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Letter

Dual similarity transformations and integrable reductions of matrix mKdV models

Wen-Xiu Ma [a,b,c,d,*]

[a] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
[b] Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[c] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
[d] Material Science Innovation and Modelling, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa

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ABSTRACT

This work aims to construct dual similarity transformations and explore integrable reductions of matrix modified Korteweg–de Vries (mKdV) models. Starting from the zero-curvature formulation, the study employs similarity transformations as the principal tool. Four representative scenarios of reduced Ablowitz–Kaup–Newell–Segur matrix spectral problems are analyzed, providing concrete examples of reduced matrix mKdV integrable models derived through dual similarity transformations.

1. Introduction

The construction of integrable models often begins with the formulation of Lax pairs [1,2], where the spectral matrices are derived from matrix Lie algebras [3,4]. These Lax pairs give rise to infinitely many commuting symmetries and conservation laws, which are closely linked to underlying bi-Hamiltonian structures [5,6]. Furthermore, the inverse scattering transform provides a powerful method for solving the associated Cauchy problems [7,8].

The matrix Ablowitz–Kaup–Newell–Segur (AKNS) spectral problems offer a universal framework for generating a wide class of integrable models, including the nonlinear Schrödinger (NLS) equation and the modified Korteweg–de Vries (mKdV) equation. Similarity transformations have been extensively used to derive reduced integrable models [9–11], including nonlocal models involving reflection points [12]. In particular, applying a pair of similarity transformations leads to a novel class of reduced integrable models [13]. The main challenge lies in carefully balancing the reductions imposed on the potentials by the two transformations to preserve the invariance of the zero-curvature equations [14]. A complete classification of such lower-order integrable models associated with the matrix AKNS spectral problems has identified three types of nonlocal NLS equations and two types of nonlocal mKdV equations [15].

Moreover, various powerful methods have been developed to study reduced integrable models, especially for constructing soliton solutions. The inverse scattering transform continues to be an effective approach for solving the Cauchy problems of nonlocal integrable models [16,17]. Classical techniques such as the Hirota bilinear method, Bäcklund transformations, Darboux transformations, and the Riemann–Hilbert method have also proven to be highly effective. Moreover, several innovative mathematical frameworks have been introduced to explore nonlocal reduced integrable models (see, e.g., [15], [18–23]).

In this work, we aim to develop dual similarity transformations and reduced integrable mKdV models, based on the matrix AKNS spectral problems. We begin by formulating two consistent similarity transformations and then apply them to matrix spectral problems to derive reduced

* Correspondence to: Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA.
E-mail address: mawx@cas.usf.edu.

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integrable models. In Section 2, we lay the groundwork for the subsequent analysis by revisiting the matrix AKNS spectral problems and their associated integrable mKdV models, and we propose a general framework for implementing pairs of similarity transformations and constructing reduced mKdV integrable models. In Section 3, we explore four application scenarios within this generating scheme, each employing distinct sets of dual similarity transformations. These examples of mKdV integrable models illustrate the richness and diversity of reduced matrix AKNS integrable models. The final section provides a summary of our results, along with concluding remarks.

2. Reduced mKdV integrable models via dual similarity transformations

2.1. On the matrix AKNS integrable hierarchies: a revisit

Let m and n be two natural numbers. In the AKNS framework, the vector dependent variable $u = u(p,q)$ consists of two matrix potentials, defined as follows:

$$p = p(x,t) = (p_{jk})_{m\times n},\ q = q(x,t) = (q_{kj})_{n\times m}. \tag{2.1}$$

For each $r \geq 0$, we introduce a pair of standard matrix AKNS spectral problems:

$$-i\phi_x = U\phi,\ -i\phi_t = V^{[r]}\phi, \tag{2.2}$$

where the Lax pairs are given by

$$U = U(u,\lambda) = \lambda\Lambda + P, \tag{2.3}$$

and

$$V^{[r]} = V^{[r]}(u,\lambda) = \lambda^r\Omega + Q^{[r]}, \tag{2.4}$$

with

$$\Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix},\ P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.5}$$

and

$$\Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix},\ Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}. \tag{2.6}$$

Here, I_k denotes the identity matrix of size k, λ stands for the spectral parameter, α_1,α_2 and β_1,β_2 are two pairs of distinct arbitrary constants. In addition, $Q^{[0]}$ is taken as the zero matrix of order $(m+n)$. To solve the stationary zero-curvature equation

$$W_x = i[U,W], \tag{2.7}$$

we start with the initial data $W^{[0]} = \Omega$, and consider the following Laurent series expansion:

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]} = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}, \tag{2.8}$$

which yields a unique solution in form of a Laurent series. Such series expansion palys a crucial role in constructing hierarchies of integrable models (see, e.g., [24,25]).

