Qualitative Theory of Dynamical Systems (2024) 23:177
https://doi.org/10.1007/s12346-024-01036-6

General Solution to a Nonlocal Linear Differential Equation of First-Order

Wen-Xiu Ma[1,2,3,4]

Received: 25 January 2024 / Accepted: 4 April 2024

Abstract
The aim of this paper is to construct the general solution to a nonlocal linear differential equation of first-order, either homogeneous or inhomogeneous, together with its stability analysis. The success lies in decomposing functions into their even and odd parts, which presents an innovative approach to nonlocal equations. Our analysis also exhibits an unusual solution phenomenon occurring in nonlocal models.

Keywords Nonlocal differential equation · Inhomogeneous equation · General solution

Mathematics Subject Classification 34A05 · 34A30 · 34A99

1 Introduction

Nonlocal differential equations have many applications in the physical sciences and engineering [1, 2]. One typical example of nonlocal dynamics is pantograph modeling, which has a long history in pantograph mechanics and pantograph transport [3]. In 1821, Professor William Wallace invented the eidograph to improve upon the practical utility of the pantograph [4]. Unsupervised machine learning in artificial intelligence deals with a nonlocal superposition [5]. As a property of the universe that is independent of our description of nature, the notion of non-locality in quantum mechanics was introduced in the context of the EPR controversy on the phenomenon of entanglement between quantum systems [6].

✉ Wen-Xiu Ma
mawx@cas.usf.edu

1 Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China

2 Department of Mathematics, King Abdulaziz University, 21589 Jeddah, Saudi Arabia

3 Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA

4 Material Science Innovation and Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, 2735 Mmabatho, South Africa

Published online: 28 April 2024

One class of nonlocal equations consists of delay differential equations (DDEs) [7, 8]. DDE models have been introduced to analyze ultradian oscillations of insulin and glucose [9, 10], and to describe physiological control systems regarding dynamical respiratory and hematopoietic diseases [11]. DDEs contain discrete delay equations, for example,

$$x'(t) = f(t, x(t), x(t-a)), \tag{1.1}$$

where $a > 0$ (see, e.g., [12]), and pantograph equations, for example,

$$x'(t) = f(t, x(t), x(\lambda t)), \tag{1.2}$$

where $0 < \lambda < 1$ (see, e.g., [13]). Throughout our discussion, t stands for the independent variable, x denotes the dependent variable, and f is a given function.

Motivated by recent studies on nonlocal integrable partial differential equations (see, e.g., [14, 15]), we would like to consider another class of nonlocal differential equations, which involve the value of the dependent variable at the inverse of the independent variable with respect to a group operation. Two examples of such equations of first-order are

$$x'(t) = f(t, x(t), x(-t)), \tag{1.3}$$

and

$$x'(t) = f(t, x(t), x(t^{-1})). \tag{1.4}$$

In this article, we would like to solve the nonlocal linear differential equation of first-order:

$$x'(t) = \lambda x(t) + \mu x(-t) + f(t),\ t \in \mathbb{R}, \tag{1.5}$$

where λ and μ are arbitrary real constants and f is assumed to be continuous. Obviously, this equation possesses the standard superposition principle. Therefore, we will solve the homogeneous equation first and then construct a particular solution to the inhomogeneous counterpart. The novelty is to decompose a function into its even and odd parts so that the nonlocal equation is transformed into a local one to solve. Our general solution will also show that an unusual phenomenon in existence and the uniqueness of solutions occurs in the nonlocal case. The conclusion is given in the last section.

