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The Commutative Property of Reciprocal Transformations and Dimensional Deformations

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Abstract
The paper aims to analyze the commutative property of reciprocal transformations and dimensional deformations using conservation laws. First, a geometric proof of the commutative property of reciprocal transformations is presented, based on the coordinate-free property of the exponential map. Second, it is shown that the deformation algorithm does not always keep the commutative property. Illuminating examples are provided.

Keywords Generalized vector field · Symmetry · The exponential map

Mathematics Subject Classification 35Q51 · 37KQ05 · 37K10

1 Introduction

Integrable equations possess infinitely many symmetries [1], and the corresponding Lax pairs of matrix spectral problems guarantee the commutativity of those symmetries under the Lie bracket of evolutionary vector fields [2]. Symmetries generate diverse solution manifolds, and can be used to show a kind of integrable characteristics, based on the Liouville–Arnold theorem [3, 4]. The complete integrability can be explored through verifying if squared eigenfunctions of matrix spectral problems form a complete set of basis vectors in a normed space [5].

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It is known that reciprocal transformations [6]—an interesting topic in soliton theory—transform integrable equations by changes of coordinates [7], which yield auto-Bäcklund transformations [8]. Recently, similar to reciprocal transformations, Lou et al. [9] proposed a deformation algorithm to deform given integrable equations from lower dimensions to higher dimensions, and they observed a phenomenon that the deformed equations could commute, while the original equations do. Such an interesting phenomenon occurs in all (1+1)-dimensional integrable equations in the literature [9].

This paper aims to present a proof of the commutative property of reciprocal transformations by a geometric means. The key idea is to transform the property into the coordinate-free property of the exponential map. Analysis is also given for the deformation algorithm, which shows that the commutative property might not be kept after making dimensional deformations. Illuminating examples are provided.

2 The Commutative Property of Reciprocal Transformations

2.1 Reciprocal Transformations

Let x be a vector of space variables, t, the time variable, and u, a vector of dependent variables. We consider a partial differential equation

$$E(x, t, u^{(m)}) = 0, \tag{2.1}$$

where E is assumed to be sufficiently differentiable with respect to the indicated variables, $m \in \mathbb{N}$, and $u^{(m)}$ denotes the set of partial derivatives of u up to order m with respect to x and t, as in the study of symmetries of evolution equations [10].

In what follows, we consider the (1+1)-dimensional case, and assume that the vector E has the same dimension as the vector u so that E can define a characteristic of an evolutionary vector field.

A reciprocal transformation is defined through using a conservation law. So, let us have a conservation law of (2.1):

$$(\rho_E)_t = (J_E)_x,\ \ \rho_E = \rho_E(x, t, u^{(m)}),\ \ J_E = J_E(x, t, u^{(m)}), \tag{2.2}$$

where ρ_E and J_E are assumed to be sufficiently differentiable with respect to the indicated variables, too. A reciprocal transformation from (x, t) to (x', t') is defined by

$$\frac{\partial}{\partial x} = \frac{1}{\rho'_E}\frac{\partial}{\partial x'},\ \ \frac{\partial}{\partial t} = \frac{J'_E}{\rho'_E}\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}, \tag{2.3}$$

where

$$\rho'_E = \rho'_E\left(x', t', u'^{(m)}\right) = \frac{1}{\rho_E\left(x, t, u^{(m)}\right)},$$

$$J'_E = J'_E\left(x', t', u'^{(m)}\right) = \frac{J_E\left(x, t, u^{(m)}\right)}{\rho_E\left(x, t, u^{(m)}\right)}, \tag{2.4}$$

with $u'^{(m)}$ denoting the set of partial derivatives of u up to order m with respect to x' and t'. The original conservation law becomes a new conservation law:

$$(\rho'_E)_{t'} = (J'_E)_{x'}, \tag{2.5}$$

in the new coordinates, and obviously, the new reciprocal transformation using this new conservation law is exactly the inverse transformation of the above reciprocal transformation, since we have

$$(\rho'_E)'_E = \frac{1}{\rho'_E} = \rho_E,\ (J'_E)'_E = \frac{J'_E}{\rho'_E} = \frac{J_E}{\rho_E}\rho_E = J_E. \tag{2.6}$$

The above reciprocal transformation (2.3) is equivalent to

$$dx' = \rho_E dx + J_E dt,\ dt' = dt, \tag{2.7}$$

whose compatibility condition is exactly the conservation law (2.2). The transformed equation of (2.1) is defined by

$$E'\left(x, t, u^{(m)}\right) = E\left(x', t', u'^{(m)}\right) = 0. \tag{2.8}$$

Note that the original equations do not need to be of evolutionary type.