The zero-curvature equations:

$$U_t - V^{[r]}_x + i[U,V^{[r]}] = 0,\ r \geq 0, \tag{2.9}$$

guarantee the compatibility of the two matrix spectral problems in (2.2). Given the specific forms of U and $V^{[r]}$ in (2.3) and (2.4), these equations generate the matrix AKNS hierarchy of integrable models:

$$p_t = i\alpha b^{[r+1]},\ q_t = -i\alpha c^{[r+1]},\ r \geq 0, \tag{2.10}$$

where $\alpha = \alpha_1 - \alpha_2$. The simplest case, with $m = n = 1$, recovers the classical AKNS integrable hierarchy with scalar potentials p and q [26]. Each system in the matrix AKNS integrable hierarchy admits a bi-Hamiltonian structure, along with infinitely many symmetries and conserved quantities (see, e.g., [27–29]).

When $r = 2s+1,\ s \geq 1$, the above matrix AKNS integrable hierarchy (2.10) reduces to the matrix mKdV integrable hierarchy. In particular, for $s = 1$, we obtain the first nonlinear integrable model – the matrix mKdV integrable model:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp),\ q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \tag{2.11}$$

where $\beta = \beta_1 - \beta_2$. The corresponding Lax matrix $V^{[3]}$ is given by

$$V^{[3]} = \lambda^3\Omega + \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P,P_x] + P_{xx} + 2P^3), \tag{2.12}$$

where $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. These equations serve as our basic objects for the subsequent analysis. We point out that many other significant examples of higher-order matrix AKNS integrable models can similarly be derived (see, e.g., [30]).

2.2. Dual similarity transformations

We focus on the case where

$$m = n,\ \alpha_1 = -\alpha_2 = \frac{1}{2},\ \beta_1 = -\beta_2 = -\frac{1}{2}, \tag{2.13}$$

which results in two square potential matrices, p and q. To introduce dual similarity transformations, we start by taking two constant, invertible and square matrices of order n, denoted Δ_1 and Δ_2, and two constant, invertible and symmetric square matrices of order n, denoted Σ_1 and Σ_2. We then define two invertible constant square matrices of order $2n$ as follows, as done in [14,31,33]:

$$\Delta = \begin{bmatrix} 0 & \Delta_1 \\ \Delta_2 & 0 \end{bmatrix},\ \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}. \tag{2.14}$$

Since both Δ and Σ satisfy the following similarity properties

$$\Delta\Lambda\Delta^{-1} = -\Sigma\Lambda\Sigma^{-1} = -\Lambda,\ \Delta\Omega\Delta^{-1} = -\Sigma\Omega\Sigma^{-1} == -\Omega, \tag{2.15}$$

where Λ and Ω are defined as in (2.5) and (2.6), and assuming that A^T stands for the matrix transpose of a matrix A, we propose the following dual similarity transformations:

$$\Delta U(\lambda)\Delta^{-1} = -U^T(\lambda) = -(U(\lambda))^T,\ \Sigma U(\lambda)\Sigma^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T, \tag{2.16}$$

whose constant terms correspond to the identities in (2.15). It will be shown later that the original zero-curvature equations of the mKdV models remain invariant under each of these dual similarity transformations.

Obviously, the dual similarity transformations lead to the following relations for the potential matrix P:

$$\Delta P\Delta^{-1} = -P^T,\ \Sigma P\Sigma^{-1} = -P^T. \tag{2.17}$$

These transformations give rise to the following pairs of constraints for the two matrix potentials p and q:

$$p^T = -\Delta_2 p\Delta_1^{-1},\ q^T = -\Delta_1 q\Delta_2^{-1}, \tag{2.18}$$

and

$$p^T = -\Sigma_2 q\Sigma_1^{-1},\ q^T = -\Sigma_1 p\Sigma_2^{-1}, \tag{2.19}$$

respectively. Since Σ is symmetric, the two constraints in each of (2.19) are compatible. To make compatible constraints in (2.18), we impose the following sufficient condition:

$$\Delta_1^{-1}\Sigma_1 = \Sigma_2^{-1}\Delta_2, \tag{2.20}$$

under which the two constraints in (2.18) imply each other.

Therefore, under the symmetric condition of Σ and the condition given in (2.20), the dual similarity transformations in (2.16) together generate the reduced AKNS matrix spectral problems:

$$-i\phi_x = U\phi,\ U = \begin{bmatrix} \frac{1}{2}\lambda I_n & p \\ -\Sigma_2^{-1}p^T\Sigma_1 & -\frac{1}{2}\lambda I_n \end{bmatrix}, \tag{2.21}$$

where p must satisfy the first constraint in (2.18), i.e., $p^T = -\Delta_2 p\Delta_1^{-1}$, or equivalently, the other reduced AKNS matrix spectral problems:

$$-i\phi_x = U\phi,\ U = \begin{bmatrix} \frac{1}{2}\lambda I_n & -\Sigma_1^{-1}q^T\Sigma_2 \\ q & -\frac{1}{2}\lambda I_n \end{bmatrix}, \tag{2.22}$$

where q must satisfy the second constraint in (2.18), i.e., $q^T = -\Delta_1 q\Delta_2^{-1}$.