2 General Solution to the Homogeneous Equation

We first solve the homogeneous nonlocal differential equation

$$x'(t) = \lambda x(t) + \mu x(-t),\ t \in \mathbb{R}, \tag{2.1}$$

where λ and μ are arbitrary real constants. Let us take the decomposition

$$x(t) = y(t) + z(t), \tag{2.2}$$

where y is even and z is odd. Then, by combining even and odd functions in the resulting equation, the homogeneous nonlocal Eq. (2.1) becomes

$$\begin{cases} y'(t) = (\lambda - \mu)z(t), \\ z'(t) = (\lambda + \mu)y(t). \end{cases} \tag{2.3}$$

This is local. The advantage of decomposing a function into its even and odd parts is that a nonlocal equation is transformed into a local system. Obviously, y and z solve the same second-order differential equation:

$$y''(t) = (\lambda^2 - \mu^2)y(t), \tag{2.4}$$

and

$$z''(t) = (\lambda^2 - \mu^2)z(t). \tag{2.5}$$

Below, we present the general solution to the homogeneous nonlocal Eq. (2.1) in each of the following four cases.

Case 1: $\lambda + \mu = 0$: In this case, by (2.3), we have the odd part $z = 0$, and thus $y' = 0$, which yields the even part $y = c_1$, where c_1 is an arbitrary constant. This gives the general solution

$$x(t) = c_1. \tag{2.6}$$

Case 2: $\lambda - \mu = 0$: In this case, again by (2.3), we have the even part $y = c_1$, and thus $z' = 2\lambda c_1$, which leads to the odd part $z = 2\lambda c_1 t$, where c_1 is an arbitrary constant. This implies the general solution

$$x(t) = c_1(2\lambda t + 1). \tag{2.7}$$

Case 3: $\lambda^2 - \mu^2 > 0$: Let us introduce

$$\nu = \sqrt{\lambda^2 - \mu^2}. \tag{2.8}$$

Then by (2.4),

$$y(t) = c_1 \mathrm{e}^{\nu t} + c_2 \mathrm{e}^{-\nu t},$$

where c_1 and c_2 are arbitrary constants. Since y is even, we obtain

$$y(t) = c_1 \mathrm{e}^{\nu t} + c_1 \mathrm{e}^{-\nu t} = \frac{1}{2} c_1 \cosh(\nu t),$$

and further by (2.3), we have

$$z(t) = \frac{\nu}{2(\lambda - \mu)} c_1 \sinh(\nu t).$$

Accordingly, we obtain the general solution

$$x(t) = \frac{1}{2}c_1 \left[\cosh(\nu t) + \frac{\nu}{\lambda - \mu} \sinh(\nu t) \right], \tag{2.9}$$

where ν is defined by (2.8).

Case 4: $\lambda^2 - \mu^2 < 0$: Similarly, we introduce

$$\nu = \sqrt{\mu^2 - \lambda^2}. \tag{2.10}$$

Again by (2.4),

$$y(t) = c_1 \cos(\nu t) + c_2 \sin(\nu t),$$

where c_1 and c_2 are arbitrary constants. Since y is even, we get

$$y(t) = c_1 \cos(\nu t),$$

and further by (2.3), we have

$$z(t) = -\frac{\nu}{\lambda - \mu} c_1 \sin(\nu t).$$

Finally, we obtain the general solution

$$x(t) = c_1 \left[\cos(\nu t) - \frac{\nu}{\lambda - \mu} \sin(\nu t) \right], \tag{2.11}$$

where ν is defined by (2.10).

To summarize, the general solution to the homogeneous nonlocal differential Eq. (2.1) is given by (2.6), (2.7), (2.9) or (2.11), depending on the four cases of the two coefficients. It contains one arbitrary constant, and thus, the dimension of the solution space of the nonlocal differential Eq. (2.1) is one.

3 Particular Solution to the Inhomogeneous Counterpart

Clearly, the inhomogeneous nonlocal Eq. (1.5) possesses the superposition principle:

$$x = x_g + x_p, \tag{3.1}$$

where x_g is the general solution to its homogeneous counterpart and x_p is a particular solution. Thus, we only need to find a particular solution to the inhomogeneous Eq. (1.5).

