1 and Σ_2 be two constant, invertible, symmetric matrices of orders m and n , respectively, and let Δ_1 and Δ_2 be two other constant, invertible matrices of the same respective orders. Assume that either p satisfies

$$\Delta_1 p = p \Delta_2, \quad \Sigma_1^{-1} \Delta_1^T \Sigma_1 p = p \Sigma_2^{-1} \Delta_2^T \Sigma_2,$$

or alternatively, q satisfies

$$q \Delta_1 = \Delta_2 q, \quad q \Sigma_1^{-1} \Delta_1^T \Sigma_1 = \Sigma_2^{-1} \Delta_2^T \Sigma_2 q.$$

Then:

- All simplified mKdV equations in the heirarchy

$$p_t = i(\alpha_1 - \alpha_2) b^{[2s+2]}|_{q=-\Sigma_2^{-1} p^T \Sigma_1}, \quad s \geq 0,$$

commute, and each admits a Lax pair consisting of

$$U = \begin{bmatrix} \alpha_1 \lambda I_m & p \\ -\Sigma_2^{-1} p^T \Sigma_1 & \alpha_2 \lambda I_n \end{bmatrix}, \quad V^{[2s+1]}|_{q=-\Sigma_2^{-1} p^T \Sigma_1}, \quad s \geq 0.$$

- Similarly, all simplified mKdV equations in the heirarchy

$$q_t = -i(\alpha_1 - \alpha_2) c^{[2s+2]}|_{p=-\Sigma_1^{-1} q^T \Sigma_2}, \quad s \geq 0,$$

commute, and each possesses a Lax pair formed by

$$U = \begin{bmatrix} \alpha_1 \lambda I_m & -\Sigma_1^{-1} q^T \Sigma_2 \\ q & \alpha_2 \lambda I_n \end{bmatrix}, \quad V^{[2s+1]}|_{p=-\Sigma_1^{-1} q^T \Sigma_2}, \quad s \geq 0..$$

Therefore, each of the reduced hierarchies (2.25) and (2.26) defines an integrable hierarchy, where every equation serves as a symmetry of the others. Moreover, one can furnish Hamiltonian structures using the trace identity and derive a hierarchy of conserved functionals.

3 Three Representative Cases

In this section, we formulate three sets of dual group reductions and compute the corresponding reduced mKdV integrable hierarchies to illustrate the preceding analyses. It is worth noting that the arbitrariness of the matrices $\Delta_1, \Delta_2, \Sigma_1,$ and Σ_2 allows for a wide variety of new integrable mKdV models constructed via dual group reductions. Our analysis focuses on three representative cases to demonstrate the effectiveness of the proposed approach.

3.1 Case of $m = 1$ and $n = 3$

As a first case, we select the following matrix blocks:

$$\Delta_1 = 1, \Delta_2 = \begin{bmatrix} 0 & 0 & \delta_1 \\ 0 & 1 & 0 \\ \frac{1}{\delta_1} & 0 & 0 \end{bmatrix}; \Sigma_1 = \sigma_1\sigma_2, \Sigma_2 = \begin{bmatrix} 0 & 0 & \sigma_2 \\ 0 & 1 & 0 \\ \sigma_2 & 0 & 0 \end{bmatrix}; \quad (3.1)$$

where δ_1, σ_1 and σ_2 are all arbitrary non-zero constants. Under this choice, we obtain the two matrix potentials:

$$p = (p_1, p_2, \delta_1 p_1), q = -\sigma_1(\delta_1 p_1, \sigma_2 p_2, p_1)^T, \quad (3.2)$$

and the reduced matrix spectral problem becomes

$$-i\phi_x = U|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi = \begin{bmatrix} \alpha_1\lambda & p_1 & p_2 & \delta_1 p_1 \\ -\delta_1\sigma_1 p_1 & \alpha_2\lambda & 0 & 0 \\ -\sigma_1\sigma_2 p_2 & 0 & \alpha_2\lambda & 0 \\ -\sigma_1 p_1 & 0 & 0 & \alpha_2\lambda \end{bmatrix} \phi. \quad (3.3)$$