2.3. Reduced matrix mKdV integrable models

Note that we have prescribed the initial data:

$$W^{[0]} = \Omega = \begin{bmatrix} -\frac{1}{2}I_n & 0 \\ 0 & \frac{1}{2}I_n \end{bmatrix}, \tag{2.23}$$

for the Laurent series solution W. Under the similarity transformations given in (2.16), and by the uniqueness of solutions to the stationary zero-curvature equation, the solution W, determined by (2.8), satisfies

$$\Delta W(\lambda)\Delta^{-1} = -W^T(\lambda) = -(W(\lambda))^T,\ \Sigma W(\lambda)\Sigma^{-1} = W^T(-\lambda) = (W(-\lambda))^T, \tag{2.24}$$

since the solutions in each pair share the same initial values:

$$\Delta W(\lambda)\Delta^{-1}|_{\lambda=\infty} = -(W(\lambda))^T|_{\lambda=\infty} = -\Omega,\ \Sigma W(\lambda)\Sigma^{-1}|_{\lambda=\infty} = (W(-\lambda))^T|_{\lambda=\infty} = \Omega. \tag{2.25}$$

Therefore, for all $s \ge 0$, we have:

$$\begin{cases} \Delta V^{[2s+1]}(\lambda)\Delta^{-1} = -V^{[2s+1]T}(\lambda) = -(V^{[2s+1]}(\lambda))^T, \\ \Sigma V^{[2s+1]}(\lambda)\Sigma^{-1} = -V^{[2s+1]T}(-\lambda) = -(V^{[2s+1]}(-\lambda))^T, \end{cases} \tag{2.26}$$

which leads to the following invariance property:

$$\Delta(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Delta^{-1} = -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda))^T, \tag{2.27}$$

and similarly,

$$\Sigma(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Sigma^{-1} = -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(-\lambda))^T. \tag{2.28}$$

Consequently, the matrix AKNS integrable models given in (2.10) with $r = 2s+1$ reduce to the following integrable mKdV models:

$$p_t = 2ib^{[2s+2]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}, s \ge 0, \tag{2.29}$$

where p satisfies the first constraint in (2.18), or equivalently,

$$q_t = -2ic^{[2s+2]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2},\ s \ge 0, \tag{2.30}$$

where q satisfies the second constraint in (2.18). In our formulation, the symmetric condition on Σ and the constraint in (2.20) are essential.

Moreover, the matrix spectral problems, consisting of (2.21) and

$$-i\phi_t = V^{[2s+1]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi,\ s \ge 0, \tag{2.31}$$

provide Lax pairs for the reduced integrable mKdV hierarchy (2.29). Alternatively, the matrix spectral problems, consisting of (2.22) and

$$-i\phi_t = V^{[2s+1]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}\phi,\ s \ge 0, \tag{2.32}$$

provide Lax pairs for the reduced integrable mKdV hierarchy (2.30).

As a direct consequence of the Lax operator algebras (see, e.g., [34]), these reduced integrable models possess infinitely many commuting symmetries. It is worth noting that since Δ_1, Δ_2 and Σ_1 are arbitrary, choosing specific forms for these matrices enables the construction of a variety of integrable mKdV models. These models serve as explicit examples of reduced matrix AKNS systems. However, for $r = 2s,\ s \ge 0$, the similarity properties observed in (2.26) no longer hold, and thus such reductions are not applicable in this case.

3. Representative scenarios

In this section, we explore four distinct scenarios by selecting four sets of dual similarity transformations. Each scenario provides illustrative examples of reduced matrix AKNS spectral problems and their associated mKdV integrable models. We focus on the cases where $m = n = 2$ and $m = n = 3$, with the spectral matrix given by

$$U = U(u,\lambda) = \begin{bmatrix} \frac{1}{2}\lambda I_2 & p \\ q & -\frac{1}{2}\lambda I_2 \end{bmatrix}, \tag{3.1}$$

or

$$U = U(u,\lambda) = \begin{bmatrix} \frac{1}{2}\lambda I_3 & p \\ q & -\frac{1}{2}\lambda I_3 \end{bmatrix}, \tag{3.2}$$

where the potential p satisfied the first constraint in (2.18) and q is determined by either the first or the second constraint in (2.19).