Similarly, we express

$$f(t) = g(t) + h(t), \tag{3.2}$$

where g is even and h is odd. Then, the inhomogeneous Eq. (1.5) becomes

$$\begin{cases} y'(t) = (\lambda - \mu)z(t) + h(t), \\ z'(t) = (\lambda + \mu)y(t) + g(t), \end{cases} \tag{3.3}$$

which is local. It further follows that y and z solve the second-order inhomogeneous differential equations:

$$y''(t) = (\lambda^2 - \mu^2)y(t) + \xi(t), \;\; \xi(t) = (\lambda - \mu)g(t) + h'(t), \tag{3.4}$$

and

$$z''(t) = (\lambda^2 - \mu^2)z(t) + \eta(t), \;\; \eta(t) = (\lambda + \mu)h(t) + g'(t), \tag{3.5}$$

where f is assumed to be differentiable.

In what follows, we will construct a particular solution by solving the above local inhomogeneous linear system (3.3) in each of the four cases.

Case 1 - $\lambda + \mu = 0$: Since $z' = g$, we have

$$z_p = \int_0^t g(s)ds,$$

and then by (3.3), we can get

$$y_p = 2\lambda \int_0^t \int_0^s g(r)drds + \int_0^t h(s)ds.$$

Therefore, a particular solution to (1.5) can be taken as

$$x_p = y_p + z_p = 2\lambda \int_0^t \int_0^s g(r)drds + \int_0^t f(s)ds, \tag{3.6}$$

where g is the even part of f.

Case 2 - $\lambda - \mu = 0$: Since $y' = h$, we can have

$$y_p = \int_0^t h(s)ds,$$

and again by (3.3), we obtain

$$z_p = 2\lambda \int_0^t \int_0^s h(r)drds + \int_0^t g(s)ds.$$

Thus, a particular solution to (1.5) can be given by

$$x_p = y_p + z_p = 2\lambda \int_0^t \int_0^s h(r)drds + \int_0^t f(s)ds, \tag{3.7}$$

where h is the odd part of f.

Case 3 - $\lambda^2 - \mu^2 > 0$: By solving the local linear system (2.3), we find its fundamental matrix solution:

$$U_1(t, t_0) = \begin{bmatrix} \cosh(\nu(t-t_0)) & \frac{\lambda-\mu}{\nu}\sinh(\nu(t-t_0)) \\ \dfrac{\nu}{\lambda-\mu}\sinh(\nu(t-t_0)) & \cosh(\nu(t-t_0)) \end{bmatrix}, \tag{3.8}$$

where ν is defined by (2.8). By the variation of parameters, we obtain a particular solution to the inhomogeneous system (3.3):

$$(y_p, z_p)^T = \int_0^t U_1(t, s)(h(s), g(s))^T ds. \tag{3.9}$$

This generates a particular solution to the inhomogeneous Eq. (1.5):

$$\begin{aligned} x_p = y_p + z_p = \frac{\lambda-\mu}{\nu}\int_0^t \sinh(\nu(t-s))g(s)ds \\ + \frac{\nu}{\lambda-\mu}\int_0^t \sinh(\nu(t-s))h(s)ds + \int_0^t \cosh(\nu(t-s))f(s)ds. \end{aligned} \tag{3.10}$$

Further, we can express this particular solution in terms of f:

$$x_p = \frac{\lambda}{\nu}\int_0^t \sinh(\nu(t-s))f(s)ds - \frac{\mu}{\nu}\int_0^t \sinh(\nu(t-s))f(-s)ds + \int_0^t cosh(\nu(t-s))f(s)ds, \tag{3.11}$$

where ν is defined by (2.8).