2.2 The Commutative Property

Let us consider another partial differential equation

$$F\left(x, t, u^{(n)}\right) = 0, \tag{2.9}$$

where F is assumed to be sufficiently differentiable with respect to the indicated variables and has the same dimension as u as well, $n \in \mathbb{N}$, and $u^{(n)}$ is defined as before. Associated with a conservation law of it:

$$(\rho_F)_t = (J_F)_x,\ \rho_F = \rho_F\left(x, t, u^{(n)}\right),\ J_F = J_F\left(x, t, u^{(n)}\right), \tag{2.10}$$

where ρ_F and J_F are also assumed to be sufficiently differentiable with respect to the indicated variables, a reciprocal transformation from (x, t) to (x'', t'') is similarly given by

$$\frac{\partial}{\partial x} = \frac{1}{\rho'_F}\frac{\partial}{\partial x''},\ \frac{\partial}{\partial t} = \frac{J'_F}{\rho'_F}\frac{\partial}{\partial x''} + \frac{\partial}{\partial t''}, \tag{2.11}$$

where

$$\rho'_F = \rho_F\left(x'', t'', u''^{(n)}\right) = \frac{1}{\rho_F\left(x, t, u^{(n)}\right)},$$
$$J'_F = J_F\left(x'', t'', u''^{(n)}\right) = \frac{J_F\left(x, t, u^{(n)}\right)}{\rho_F\left(x, t, u^{(n)}\right)}, \tag{2.12}$$

where $u''^{(n)}$ denotes the set of partial derivatives of u up to order n with respect to x'' and t''. The transformed equation of (2.9) reads

$$F'\left(x, t, u^{(n)}\right) = F\left(x'', t'', u''^{(n)}\right) = 0. \tag{2.13}$$

Let v be a generalized vector field on a jet space, and exp be the exponential map, i.e.,

$$\frac{d}{d\varepsilon}\exp(\varepsilon v)w = v\big|_{u=\exp(\varepsilon v)w}, \tag{2.14}$$

for all w and small ε (see, e.g., [10] for details). For example, if $v = u_x$, then $(\exp(\varepsilon v)w)(x, t) = w(x + \varepsilon, t)$.

Noting the above definition of the transformed equations and the coordinate-free property of the exponential map on a jet space, we have

$$\exp\left(\varepsilon E'(x, t, u^{(m)})\right) = \exp\left(\varepsilon E(x', t', u'^{(m)})\right) = \exp\left(\varepsilon E(x, t, u^{(m)})\right),$$
$$\exp\left(\varepsilon F'(x, t, u^{(n)})\right) = \exp\left(\varepsilon F(x'', t'', u''^{(n)})\right) = \exp\left(\varepsilon F(x, t, u^{(n)})\right),$$

where ε is a small parameter. Therefore, if the original Eqs. (2.1) and (2.9) commute under the Lie bracket of evolutionary vector fields:

$$[E, F] := \frac{d}{d\varepsilon}\big[E(u + \varepsilon F) - F(u + \varepsilon E)\big]\big|_{\varepsilon=0} = 0, \tag{2.15}$$

i.e., we have

$$\exp\left(\varepsilon E(x, t, u^{(m)})\right)\exp\left(\delta F(x, t, u^{(n)})\right)$$
$$= \exp\left(\delta F(x, t, u^{(n)})\right)\exp\left(\varepsilon E(x, t, u^{(m)})\right),$$

then we see that

$$\exp\left(\varepsilon E'(x, t, u^{(m)})\right)\exp\left(\delta F'(x, t, u^{(n)})\right)$$
$$= \exp\left(\delta F'(x, t, u^{(n)})\right)\exp\left(\varepsilon E'(x, t, u^{(m)})\right), \tag{2.16}$$

where ε and δ are small parameters.