Example 3.1. We begin by introducing a set of dual similarity transformations. Specifically, we consider the following pairs of matrices:

$$\Delta_1 = \begin{bmatrix} 0 & \delta_1 \\ \delta_2 & \delta_3 \end{bmatrix},\ \Delta_2 = \begin{bmatrix} 0 & -\delta_2 \\ -\delta_1 & -\delta_3 \end{bmatrix};\ \Sigma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix},\ \Sigma_2 = \begin{bmatrix} 0 & -\frac{\delta_1\delta_2}{\sigma_1} \\ -\frac{\delta_1\delta_2}{\sigma_1} & -\frac{2\delta_1\delta_3}{\sigma_1} \end{bmatrix}; \tag{3.3}$$

where $\delta_1, \delta_2, \sigma_1$ and σ_2 are arbitrary nonzero constants, while δ_3 is arbitrary but not necessarily nonzero. Under these choices, the dual similarity transformations in (2.16) yield the expressions for p and q:

$$p = \begin{bmatrix} p_3 & p_1 \\ p_2 & \frac{\delta_3 p_2 + \delta_1 p_3}{\delta_2} \end{bmatrix}, \tag{3.4}$$

$$q = \begin{bmatrix} \frac{\sigma_1^2(\delta_1 p_3 - \delta_3 p_2)}{\delta_1\delta_2^2} & \frac{\sigma_1^2(\delta_2 p_1 - 2\delta_3 p_3)}{\delta_1\delta_2^2} \\ \frac{\sigma_1^2 p_2}{\delta_1\delta_2} & \frac{\sigma_1^2 p_3}{\delta_1\delta_2} \end{bmatrix}. \tag{3.5}$$

Consequently, the corresponding reduced matrix mKdV integrable model system, with $u=(p_1,p_2,p_3)^T$, is given by

$$\begin{cases} p_{1,t}=p_{1,xxx}+\dfrac{6\sigma_1^2}{\delta_1\delta_2^3}\big[\delta_2(\delta_2p_1p_2+\delta_1p_3^2)p_{1,x}+\delta_3(\delta_2p_1-\delta_3p_3)p_3p_{2,x}+(2\delta_1\delta_2p_1-\delta_3^2p_2-2\delta_1\delta_3p_3)p_3p_{3,x}\big], \\ p_{2,t}=p_{2,xxx}+\dfrac{6\sigma_1^2}{\delta_1\delta_2^2}\big[(\delta_2p_1p_2-\delta_3p_2p_3+\delta_1p_3^2)p_{2,x}+2\delta_1p_2p_3p_{3,x}\big], \\ p_{3,t}=p_{3,xxx}+\dfrac{6\sigma_1^2}{\delta_1\delta_2^2}\big[\delta_2p_2p_3p_{1,x}+(\delta_2p_1-\delta_3p_3)p_3p_{2,x}+(\delta_2p_1p_2-2\delta_3p_2p_3+\delta_1p_3^2)p_{3,x}\big], \end{cases} \tag{3.6}$$

where the constants $\delta_1,\delta_2,\sigma_1$ and σ_2 are arbitrary but nonzero, while δ_3 is arbitrary and may vanish.

By talking

$$\delta_1=-\rho,\ \delta_2=-1,\ \delta_3=\sigma_1=\sigma_2=1, \tag{3.7}$$

the system simplifies to the following mKdV integrable model system:

$$\begin{cases} p_{1,t}=p_{1,xxx}+6[(\rho p_1p_2+p_3^2)p_{1,x}-\rho(p_1+p_3)p_3p_{2,x}+(2p_1-\rho p_2+2p_3)p_3p_{3,x}], \\ p_{2,t}=p_{2,xxx}+6[(\rho(p_1+p_3)p_2+p_3^2)p_{2,x}+2p_2p_3p_{3,x}], \\ p_{3,t}=p_{3,xxx}+6[\rho p_2p_3p_{1,x}+\rho(p_1+p_3)p_3p_{2,x}+(\rho p_1p_2+2\rho p_2p_3+p_3^2)p_{3,x}], \end{cases} \tag{3.8}$$

where $\rho=\pm1$.

Example 3.2. Next, we formulate the second scenario and select the following specific pairs of matrices:

$$\Delta_1=\begin{bmatrix}\delta_1 & \delta_3\\ 0 & \delta_2\end{bmatrix},\ \Delta_2=\begin{bmatrix}-\delta_1 & 0\\ -\delta_3 & -\delta_2\end{bmatrix};\ \Sigma_1=\begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix},\ \Sigma_2=\begin{bmatrix}-\dfrac{\delta_1^2}{\sigma_1} & -\dfrac{\delta_1\delta_3}{\sigma_1}\\ -\dfrac{\delta_1\delta_3}{\sigma_1} & -\dfrac{\delta_3^2}{\sigma_1}-\dfrac{\delta_2^2}{\sigma_2}\end{bmatrix}; \tag{3.9}$$

where $\delta_1,\delta_2,\sigma_1$, and σ_2 are arbitrary nonzero constants, while δ_3 is arbitrary but not necessarily nonzero. In this manner, the dual similarity transformations in (2.16) generate the expressions for p and q:

$$p=\begin{bmatrix} p_1 & p_3\\ \dfrac{\delta_1}{\delta_2}p_3-\dfrac{\delta_3}{\delta_2}p_1 & p_2\end{bmatrix}, \tag{3.10}$$