Case 4 - $\lambda^2 - \mu^2 < 0$: Similarly solving the local linear system (2.3) yields its fundamental matrix solution:

$$U_2(t, t_0) = \begin{bmatrix} \cos(\nu(t-t_0)) & \dfrac{\lambda-\mu}{\nu}\sin(\nu(t-t_0)) \\ -\dfrac{\nu}{\lambda-\mu}\sin(\nu(t-t_0)) & \cos(\nu(t-t_0)) \end{bmatrix}, \tag{3.12}$$

where ν is defined by (2.10). Again by the variation of parameters, we get a particular solution to the inhomogeneous system (3.3):

$$(y_p, z_p)^T = \int_0^t U_2(t, s)(h(s), g(s))^T ds. \tag{3.13}$$

This yields a particular solution to the inhomogeneous Eq. (1.5):

$$\begin{aligned} x_p = y_p + z_p = \frac{\lambda-\mu}{\nu}\int_0^t \sin(\nu(t-s))g(s)ds \\ - \frac{\nu}{\lambda-\mu}\int_0^t \sin(\nu(t-s))h(s)ds + \int_0^t \cos(\nu(t-s))f(s)ds. \end{aligned} \tag{3.14}$$

Finally, we can formulate this particular solution in terms of f:

$$x_p = \frac{\lambda}{\nu}\int_0^t \sin(\nu(t-s))f(s)ds - \frac{\mu}{\nu}\int_0^t \sin(\nu(t-s))f(-s)ds + \int_0^t \cos(\nu(t-s))f(s)ds, \tag{3.15}$$

where ν is defined by (2.10).

To conclude, a particular solution to the inhomogeneous nonlocal differential Eq. (1.5) can be determined by (3.6), (3.7), (3.11) or (3.15), depending on the four particular situations of the two coefficients.

4 Particular Solution by Solving the Second-Order Equation

Let us now construct a particular solution to the inhomogeneous nonlocal Eq. (1.5), where $\lambda^2 \neq \mu^2$, by solving the second-order inhomogeneous Eq. (3.4).

Let us first consider the case of $\lambda^2 - \mu^2 > 0$. In this case, note that the homegeneous counterpart (2.4) has two linearly independent solutions $y_1 = \mathrm{e}^{\nu t}$ and $y_2 = \mathrm{e}^{-\nu t}$, and their Wronskian is

$$W(y_1(t), y_2(t)) = -2\nu.$$

Therefore, by the variation of parameters, we can work out a particular solution to the inhomogeneous Eq. (3.4):

$$\begin{aligned} y_p &= -y_1(t)\int_0^t \frac{y_2(s)\xi(s)}{W(y_1(s), y_2(s))}ds + y_2(t)\int_0^t \frac{y_1(s)\xi(s)}{W(y_1(s), y_2(s))}ds \\ &= \frac{1}{\nu}\int_0^t \sinh(\nu(t-s))\xi(s)ds. \end{aligned} \tag{4.1}$$

Then by (3.3), we get

$$z_p = \frac{1}{\lambda-\mu}(y_p' - h) = \frac{1}{\lambda-\mu}\Big[\int_0^t \cosh(\nu(t-s))\xi(s)ds - h\Big]. \tag{4.2}$$

Consequently, we obtain a particular solution

$$x_p = y_p + z_p = \frac{1}{\nu}\int_0^t \sinh(\nu(t-s))\xi(s)ds + \frac{1}{\lambda-\mu}\Big[\int_0^t \cosh(\nu(t-s))\xi(s)ds - h\Big]. \tag{4.3}$$

By virtue of the expression of ξ in (3.4), a direct computation can show that this engenders the particular solution in (3.10), and further, the solution is formulated as (3.11), in terms of f.

Let us second consider the case of $\lambda^2 - \mu^2 < 0$. In this case, similarly note that the homegeneous counterpart (2.4) has two linearly independent solutions $y_1 = \sin(\nu t)$

and $y_2 = \cos(\nu t)$, and their Wronskian is

$$W(y_1(t), y_2(t)) = -\nu.$$

Thus, by the variation of parameters, we can work out a particular solution to the inhomogeneous Eq. (3.4):

$$y_p = \frac{1}{\nu}\int_0^t \sin(\nu(t-s))\xi(s)ds, \tag{4.4}$$

and again by (3.3), we have

$$z_p = \frac{1}{\lambda-\mu}(y_p' - h) = \frac{1}{\lambda-\mu}\Big[\int_0^t \cos(\nu(t-s))\xi(s)ds - h\Big]. \tag{4.5}$$