This exactly means that the two evolutionary vector fields with the transformed characteristics E' and F' commute in the local coordinates $(x, t, u^{(\max\{m,n\})})$, namely, we have

$$[E', F'] = \frac{d}{d\varepsilon}\big[E'(u + \varepsilon F') - F'(u + \varepsilon E')\big]\big|_{\varepsilon=0} = 0, \tag{2.17}$$

and thus, the evolutionary vector field with the transformed characteristic F' in (2.13) (or E' in (2.8)) is a symmetry of the transformed Eq. (2.8) (or (2.13)).

We summarize the above result as the following theorem.

Theorem 2.1 *If two partial differential Eqs.* (2.1) *and* (2.9) *commute, i.e., the commutative property* (2.15) *holds, then two transformed Eqs.* (2.8) *and* (2.13) *defined through the reciprocal transformation commute, too, i.e., the commutative property* (2.17) *holds.*

Example 2.1 Consider a pair of equations, each of which is the nonlinear transport equation

$$E = F = u_t - uu_x = 0, \tag{2.18}$$

as an example. Associated with two conservation laws of it:

$$\rho_E = u,\ J_E = \frac{1}{2}u^2;\ \rho_F = \frac{1}{2}u^2,\ J_F = \frac{1}{3}u^3, \tag{2.19}$$

the corresponding two commuting transformed equations are

$$E' = u_t - K = 0,\ K = u_x - \tfrac{1}{2}uu_x, \tag{2.20}$$

and

$$F' = u_t - S = 0,\ S = \frac{2}{u}u_x + \frac{2}{3}uu_x, \tag{2.21}$$

respectively. The commutativity between those two deformed equations can also be proved directly by checking that $[K, S] = 0$.

Additionally, we point out that there are shock wave solutions to the two transformed equations.

3 Dimensional Deformations and Related Commutativity

3.1 A Formulation of the Deformation Algorithm

We would like to formulate a general procedure for deforming partial differential equations, based on conservation laws, from lower dimensions to higher dimensions [9].

Let us consider the partial differential Eq. (2.1). The starting point is to take a set of conservation laws of (2.1):

$$(\rho_{i,E})_t = (J_{i,E})_x,\ \rho_{i,E} = \rho_{i,E}(x,t,u),\ J_{i,E} \\ = J_{i,E}(x,t,u,u_x) = J_{i,E}(x,t,u^{(m)}),\ 1 \le i \le k, \tag{3.1}$$

where $\rho_{i,E}$ and $J_{i,E}$ are assumed to be sufficiently differentiable with respect to the indicated variables, $k \in \mathbb{N}$, and $u^{(m)}$ is defined as before. Let us set

$$y = (y_1, \cdots y_k),\ y' = (y'_1, \cdots, y'_k), \tag{3.2}$$

and introduce a deformation from (x, t, y) to (x', t', y') by

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} + \sum_{i=1}^{k} \rho'_{i,E} \frac{\partial}{\partial y_i},\ \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \sum_{i=1}^{k} J'_{i,E} \frac{\partial}{\partial y_i},\ \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \tag{3.3}$$

where for $1 \le i \le k$, one has

$$\begin{cases} \rho'_{i,E} = \rho'_{i,E}(x,t,y,u) = \rho_{i,E}(x',t',u), \\ J'_{i,E} = J'_{i,E}(x,t,y,u,u_x,u_y) = J'_{i,E}(x,t,y,u^{(m)}) \\ \qquad = J_{i,E}(x',t',u,u_{x'}) = J_{i,E}(x',t',u'^{(m)}), \end{cases} \tag{3.4}$$

with $u^{(m)}$ denoting the set of partial derivatives of u up to order m with respect to x, t and y, and $u'^{(m)}$, the set of partial derivatives of u up to order m with respect to x' and t'. This is equivalent to

$$dy_i = dy'_i + \rho'_{i,E}dx' + J'_{i,E}dt',\ dx = dx',\ dt = dt', \tag{3.5}$$

where $1 \le i \le k$. Obviously, $\left[\frac{\partial}{\partial x'}, \frac{\partial}{\partial t'}\right] = 0$ if and only if

$$\left(\rho'_{i,E}\right)_{t'} = \left(J'_{i,E}\right)_{x'},\ 1 \le i \le k. \tag{3.6}$$

The dimensionally deformed equation of (2.1) is defined by

$$E'\left(x,t,y,u^{(m)}\right) = E\left(x',t',u'^{(m)}\right) = 0. \tag{3.7}$$

This determines a subvariety of the new mth jet space with the variables x, t, y, u.