and

$$q=\begin{bmatrix} \dfrac{\sigma_1[(\delta_2^2\sigma_1+\delta_3^2\sigma_2)p_1-\delta_1\delta_3\sigma_2p_3]}{\delta_1^2\delta_2^2} & -\dfrac{\sigma_2\left\{(\delta_2^2\delta_3\sigma_1+\delta_3^3\sigma_2)p_1+\delta_1[\delta_2\delta_3\sigma_2p_2-(\delta_2^2\sigma_1+\delta_3^2\sigma_2)p_3]\right\}}{\delta_1^2\delta_2^3}\\ \dfrac{\sigma_1\sigma_2(\delta_1p_3-\delta_3p_1)}{\delta_1\delta_2^2} & \dfrac{\sigma_2^2[\delta_3^2p_1+\delta_1(\delta_2p_2-\delta_3p_3)]}{\delta_1\delta_2^3}\end{bmatrix}. \tag{3.11}$$

It is now straightforward to observe that the corresponding reduced matrix mKdV integrable model system, with $u=(p_1,p_2,p_3)^T$, is given by:

$$\begin{cases} p_{1,t}=p_{1,xxx}+\dfrac{6}{\delta_1^2\delta_2^4}\Big\{\big[(\delta_2^2\sigma_1+\delta_3^2\sigma_2)^2p_1^2+\delta_1\delta_3\sigma_2(\delta_2\delta_3p_2\sigma_2-3\delta_2^2\sigma_1p_3-2\delta_3^2\sigma_2p_3)p_1+\delta_1^2\sigma_2(\delta_2^2\sigma_1p_3+\delta_3^2\sigma_2p_3-\delta_2\delta_3\sigma_2p_2)p_3\big]p_{1,x} \\ \qquad -\delta_1\sigma_2(\delta_3p_1-\delta_1p_3)\big[(\delta_2^2\sigma_1+\delta_3^2\sigma_2)p_1+\delta_1\sigma_2(\delta_2p_2-\delta_3p_3)\big]p_{3,x}\Big\}, \\ p_{2,t}=p_{2,xxx}+\dfrac{3}{\delta_1^2\delta_2^4}\Big\{\delta_3\big[(\delta_2^2\sigma_1+\delta_3^2\sigma_2)p_1+\delta_1\sigma_2(\delta_2p_2-\delta_3p_3)\big](\delta_3\sigma_2p_2-\delta_2\sigma_1p_3)p_{1,x} \\ \qquad +\sigma_2\big[\delta_3^2(\delta_2^2\sigma_1+\delta_3^2\sigma_2)p_1^2+\delta_1\delta_3(3\delta_2\delta_3\sigma_2p_2-3\delta_2^2\sigma_1p_3-2\delta_3^2\sigma_2p_3)p_1+\delta_1^2(2\delta_2^2\sigma_2p_2^2-3\delta_2\delta_3\sigma_2p_2p_3+2\delta_2^2\sigma_1p_3^2+\delta_3^2\sigma_2p_3^2)\big]p_{2,x} \\ \qquad -\big[(\delta_2^2\sigma_1+\delta_3^2\sigma_2)p_1+\delta_1\sigma_2(\delta_2p_2-\delta_3p_3)\big]\big[\delta_2\delta_3\sigma_1p_1+\delta_1(\delta_3\sigma_2p_2-2\delta_2\sigma_1p_3)\big]p_{3,x}\Big\}, \\ p_{3,t}=p_{3,xxx}+\dfrac{3}{\delta_1^2\delta_2^3}\Big\{\big[(\delta_2^2\sigma_1+\delta_3^2\sigma_2)p_1+\delta_1\sigma_2(\delta_2p_2-\delta_3p_3)\big](\delta_2\sigma_1p_3-\delta_3\sigma_2p_2)p_{1,x} \\ \qquad -\sigma_2(\delta_3p_1-\delta_1p_3)\big[(\delta_2^2\sigma_1+\delta_3^2\sigma_2)p_1+\delta_1\sigma_2(\delta_2p_2-\delta_3p_3)\big]p_{2,x}+\big[\delta_2\sigma_1(\delta_2^2\sigma_1+\delta_3^2\sigma_2)p_1^2 \\ \qquad +\delta_1\delta_3\sigma_2(\delta_3\sigma_2p_2-3\delta_2\sigma_1p_3)p_1+\delta_1^2\sigma_2(2\delta_2\sigma_1p_3^2-\delta_3\sigma_2p_2p_3+\delta_2\sigma_2p_2^2)\big]p_{3,x}\Big\}, \end{cases} \tag{3.12}$$

where $\delta_1,\delta_2,\sigma_1$ and σ_2 are arbitrary nonzero constants, and δ_3 is arbitrary, though not necessarily nonzero.