Consequently, we arrive at a particular solution,

$$x_p = y_p + z_p = \frac{1}{\nu}\int_0^t \sin(\nu(t-s))\xi(s)ds + \frac{1}{\lambda-\mu}\Big[\int_0^t \cos(\nu(t-s))\xi(s)ds - h\Big]. \tag{4.6}$$

By using the expression of ξ in (3.4) now, this is simplified into the particular solution in (3.14), and further, the one in (3.15), expressed in terms of f.

Note that the second-order inhomogeneous equation needs the differentiability of f, but the final particular solution only needs the continuity of f.

5 Stability of Solutions

Obviously, the Lyapunov stability properties of a solution to the inhomogeneous nonlocal Eq. (1.5) are the same as the ones of the zero equilibrium of the corresponding homogeneous counterpart (2.1). Therefore, we only analyze the Lyapunov stability properties of the zero equilibrium below.

Case 1: $\lambda + \mu = 0$. In this case, the general solution is given by (2.6). Clearly, the solution $x(t, t_0, x_0)$ with $x(t_0) = x_0 \in \mathbb{R}$ is defined by $x(t, t_0, x_0) = x_0$, and thus, the zero equilibrium is uniformly stable for $t_0 \geq 0$ but not asymptotically stable.

Case 2: $\lambda - \mu = 0$ but $\lambda \neq 0$. The sub-case of $\lambda = 0$ is already discussed in Case 1. Under this condition, the general solution is given by (2.7). The solution is not bounded when $c_1 \neq 0$, and thus, the zero equilibrium is unstable.

Case 3: $\lambda^2 - \mu^2 > 0$. In this case, the general solution is given by (2.9). The solution is not bounded when $c_1 \neq 0$, and thus, the zero equilibrium is unstable.

Case 4: $\lambda^2 - \mu^2 < 0$. In this case, the general solution is given by (2.11). On one hand, the solution is bounded, but if $c_1 \neq 0$, it never goes to zero when $t \to \infty$. On the other hand, note that the solution $x(t, t_0, x_0)$ with $x(t_0) = x_0 \in \mathbb{R}$ is determined by

$$x(t, t_0, x_0) = x_0 \frac{\cos(\nu t) - \frac{\nu}{\lambda-\mu}\sin(\nu t)}{\cos(\nu t_0) - \frac{\nu}{\lambda-\mu}\sin(\nu t_0)}, \tag{5.1}$$

for $t_0 \in [0, t_1]$, on which

$$\cos(\nu t) - \frac{\nu}{\lambda - \mu} \sin(\nu t) > 0. \tag{5.2}$$

Then, we can easily see that the zero equilibrium is uniformly stable for $t_0 \in [0, t_1]$ but not asymptotically stable. This is different from the uniform stability in the local case, for which there is no restriction on t_0.

6 An Unusual Solution Phenomenon

Let us illustrate the existence and uniqueness of solutions to Cauchy problems for the nonlocal Eq. (2.1).

In view of the general solution in (2.7), we can easily see that the Cauchy problem on $\mathbb{R}$:

$$\begin{cases} x'(t) = -x(t) - x(-t), \\ x(\frac{1}{2}) = x_0 \neq 0, \end{cases} \tag{6.1}$$

has no solution, but the Cauchy problem on $\mathbb{R}$:

$$\begin{cases} x'(t) = -x(t) - x(-t), \\ x(\frac{1}{2}) = 0, \end{cases} \tag{6.2}$$

has infinitely many solutions

$$x(t) = c_1(-2t + 1), \tag{6.3}$$

where c_1 is an arbitrary constant.