The above deformation algorithm goes from lower dimensions of (x, t) to higher dimensions of (x, t, y), and the change of the spatial dimension depends on the number of conservation laws. Clearly, the original equation need not be of evolutionary type. The algorithm itself is easy to implement in practical applications, since the adopted conserved densities depend only on u, but there is no guarantee in the algorithm, which generates new coordinates. The existence of new coordinates in (3.5) needs a dimensional reduction, since one generally does not have that $\left[\frac{\partial}{\partial y_i'}, \frac{\partial}{\partial x'}\right] = 0$ and $\left[\frac{\partial}{\partial y_i'}, \frac{\partial}{\partial t'}\right] = 0$, $1 \leq i \leq k$; and such a reduction makes the deformed equation to go back to the original dimension.

3.2 On the Commutativity of the Deformation Algorithm

Let us assume that the second Eq. (2.9) possesses another set of conservation laws:

$$(\rho_{j,F})_t = (J_{j,F})_x,\ \rho_{j,F} = \rho_{j,F}(x,t,u),\ J_{j,F} = J_{j,F}(x,t,u,u_x) = J_{j,F}(x,t,u^{(n)}),\ 1 \leq j \leq l, \tag{3.8}$$

where $\rho_{j,F}$ and $J_{j,F}$ are assumed to be sufficiently differentiable with respect to the indicated variables as well, $l \in \mathbb{N}$, and $u^{(n)}$ is defined as before. Similarly, let us set

$$z = (z_1, \cdots z_l),\ z'' = (z_1'', \cdots, z_l''), \tag{3.9}$$

and make a deformation from (x, t, z) to (x'', t'', z'') by

$$\frac{\partial}{\partial x''} = \frac{\partial}{\partial x} + \sum_{j=1}^{l} \rho_{j,F}'' \frac{\partial}{\partial z_j},\ \frac{\partial}{\partial t''} = \frac{\partial}{\partial t} + \sum_{j=1}^{l} J_{j,F}'' \frac{\partial}{\partial z_j},\ \frac{\partial}{\partial z''} = \frac{\partial}{\partial z}, \tag{3.10}$$

where for $1 \leq j \leq l$, one has

$$\begin{cases} \rho_{j,F}'' = \rho_{j,F}''(x,t,z,u) = \rho_{j,F}(x'',t'',u), \\ J_{j,F}'' = J_{j,F}''(x,t,z,u_x,u_z) = J_{j,F}''(x,t,z,u^{(n)}) \\ \qquad = J_{j,F}(x'',t'',u,u_{x''}) = J_{j,F}(x'',t'',u''^{(n)}), \end{cases} \tag{3.11}$$

with $u^{(n)}$ denoting the set of partial derivatives of u up to order n with respect to x, t and z, and $u''^{(n)}$ denoting the set of partial derivatives of u up to order n with respect to x'' and t''. This deformation is equivalent to

$$dz_j = dz_j'' + \rho_{j,F}'' dx'' + J_{j,F}'' dt'',\ dx = dx'',\ dt = dt'', \tag{3.12}$$

where $1 \leq j \leq l$. The dimensionally deformed equation of (2.9) is determined by

$$F'(x,t,z,u^{(n)}) = F(x'',t'',u''^{(n)}) = 0, \tag{3.13}$$

which determines another subvariety of the new nth jet space with the variables (x, t, z).

Note that in the above two deformation processes, the coordinates of $y_i,\ 1 \le i \le k$, and $z_j,\ 1 \le j \le l$, could have a common subset, i.e., it could happen that $\{y_i | 1 \le i \le k\} \cap \{z_j | 1 \le j \le l\} \ne \emptyset$.

We assume again that the two original Eqs. (2.1) and (2.9) commute, i.e., we have (2.15). Therefore, the exponential map of the evolutionary vector field with the characteristic E (or F) transforms solutions of the Eq. (2.9) (or (2.1)) to other solutions.