By substituting the chosen forms of p and q from (3.2), we find that the resulting third-order reduced integrable model in (2.25) corresponds precisely to the following integrable mKdV system:

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3} \{ p_{1,xxx} - 3\sigma_1 [(4\delta_1 p_1^2 + \sigma_2 p_2^2) p_{1,x} + \sigma_2 p_1 p_2 p_{2,x}] \}, \\ p_{2,t} = -\frac{\beta}{\alpha^3} \{ p_{2,xxx} - 6\sigma_1 [\delta_1 p_1 p_2 p_{1,x} + (\delta_1 p_1^2 + \sigma_2 p_2^2) p_{2,x}] \}, \end{cases} \quad (3.4)$$

where the constants $\alpha, \beta, \delta_1, \sigma_1, \sigma_2$ are all arbitrary but non-zero. The flexibility in choosing these parameters enables the construction of a wide variety of integrable mKdV models, including both focusing and defocusing types. Taking $\alpha = -\beta = \delta_1 = -\sigma_1 = \sigma_2 = 1$ reduces to the first example of new integrable mKdV models presented in (1.2), with the corresponding Lax pair for the integrable model (1.2) given

by

$$U = \begin{bmatrix} \alpha_1\lambda & p_1 & p_2 & p_1 \\ p_1 & (\alpha_1 - 1)\lambda & 0 & 0 \\ p_2 & 0 & (\alpha_1 - 1)\lambda & 0 \\ p_1 & 0 & 0 & (\alpha_1 - 1)\lambda \end{bmatrix}, \tag{3.5}$$

and

$$V = \begin{bmatrix} \beta_1\lambda^3 + 2\lambda(p_1^2 + p_2^2) & V_{12} & V_{13} & V_{14} \\ V_{21} & (\beta_1 + 1)\lambda^3 - \lambda p_1^2 & V_{23} & -\lambda p_1^2 \\ V_{31} & V_{32} & (\beta_1 + 1)\lambda^3 - \lambda p_2^2 & V_{34} \\ V_{41} & -\lambda p_1^2 & V_{43} & (\beta_1 + 1)\lambda^3 - \lambda p_1^2 \end{bmatrix}, \tag{3.6}$$

where α_1 and β_1 are arbitrary constants, and

$$\begin{cases} V_{12} = V_{14} = -\lambda^2 p_1 + i\lambda p_{1,x} + p_{1,xx} + 4p_1^3 + 2p_1 p_2^2, \\ V_{13} = -\lambda^2 p_2 + i\lambda p_{2,x} + p_{2,xx} + 4p_1^2 p_2 + 2p_2^3, \\ V_{21} = V_{41} = -\lambda^2 p_1 - i\lambda p_{1,x} + p_{1,xx} + 4p_1^3 + 2p_1 p_2^2, \\ V_{23} = V_{43} = -\lambda p_1 p_2 - i p_{1,x} p_2 + i p_1 p_{2,x}, \\ V_{31} = -\lambda^2 p_2 - i\lambda p_{2,x} + p_{2,xx} + 4p_1^2 p_2 + 2p_2^3, \\ V_{32} = V_{34} = -\lambda p_1 p_2 + i p_{1,x} p_2 - i p_1 p_{2,x}. \end{cases} \tag{3.7}$$