Noting that the nonlocal Eq. (2.1) has constant coefficients, the corresponding vector field function, $f(x_1, x_2) = \lambda x_1 + \mu x_2$, is smooth in $\mathbb{R}^2$. But its Cauchy problems have some specific solution situations pointed out above, and thus, the existence and uniqueness theorem does not hold in all cases of its coefficients. This is very different from local differential equations, and many solution techniques in the local case (see, e.g., [16–24]) are useful but cannot be directly applied to nonlocal differential equations. The nonlocality does bring difficulties in determining the existence and uniqueness of solutions, and we need to apply innovative thinking to gain insight into the difficulties and to find innovative approaches to establishing local and global well-posedness results for nonlocal differential equations.

7 Concluding Remarks

We have presented the general solution to a nonlocal linear differential equation of first-order in (1.5), together with stability analysis and unusual existence and uniqueness results. The general solution involves an arbitrary constant, and thus, the dimension of the solution space of the corresponding homogeneous equation is one. The key to success is to use the decomposition of functions into their even and odd parts to remove the nonlocality. Such an idea could be applied to other similar nonlocal differential equations.

There is another similar type of nonlocal linear differential equation of first-order:

$$x'(t) = \lambda x(t) + \mu x(t^{-1}) + f(t),\ t > 0,$$

where λ and μ are arbitrary real constants, and f is continuous. The coordinate t^{-1} is the inverse of t with respect to the multiplication, while the coordinate $-t$ in (1.5) is the inverse of t with respect to the addition. The above nonlocal equation seems much harder to solve. We expect that there will be some effective way to present its general solution.

Recently, various nonlocal integrable partial differential equations have been formulated, through conducting one group reduction (see, e.g., [15]) and two group reductions (see, e.g., [25, 26]) of matrix spectral problems. Soliton solutions have been generated for nonlocal nonlinear Schrödinger equations (see, e.g., [26–28]) and nonlocal modified Korteweg-de Vries equations (see, e.g., [25, 29]) by the Riemann-Hilbert technique. We point out that the same idea of decomposing functions into their even and odd parts can be used to transform those nonlocal integrable equations into local equations to study.

Acknowledgements The work was supported in part by National Natural Science Foundation of China (12271488, 11975145 and 11972291), and the Ministry of Science and Technology of China (G2021016032L and G2023016011L). The author would also like to thank Arthur Danielyan and Sherwin Kouchekian for their valuable discussions.

Author Contributions WX wrote the main manuscript text, made the computations, and reviewed the manuscript.

Data Availibility All data generated or analyzed during this study are included in this published article.

Declarations

Conflict of interest The author declares that there is no known Conflict of interest that could have appeared to influence this work.

References

1. Kuang, Y.: Delay Differential Equations with Applications in Population Dynamics. Mathematics in Science and Engineering, vol. 191. Academic Press, Boston (1993)
2. Jean-Pierre, R.: Time delay systems: an overview of some recent advances and open problems. Automatica **39**, 1667–1694 (2003)
3. Shapira, A., Tyomkyn, M.: Quasirandom graphs and the pantograph equation. Am. Math. Mon. **128**, 630–639 (2021)
4. Wallace, W.: Account of the invention of the pantograph, and a description of the eidograph. Trans. R. Soc. Edinb. **13**, 418–439 (1836)
5. LeCun, Y., Bengio, Y., Hinton, G.: Deep learning. Nature **521**, 436–444 (2015)
6. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777–780 (1935)