We would like to show that the deformation algorithm might not keep the commutative property of the original equations. That is to say, the deformation algorithm might not deform symmetries of the original equation to symmetries of the deformed equation, i.e., we might not have the commutative property (2.17), which can be seen from the following example.

Example 3.1 Let us first consider the pair of equations, both of which are the nonlinear transport Eq. (2.18), again. Based on each of the two conservation laws in (2.19), we have the corresponding two deformed equations

$$E' = u_t - K,\ K = uu_x + \frac{1}{2}u^2 u_y, \tag{3.14}$$

and

$$F' = u_t - S,\ S = uu_x + \frac{1}{6}u^3 u_y. \tag{3.15}$$

Obviously, they are commuting. Equivalently, K and S commute.

Let us second consider a pair of equations, both of which are the KdV equation:

$$E = F = u_t - 6uu_x - u_{xxx} = 0. \tag{3.16}$$

This KdV equation possesses the following two conservation laws:

$$\rho_{E,t} = J_{E,x},\ \rho_E = u,\ J_E = 3u^2 + u_{xx}.$$

and

$$\rho_{F,t} = J_{F,x}, \rho_F = u^2,\ J_F = 4u^3 - u_x^2 + 2uu_{xx}.$$

Based on those two conservation laws, we have that

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} + u\frac{\partial}{\partial y},\ \frac{\partial}{\partial x''} = \frac{\partial}{\partial x} + u^2\frac{\partial}{\partial y},$$

where we take $z = y$. Then by a Maple computation, we can see that the corresponding two deformed equations are

$$\begin{aligned} E' &:= u_t + (3u^2 + u_{x'x'})u_y - 6uu_{x'} - u_{x'x'x'} \\ &= u_t - (u^3 u_{yyy} + 3u^2 u_y u_{yy} + 3uu_y u_{xy} + 3uu_x u_{yy} \\ &\quad + 3u^2 u_{xyy} + 3u^2 u_y + 3u_x u_{xy} + 3uu_{xxy} + 6uu_x + u_{xxx}) = 0, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}F' &:= u_t + (4u^3 - u_{x''}^2 + 2uu_{x''x''})u_y - 6uu_{x''} - u_{x''x''x''} \\ &= u_t - (u^6 u_{yyy} + 6u^5 u_y u_{yy} + 3u^4 u_y^3 + 3u^4 u_{xyy} \\ &\quad + 6u^3 u_x u_{yy} + 6u^2 u_x u_y^2 + 6u^3 u_y u_{xy} + 2u^3 u_y \\ &\quad + 3u^2 u_{xxy} + 3u_x^2 u_y + 6uu_x u_{xy} + 6uu_x + u_{xxx}) = 0, \end{aligned} \tag{3.18}$$

and they do not commute, i.e., we have

$$[E', F'] = \frac{d}{d\varepsilon}\big[E'(u + \varepsilon F') - F'(u + \varepsilon E')\big]\big|_{\varepsilon=0} \neq 0. \tag{3.19}$$

We also summarize the above result as the following theorem.

Theorem 3.1 *The deformation algorithm above does not keep the commutative property, i.e., two transformed Eqs.* (3.7) *and* (3.13) *defined through the deformation algorithm might not commute, even if the commutative property* (2.15) *holds.*

4 Concluding Remarks

The commutative property of reciprocal transformations and dimensional deformations was analyzed. A proof for the commutative property in the case of reciprocal transformations was given by a geometric means, whose key is to use the exponential map to express the commutativity of evolutionary vector fields. Illuminating examples were made for the nonlinear transport equation and the KdV equation.

It would be very interesting to explore similar theories for nonlocal integrable equations (see, e.g., [11–14]). It would also be important to search for exact nonlinear wave solutions, including lump solutions [15, 16], complexitons [17] and rogue waves [18, 19], to new model equations generated by reciprocal transformations and dimensional deformations (see, e.g., [8, 9]). Particularly, are there any shock wave solutions to the (2+1)-dimensional equations (3.14) and (3.15)?

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Declarations

Conflict of interest The author declares that there is no known competing interest that could have appeared to influence this work.

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