3.2 Case of $m = 2$ and $n = 2$

As a second case, we consider the following two pairs of matrix blocks:

$$\Delta_1 = \Delta_2 = \begin{bmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{bmatrix}; \quad \Sigma_1 = \delta_1 \sigma_1 \sigma_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Sigma_2 = \sigma_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \tag{3.8}$$

where all $\delta_1, \delta_2, \sigma_1$ and σ_2 are arbitrary but non-zero constants. By making these selections, we arrive at:

$$p = \begin{bmatrix} p_1 & p_2 \\ \frac{\delta_2}{\delta_1} p_2 & p_1 \end{bmatrix}, \quad q = -\sigma_1 \begin{bmatrix} \delta_1 p_1 & \delta_1 p_2 \\ \delta_2 p_2 & \delta_1 p_1 \end{bmatrix}. \tag{3.9}$$

As a result, the reduced matrix spectral problem assumes the form:

$$-i\phi_x = U|_{q=-\Sigma_2^{-1}p^T \Sigma_1} \phi = \begin{bmatrix} \alpha_1\lambda & 0 & p_1 & p_2 \\ 0 & \alpha_1\lambda & \frac{\delta_2}{\delta_1} p_2 & p_1 \\ -\sigma_1 \delta_1 p_1 & -\sigma_1 \delta_1 p_2 & \alpha_2 \lambda & 0 \\ -\sigma_1 \delta_2 p_2 & -\sigma_1 \delta_1 p_1 & 0 & \alpha_2 \lambda \end{bmatrix} \phi. \tag{3.10}$$

The resulting class of reduced coupled integrable mKdV models in (2.25) is expressed as:

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3} \{ p_{1,xxx} - 6\sigma_1 [(\delta_1 p_1^2 + \delta_2 p_2^2) p_{1,x} + 2\delta_2 p_1 p_2 p_{2,x}] \}, \\ p_{2,t} = -\frac{\beta}{\alpha^3} \{ p_{2,xxx} - 6\sigma_1 [2\delta_1 p_1 p_2 p_{1,x} + (\delta_1 p_1^2 + \delta_2 p_2^2) p_{2,x}] \}, \end{cases} \tag{3.11}$$

where the constants $\alpha, \beta, \sigma_1, \delta_1, \delta_2$ are arbitrary but non-zero. This again provides a diverse set of integrable mKdV models, encompassing both focusing and defocusing types. Notably, the distribution of the constant coefficients in these models differs slightly from those in the previous class of integrable mKdV models in (3.4).

Choosing $\alpha = -\beta = -\sigma_1 = \delta_1 = \delta_2 = 1$ reduces to the second example of new integrable mKdV models presented in (1.3), with the corresponding Lax pair given by

$$U = \begin{bmatrix} \alpha_1 \lambda & 0 & p_1 & p_2 \\ 0 & \alpha_1 \lambda & p_2 & p_1 \\ p_1 & p_2 & (\alpha_1 - 1)\lambda & 0 \\ p_2 & p_1 & 0 & (\alpha_1 - 1)\lambda \end{bmatrix}, \tag{3.12}$$

and

$$V = \begin{bmatrix} \beta_1 \lambda^3 + \lambda(p_1^2 + p_2^2) & 2\lambda p_1 p_2 & V_{13} & V_{14} \\ 2\lambda p_1 p_2 & \beta_1 \lambda^3 + \lambda(p_1^2 + p_2^2) & V_{23} & V_{24} \\ V_{31} & V_{32} & (\beta_1 + 1)\lambda^3 - \lambda(p_1^2 + p_2^2) & -2\lambda p_1 p_2 \\ V_{41} & V_{42} & -2\lambda p_1 p_2 & (\beta_1 + 1)\lambda^3 - \lambda(p_1^2 + p_2^2) \end{bmatrix}, \tag{3.13}$$

where α_1 and β_1 are arbitrary constants, and

$$\begin{cases} V_{13} = V_{24} = -\lambda^2 p_1 + i\lambda p_{1,x} + p_{1,xx} + 2p_1^3 + 6p_1 p_2^2, \\ V_{14} = V_{23} = -\lambda^2 p_2 + i\lambda p_{2,x} + p_{2,xx} + 6p_1^2 p_2 + 2p_2^3, \\ V_{31} = V_{42} = -\lambda^2 p_1 - i\lambda p_{1,x} + p_{1,xx} + 2p_1^3 + 6p_1 p_2^2, \\ V_{32} = V_{41} = -\lambda^2 p_2 - i\lambda p_{2,x} + p_{2,xx} + 6p_1^2 p_2 + 2p_2^3. \end{cases} \tag{3.14}$$