When taking

$$\delta_1=\delta_2=\delta_3=-1,\ \sigma_1=\sigma_2=1, \tag{3.13}$$

the system further reduces to the following mKdV integrable model system:

$$\begin{cases} p_{1,t}=p_{1,xxx}+6\left[4p_1^2+p_1(p_2-5p_3)-(p_2-2p_3)p_3\right]p_{1,x}-6(2p_1+p_2-p_3)(p_1-p_3)p_{3,x}, \\ p_{2,t}=p_{2,xxx}+3\left[2p_1^2+2p_2^2+3(p_1-p_3)p_2-5p_1p_3+3p_3^2\right]p_{2,x}+3(2p_1+p_2-p_3)\left[(p_2-p_3)p_{1,x}-(p_1+p_2-2p_3)p_{3,x}\right], \\ p_{3,t}=p_{3,xxx}+3\left[2p_1^2+p_1p_2+p_2^2-(3p_1+p_2)p_3+2p_3^2\right]p_{3,x}-3(2p_1+p_2-p_3)\left[(p_2-p_3)p_{1,x}+(p_1-p_3)p_{2,x}\right]. \end{cases} \tag{3.14}$$

Example 3.3. Now, we examine the third scenario by selecting the following specific pairs of matrices:

$$\Delta_1=\begin{bmatrix}\delta_1&0&\delta_3\\0&\delta_2&0\\0&0&\delta_1\end{bmatrix},\ \Delta_2=\begin{bmatrix}\delta_1&0&0\\0&\delta_2&0\\\delta_3&0&\delta_1\end{bmatrix};\ \Sigma_1=\begin{bmatrix}0&0&\sigma_1\\0&\sigma_2&0\\\sigma_1&0&0\end{bmatrix},\ \Sigma_2=\begin{bmatrix}0&0&\dfrac{\delta_1^2}{\sigma_1}\\0&\dfrac{\delta_2^2}{\sigma_2}&0\\\dfrac{\delta_1^2}{\sigma_1}&0&\dfrac{2\delta_1\delta_3}{\sigma_1}\end{bmatrix}; \tag{3.15}$$

where $\delta_1,\delta_2,\sigma_1$ and σ_2 are arbitrary nonzero constants, while δ_3 is totally arbitrary. Once these matrices have been set, the dual similarity transformations described in (2.16) lead to the explicit expressions for p and q:

$$p=\begin{bmatrix}0&p_2&p_1\\-\dfrac{\delta_1p_2}{\delta_2}&0&p_3\\-p_1&-\dfrac{\delta_2p_3}{\delta_1}-\dfrac{\delta_3p_2}{\delta_1}&-\dfrac{\delta_3p_1}{\delta_1}\end{bmatrix}, \tag{3.16}$$

$$q=\begin{bmatrix}-\dfrac{\delta_3\sigma_1^2p_1}{\delta_1^3}&-\dfrac{\sigma_1\sigma_2(2\delta_3p_2+\delta_2p_3)}{\delta_1^2\delta_2}&-\dfrac{\sigma_1^2p_1}{\delta_1^2}\\\dfrac{\sigma_1\sigma_2(\delta_3p_2+\delta_2p_3)}{\delta_1\delta_2^2}&0&-\dfrac{\sigma_1\sigma_2p_2}{\delta_2^2}\\\dfrac{\sigma_1^2p_1}{\delta_1^2}&\dfrac{\sigma_1\sigma_2p_2}{\delta_1\delta_2}&0\end{bmatrix}. \tag{3.17}$$

Then, it is direct to see that the corresponding reduced matrix mKdV integrable model system, with $u=(p_1,p_2,p_3)^T$, takes the following form:

$$\begin{cases} p_{1,t}=p_{1,xxx}+\dfrac{3\sigma_1}{\delta_1^2\delta_2^2}\left\{2\left[\delta_2^2\sigma_1p_1^2+\delta_1\sigma_2p_2(\delta_3p_2+\delta_2p_3)\right]p_{1,x}\right. \\ \qquad\left.+\delta_1\sigma_2p_1\left[(2\delta_3p_2+\delta_2p_3)p_{2,x}+\delta_2p_2p_{3,x}\right]\right\}, \\ p_{2,t}=p_{2,xxx}+\dfrac{3\sigma_1}{\delta_1^2\delta_2^2}\left[(\delta_2^2\sigma_1p_1^2+4\delta_1\delta_3\sigma_2p_2^2+3\delta_1\delta_2\sigma_2p_2p_3)p_{2,x}\right. \\ \qquad\left.+\delta_2p_2(\delta_2\sigma_1p_1p_{1,x}+\delta_1\sigma_2p_2p_{3,x})\right], \\ p_{3,t}=p_{3,xxx}+\dfrac{3\sigma_1}{\delta_1^2\delta_2^2}\left\{p_3\left[\delta_2^2\sigma_1p_1p_{1,x}+\delta_1\sigma_2(2\delta_3p_2+\delta_2p_3)p_{2,x}\right]\right. \\ \qquad\left.+(\delta_2^2\sigma_1p_1^2+2\delta_1\delta_3\sigma_2p_2^2+3\delta_1\delta_2\sigma_2p_2p_3)p_{3,x}\right\}, \end{cases} \tag{3.18}$$

where $\delta_1,\delta_2,\sigma_1$ and σ_2 are arbitrary nonzero constants, while δ_3 is completely arbitrary.