7. Driver, R.D.: Ordinary and Delay Differential Equations. Applied Mathematical Sciences, vol. 20. Springer, New York (1977)
8. Erneux, T.: Applied Delay Differential Equations. Surveys and Tutorials in the Applied Mathematical Sciences, vol. 3. Springer, New York (2009)
9. Sturis, J., Polonsky, K.S., Mosekilde, E., Van Cauter, E.: Computer-model for mechanisms underlying ultradian oscillations of insulin and glucose. Am. J. Physiol. Endocrinol. Metab. **260**, E801–E809 (1991)
10. Makroglou, A., Li, J.X., Kuang, Y.: Mathematical models and software tools for the glucose-insulin regulatory system and diabetes: an overview. Appl. Numer. Math. **56**, 559–573 (2006)
11. Mackey, M.C., Glass, L.: Oscillation and chaos in physiological control systems. Science **197**, 287–289 (1977)
12. Engelborghs, K., Lemaire, V., Bélair, J., Roose, D.: Numerical bifurcation analysis of delay differential equations arising from physiological modeling. J. Math. Biol. **42**, 361–385 (2001)
13. Kato, T., McLeod, J.B.: The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$. Bull. Am. Math. Soc. **77**, 891–937 (1971)
14. Ablowitz, M.J., Musslimani, Z.H.: Integrable nonlocal nonlinear Schrödinger equation. Phys. Rev. Lett. **110**, 064105 (2013)
15. Ma, W.X.: Nonlocal PT-symmetric integrable equations and related Riemann-Hilbert problems. Partial Differ. Equ. Appl. Math. **4**, 100190 (2021)
16. Murphy, G.M.: Ordinary Differential Equations and Their Solutions. Dover Publications, Mineola (2011)
17. Ma, W.X.: N-soliton solutions and the Hirota conditions in (1+1)-dimensions. Int. J. Nonlinear Sci. Numer. Simul. **23**, 123–133 (2022)
18. Ma, W.X.: A novel kind of reduced integrable matrix mKdV equations and their binary Darboux transformations. Mod. Phys. Lett. B **36**, 2250094 (2022)
19. Yao, S.W., Hashemi, M.S., Inc, M.: A Lie group treatment on a generalized evolution Fisher type equation with variable coefficients. Results Phys. **46**, 106307 (2023)
20. Ma, W.X.: Lump waves in a spatial symmetric nonlinear dispersive wave model in (2 + 1)-dimensions. Mathematics **11**, 4664 (2023)
21. Xu, Y.J., Hashemi, M.S.: Exact solutions for porous fins under a uniform magnetic field: a novel reduction method. Case Stud. Therm. Eng. **45**, 103013 (2023)
22. Ma, W.X., Batwa, S., Manukure, S.: Dispersion-managed lump waves in a spatial symmetric KP model. East Asian J. Appl. Math. **13**, 246–256 (2023)
23. Akbulut, A., Mirzazadeh, M., Hashemi, M.S., Hosseini, K., Salahshour, S., Park, C.: Triki–Biswas model: its symmetry reduction, Nucci's reduction and conservation laws. Int. J. Mod. Phys. B **37**, 2350063 (2023)
24. Yao, S.W., Gulsen, S., Hashemi, M.S., Inc, M., Bicer, H.: Periodic Hunter–Saxton equation parametrized by the speed of the Galilean frame: its new solutions, Nucci's reduction, first integrals and Lie symmetry reduction. Results Phys. **47**, 106370 (2023)
25. Ma, W.X.: Type ($-\lambda$, $-\lambda^*$) reduced nonlocal integrable mKdV equations and their soliton solutions. Appl. Math. Lett. **131**, 108074 (2022)
26. Ma, W.X.: Soliton solutions to constrained nonlocal integrable nonlinear Schrödinger hierarchies of type ($-\lambda$, λ). Int. J. Geom. Methods Mod. Phys. **20**, 2350098 (2023)
27. Ma, W.X.: Integrable nonlocal nonlinear Schrödinger hierarchies of type $(-\lambda^*, \lambda)$ and soliton solutions. Rep. Math. Phys. **92**, 19–36 (2023)
28. Ma, W.X.: Binary Darboux transformation of vector nonlocal reverse-time integrable NLS equations. Chaos Solitons Fractals **180**, 114539 (2024)
29. Ma, W.X.: Nonlocal integrable mKdV equations by two nonlocal reductions and their soliton solutions. J. Geom. Phys. **177**, 104522 (2022)