3.3 Case of $m = 2$ and $n = 3$

Finally, we select two pairs of matrix blocks as follows:

$$\Delta_1 = \delta_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \Delta_2 = \delta_1 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \Sigma_1 = \sigma_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} \sigma_3 & 0 & \sigma_4 \\ 0 & \sigma_2 & 0 \\ \sigma_4 & 0 & \sigma_3 \end{bmatrix}; \tag{3.15}$$

where $\delta_1, \sigma_1, \sigma_2$ are arbitrary but non-zero constants, while σ_3 and σ_4 are arbitrary constants, subject to the condition $\sigma_3^2 \neq \sigma_4^2$. Then, we obtain the two reduced matrix potentials as follows:

$$p = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_3 & p_2 & p_1 \end{bmatrix}, \quad q = -\sigma_1 \begin{bmatrix} \frac{\sigma_3 p_1 - \sigma_4 p_3}{\sigma_3^2 - \sigma_4^2} & -\frac{\sigma_4 p_1 - \sigma_3 p_3}{\sigma_3^2 - \sigma_4^2} \\ \frac{p_2}{\sigma_2} & \frac{p_2}{\sigma_2} \\ -\frac{\sigma_4 p_1 - \sigma_3 p_3}{\sigma_3^2 - \sigma_4^2} & \frac{\sigma_3 p_1 - \sigma_4 p_3}{\sigma_3^2 - \sigma_4^2} \end{bmatrix}. \quad (3.16)$$

Thus, the reduced matrix spectral problem takes the form:

$$-i\phi_x = U|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi = \begin{bmatrix} \alpha_1\lambda & 0 & p_1 & p_2 & p_3 \\ 0 & \alpha_1\lambda & p_3 & p_2 & p_1 \\ -\frac{\sigma_1(\sigma_3 p_1 - \sigma_4 p_3)}{\sigma_3^2 - \sigma_4^2} & \frac{\sigma_1(\sigma_4 p_1 - \sigma_3 p_3)}{\sigma_3^2 - \sigma_4^2} & \alpha_2\lambda & 0 & 0 \\ -\frac{\sigma_1 p_2}{\sigma_2} & -\frac{\sigma_1 p_2}{\sigma_2} & 0 & \alpha_2\lambda & 0 \\ \frac{\sigma_1(\sigma_4 p_1 - \sigma_3 p_3)}{\sigma_3^2 - \sigma_4^2} & -\frac{\sigma_1(\sigma_3 p_1 - \sigma_4 p_3)}{\sigma_3^2 - \sigma_4^2} & 0 & 0 & \alpha_2\lambda \end{bmatrix} \phi. \quad (3.17)$$

The class of corresponding reduced coupled integrable mKdV models in (2.25) is as follows:

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3} p_{1,xxx} + \frac{3\beta\sigma_1}{\alpha^3\sigma_2(\sigma_3^2 - \sigma_4^2)} \{ [2\sigma_2\sigma_3 p_1^2 - 4\sigma_2\sigma_4 p_1 p_3 + (\sigma_3^2 - \sigma_4^2) p_2^2 + 2\sigma_2\sigma_3 p_3^2] p_{1,x} \\ \quad + (\sigma_3^2 - \sigma_4^2) p_2(p_1 + p_3) p_{2,x} + [-2\sigma_2\sigma_4 p_1^2 + 4\sigma_2\sigma_3 p_1 p_3 + (\sigma_3^2 - \sigma_4^2) p_2^2 - 2\sigma_2\sigma_4 p_3^2] p_{3,x} \}, \\ p_{2,t} = -\frac{\beta}{\alpha^3} p_{2,xxx} + \frac{3\beta\sigma_1}{\alpha^3\sigma_2(\sigma_3 + \sigma_4)} \{ \sigma_2 p_2(p_1 + p_3)(p_{1,x} + p_{3,x}) + [\sigma_2(p_1 + p_3)^2 + 4(\sigma_3 + \sigma_4) p_2^2] p_{2,x} \}, \\ p_{3,t} = -\frac{\beta}{\alpha^3} p_{3,xxx} + \frac{3\beta\sigma_1}{\alpha^3\sigma_2(\sigma_3^2 - \sigma_4^2)} \{ [-2\sigma_2\sigma_4 p_1^2 + 4\sigma_2\sigma_3 p_1 p_3 + (\sigma_3^2 - \sigma_4^2) p_2^2 - 2\sigma_2\sigma_4 p_3^2] p_{1,x} \\ \quad + (\sigma_3^2 - \sigma_4^2) p_2(p_1 + p_3) p_{2,x} + [2\sigma_2\sigma_3 p_1^2 - 4\sigma_2\sigma_4 p_1 p_3 + (\sigma_3^2 - \sigma_4^2) p_2^2 + 2\sigma_2\sigma_3 p_3^2] p_{3,x} \}, \end{cases} \quad (3.18)$$

where the constants $\alpha, \beta, \sigma_1, \sigma_2$ are arbitrary but non-zero, and the constants σ_3, σ_4 are arbitrary with the condition that $\sigma_3^2 \neq \sigma_4^2$. This leads to a broad class of integrable mKdV models formulated via dual group reductions. It is worth noting that δ_1 does not appear in the resulting integrable mKdV models.

If we further choose

$$\alpha = -\beta = \sigma_1 = -\sigma_2 = \sigma_4 = -1, \quad \sigma_3 = 2, \quad (3.19)$$

then the corresponding spectral matrix is

$$U = U(u, \lambda) = \begin{bmatrix} \alpha_1\lambda & 0 & p_1 & p_2 & p_3 \\ 0 & \alpha_1\lambda & p_3 & p_2 & p_1 \\ \frac{2}{3}p_1 + \frac{1}{3}p_3 & \frac{1}{3}p_1 + \frac{2}{3}p_3 & \alpha_2\lambda & 0 & 0 \\ p_2 & p_2 & 0 & \alpha_2\lambda & 0 \\ \frac{1}{3}p_1 + \frac{2}{3}p_3 & \frac{2}{3}p_1 + \frac{1}{3}p_3 & 0 & 0 & \alpha_2\lambda \end{bmatrix}, \quad (3.20)$$

where $u = (p_1, p_2, p_3)^T$ and $\alpha_1 - \alpha_2 = -1$, and the resulting coupled integrable mKdV model takes the form:

$$\begin{cases} p_{1,t} = p_{1,xxx} + (4p_1^2 + 4p_1p_3 + 3p_2^2 + 4p_3^2)p_{1,x} + 3p_2(p_1 + p_3)p_{2,x} + (2p_1^2 + 8p_1p_3 + 3p_2^2 + 2p_3^2)p_{3,x}, \\ p_{2,t} = p_{2,xxx} + 3p_2(p_1 + p_3)(p_{1,x} + p_{3,x}) + 3[(p_1 + p_3)^2 + 4p_2^2]p_{2,x}, \\ p_{3,t} = p_{3,xxx} + (2p_1^2 + 8p_1p_3 + 3p_2^2 + 2p_3^2)p_{1,x} + 3p_2(p_1 + p_3)p_{2,x} + (4p_1^2 + 4p_1p_3 + 3p_2^2 + 4p_3^2)p_{3,x}. \end{cases} \quad (3.21)$$

This provides the third concrete example of reduced integrable matrix mKdV models derived through the dual-reduction approach.