When choosing

$$\delta_1=\delta_2=-\delta_3=1,\ \sigma_1=\sigma_2=1, \tag{3.19}$$

we obtain the simplified mKdV integrable model system:

$$\begin{cases} p_{1,t}=p_{1,xxx}+6[p_1^2-p_2(p_2-p_3)]p_{1,x}-3p_1\left[(2p_2-p_3)p_{2,x}-p_2p_{3,x}\right], \\ p_{2,t}=p_{2,xxx}+3[p_1^2-p_2(4p_2-3p_3)]p_{2,x}+3p_2(p_1p_{1,x}+p_2p_{3,x}), \\ p_{3,t}=p_{3,xxx}+3[p_1^2-p_2(2p_2-3p_3)]p_{3,x}+3p_3\left[p_1p_{1,x}+(p_3-2p_2)p_{2,x}\right]. \end{cases} \tag{3.20}$$

Example 3.4. Finally, we investigate the fourth scenario by introducing the following specific matrix pairs:

$$\Delta_1=\begin{bmatrix}\delta_1&0&\delta_3\\0&\delta_2&0\\0&0&\delta_1\end{bmatrix},\ \Delta_2=\begin{bmatrix}\delta_1&0&0\\0&\delta_2&0\\\delta_3&0&\delta_1\end{bmatrix};\ \Sigma_1=\begin{bmatrix}\sigma_1&0&0\\0&\sigma_2&0\\0&0&\sigma_3\end{bmatrix},\ \Sigma_2=\begin{bmatrix}\dfrac{\delta_1^2}{\sigma_1}&0&\dfrac{\delta_1\delta_3}{\sigma_1}\\0&\dfrac{\delta_2^2}{\sigma_2}&0\\\dfrac{\delta_1\delta_3}{\sigma_1}&0&\dfrac{\delta_1^2}{\sigma_3}+\dfrac{\delta_3^2}{\sigma_1}\end{bmatrix}; \tag{3.21}$$

where once again, we define δ_1,δ_2 and σ_i, $1\le i\le 3$, as arbitrary nonzero constants. After setting these matrices, the dual similarity transformations described in (2.16) yield the explicit expressions for p and q:

$$p = \begin{bmatrix} 0 & p_2 & p_1 \\ -\frac{\delta_1 p_2}{\delta_2} & 0 & p_3 \\ -p_1 & -\frac{\delta_3 p_2 + \delta_2 p_3}{\delta_1} & -\frac{\delta_3 p_1}{\delta_1} \end{bmatrix}, \tag{3.22}$$

$$q = \begin{bmatrix} \frac{\delta_3 \sigma_1 \sigma_3 p_1}{\delta_1^3} & \frac{\sigma_2(\delta_1^2 \sigma_1 + \delta_3^2 \sigma_3) p_2 + \delta_2 \delta_3 \sigma_2 \sigma_3}{\delta_1^3 \delta_2} & \frac{\sigma_1 \sigma_3 p_1}{\delta_1^2} \\ -\frac{\sigma_1 \sigma_2 p_2}{\delta_2^2} & 0 & \frac{\sigma_2 \sigma_3 (\delta_3 p_2 + \delta_2 p_3)}{\delta_1 \delta_2^2} \\ -\frac{\sigma_1 \sigma_3 p_1}{\delta_1^2} & -\frac{\sigma_2 \sigma_3 (\delta_3 p_2 + \delta_2 p_3)}{\delta_1^2 \delta_2} & 0 \end{bmatrix}. \tag{3.23}$$

Thus, it is clear that the corresponding reduced matrix mKdV integrable model system, with $u = (p_1, p_2, p_3)^T$, takes the following form:

$$\begin{cases} p_{1,t} = p_{1,xxx} - \frac{3}{\delta_1^2 \delta_2^2} \{ \left[2\delta_2^2 \sigma_1 \sigma_3 p_1^2 + \sigma_2(\delta_1^2 \sigma_1 p_2^2 + \delta_3^2 \sigma_3 p_2^2 + 2\delta_2 \delta_3 \sigma_3 p_2 p_3 + \delta_2^2 \sigma_3 p_3^2) \right] p_{1,x} \\ \qquad + 3\sigma_2 p_1 \left[(\delta_1^2 \sigma_1 + \delta_3^2 \sigma_3) p_2 p_{2,x} + \delta_2 \delta_3 \sigma_3 p_3 p_{2,x} + \delta_2 \sigma_3 (\delta_3 p_2 + \delta_2 p_3) p_{3,x} \right] \}, \\ p_{2,t} = p_{2,xxx} - \frac{3}{\delta_1^2 \delta_2} \{ \left[\delta_2^2 \sigma_3 (\sigma_1 p_1^2 + \sigma_2 p_3^2) + 2\sigma_2 (\delta_1^2 \sigma_1 + \delta_3^2 \sigma_3) p_2^2 + 3\delta_2 \delta_3 \sigma_2 \sigma_3 p_2 p_3 \right] p_{2,x} \\ \qquad + \delta_2 \sigma_3 p_2 \left[\delta_2 \sigma_1 p_1 p_{1,x} + \sigma_2 (\delta_2 p_3 + \delta_3 p_2) p_{3,x} \right] \}, \\ p_{3,t} = p_{3,xxx} - \frac{3}{\delta_1^2 \delta_2} \{ \left[\delta_2^2 \sigma_1 \sigma_3 p_1^2 + \sigma_2 (\delta_1^2 \sigma_1 + \delta_3^2 \sigma_3) p_2^2 + 3\delta_2 \delta_3 \sigma_2 \sigma_3 p_2 p_3 + 2\delta_2^2 \sigma_2 \sigma_3 p_3^2 \right] p_{3,x} \\ \qquad + p_3 \left[\delta_2^2 \sigma_1 \sigma_3 p_1 p_{1,x} + \sigma_2 (\delta_1^2 \sigma_1 p_2 + \delta_3^2 \sigma_3 p_2 + \delta_2 \delta_3 \sigma_3 p_3) p_{2,x} \right] \}, \end{cases} \tag{3.24}$$

where $\delta_1, \delta_2, \sigma_1, \sigma_2, \sigma_3$ are arbitrary nonzero constants, while δ_3 is arbitrary but not necessarily nonzero.

When choosing

$$\delta_1 = -\delta_2 = \delta_3 = 1, \ \sigma_1 = -\sigma_2 = \sigma_3 = 1, \tag{3.25}$$

we obtain the simplified mKdV integrable model system:

$$\begin{cases} p_{1,t} = p_{1,xxx} - 3(2p_1^2 - 2p_2^2 + 2p_2 p_3 - p_3^2) p_{1,x} + 3p_1 \left[(2p_2 - p_3) p_{2,x} - (p_2 - p_3) p_{3,x} \right], \\ p_{2,t} = p_{2,xxx} - 3(p_1^2 - 4p_2^2 + 3p_2 p_3 - p_3^2) p_{2,x} - 3p_2 \left[p_1 p_{1,x} + (p_2 - p_3) p_{3,x} \right], \\ p_{3,t} = p_{3,xxx} - 3(p_1^2 - 2p_2^2 + 3p_2 p_3 - 2p_3^2) p_{3,x} - 3p_3 \left[p_1 p_{1,x} + (p_3 - 2p_2) p_{2,x} \right]. \end{cases} \tag{3.26}$$

We note that it is straightforward to compute the Lax matrix $V^{[3]}$, defined in (2.12), for each of the four scenarios. This matrix provides the temporal part of the Lax pairs for the resulting reduced mKdV integrable models. Furthermore, we can derive the entire integrable hierarchies, given by (2.29) (or (2.30)), along with their Lax pairs, determined by (2.21) and (2.31) (or (2.22) and (2.32)).

4. Concluding remarks

This letter investigates various types of dual similarity transformations and applies them to matrix AKNS spectral problems to derive reduced matrix mKdV integrable models. Four specific scenarios of such reduced integrable mKdV models are presented, along with their corresponding reduced matrix AKNS spectral problems, which give rise to integrable hierarchies. The main emphasis of this study is the formulation of appropriate dual similarity transformations that yield novel mKdV integrable models, thereby extending the applicability of the zero-curvature framework developed in previous works (see, e.g., [31–33]). One may also consider using the trace identity [6] to establish Hamiltonian structures for the presented mKdV integrable models.

The examples presented in this study demonstrate the versatility and depth of reduced Lax pairs in constructing integrable models. By applying various dual similarity transformations to the zero-curvature equations, a broad spectrum of integrable reductions can be achieved (see, e.g., [35–38]). The choice of diagonal and off-diagonal block matrices in these transformations plays a pivotal role in shaping the structure of the resulting systems. Moreover, such transformations contribute to the ongoing development of integrable models associated with higher-order matrix spectral problems, as discussed in [39–45]. Comparative analysis of these models with other integrable systems may provide valuable insights into the underlying algebraic and geometric structures.

An equally compelling direction for future research involves the investigation of rich solution phenomena, such as rogue waves, lump solutions, and soliton waves (see, e.g., [46–55]). Approaches such as the inverse scattering method and the Riemann-Hilbert approach offer promising tools for deriving these nonlinear wave solutions. In particular, the use of Darboux transformations significantly expands the possibilities for exploring fascinating nonlinear wave phenomena, with significant potential applications in applied mathematics and engineering sciences.

In conclusion, this research establishes a robust framework for the formulation and comprehensive analysis of integrable models. The integrable models developed herein provide fresh perspectives on the classification of multi-component integrable systems within the Lax pair framework, with the expectation that they will contribute meaningfully to future applications in both the physical and engineering sciences.

Declaration of competing interest

The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

All data generated or analyzed during this study are included in this published article.

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