We point out that all matrix blocks used in the above formulation of dual group reductions were obtained through symbolic computations performed using Maple. Extending the analysis to larger values of m and n presents greater challenges but also offers a broader diversity of matrix spectral problems and associated integrable models.

4 Summary and Remarks

In this work, we have developed a class of reduced matrix integrable mKdV hierarchies by applying dual group reductions to the matrix AKNS spectral problem. These reductions yield simplified models that retain the integrability of the original hierarchy, evidenced by the preservation of Lax pairs, infinitely many commuting symmetries, and conserved quantities. Three illustrative examples have been presented to demonstrate the construction and integrability of the reduced models. The flexibility in choosing structural matrices and parameters allows for a variety of integrable mKdV models, encompassing both focusing and defocusing types. Symbolic computations using Maple were employed to obtain the matrix blocks necessary for these reductions, especially in cases with multiple parameters or higher matrix dimensions.

We emphasize that the reduced matrix integrable mKdV models presented in Examples 3.1 and 3.2 include the following two specific examples, which differ from those obtained via a single group reduction in [38]:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 3(4\gamma_1 p_1^2 + \gamma_2 p_2^2)p_{1,x} + 3\gamma_2 p_1 p_2 p_{2,x}, \\ p_{2,t} = p_{2,xxx} + 6\gamma_1 p_1 p_2 p_{1,x} + 6(\gamma_1 p_1^2 + \gamma_2 p_2^2)p_{2,x}, \end{cases} \quad (4.1)$$

and

$$\begin{cases} p_{1,t} = p_{1,xxx} + 6(\gamma_1 p_1^2 + \gamma_2 p_2^2)p_{1,x} + 12\gamma_2 p_1 p_2 p_{2,x}, \\ p_{2,t} = p_{2,xxx} + 12\gamma_1 p_1 p_2 p_{1,x} + 6(\gamma_1 p_1^2 + \gamma_2 p_2^2)p_{2,x}, \end{cases} \quad (4.2)$$

where γ_1 and γ_2 are arbitrary non-zero constants, making these cases more general than those in (1.2) and (1.3). In the dual group reduction, one group reduction imposes a constraint involving the matrix potentials p and q in the AKNS matrix spectral problem, while the other engenders a compatible symmetry constraint on each matrix potential, p and q . The type of dual reductions analyzed in this work extends those previously considered in the literature (e.g., [41, 42]). Notably, the approach here does

not extend directly to the NLS integrable hierarchy, which requires different group reduction schemes. Nevertheless, one may impose three or more group reductions on the AKNS matrix spectral problem simultaneously; however, such reductions require more careful constraints on the two matrix potentials p and q , and they involve considerably more extensive computations.

The methodology and results presented here not only enrich the classification of integrable models, applicable as well to other matrix spectral problems in the literature (see, e.g., [43, 44]), but also offer a foundation for further analytical and numerical investigation. Soliton theory techniques such as the Darboux transformation, Hirota bilinear method, and Wronskian representations may be applied to construct explicit solutions of these reduced models. The resulting integrable mKdV systems are expected to support a range of interesting solution types, including rational solutions (see, e.g., [45]), lump waves (see, e.g., [46–48]), breathers and rogue waves (see, e.g., [49–52]), and multi-wave and interaction solutions (see, e.g., [53, 54]). In addition to the Darboux transformation [55, 56], the Riemann–Hilbert approach offers an alternative method for constructing soliton-type solutions, particularly for models involving multiple poles in the scattering data [57]. Future work may also explore physical interpretations, connections to nonlocal systems, and higher-order generalizations.

Acknowledgements The work was supported in part by the Ministry of Science and Technology of China (G2021016032L and G2023016011L) and the National Natural Science Foundation of China (12271488 and 11975145).

Author Contributions WX wrote the main manuscript text, made the computations, and reviewed the manuscript.

Funding Open access funding provided by North-West University.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflicts of Interest The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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