Physics Letters A 523 (2024) 129783

Contents lists available at ScienceDirect

Physics Letters A

journal homepage: www.elsevier.com/locate/pla

Letter

Integrable couplings stemming from three-dimensional unital algebras

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A R T I C L E I N F O

Communicated by B. Malomed

MSC:
37K15
35Q55

Keywords:
Integrable model
Three-dimensional algebra
Lax pair
Recursion operator
Symmetry

A B S T R A C T

This study aims to explore the connection between integrable couplings and three-dimensional unital algebras. Expansions in these algebras generate integrable couplings, including nonlinear integrable couplings, and their corresponding Lax pairs and hereditary recursion operators possess specific structures, associated with non-semisimple Lie algebras. Applications to the KdV equation are explicitly presented.

1. Introduction

Integrable equations have made significant contributions to mathematical physics, particularly in understanding phenomena like solitons and nonlinear waves. The Korteweg-de Vries (KdV) equation, nonlinear Schrödinger equation, and sine-Gordon equation stand out as prominent examples, each with its own distinct characteristics and applications across various disciplines.

Integrable couplings, on the other hand, delve deeper into the realm of integrable equations, representing a specialized subset within the zero curvature formulation. They maintain the integrability property even when coupled together, which is a remarkable feature [1]. The association of their Lax pairs with non-semisimple Lie algebras adds another layer of complexity and mathematical richness to their study, providing insights into their symmetries and solutions.

There is a body of interesting work exploring integrable couplings within the zero curvature formulation (see, e.g., [2–12]). Perturbation equations are a particular class of integrable couplings [1], and a first-order perturbation equation

$$u_t = K(u),\ \ v_t = K'(u)[v], \tag{1.1}$$

where X' denotes the Gateaux derivative of X, is associated with either of the two spectral matrices:

$$\hat{U} = \begin{bmatrix} U(u) & 0 \\ U'(u)[v] & U(u) \end{bmatrix} \text{ or } \hat{U} = \begin{bmatrix} U(u) & U'(u)[v] \\ 0 & U(u) \end{bmatrix}, \tag{1.2}$$

where U is the spectral matrix associated with the given integrable equation $u_t = K(u)$. The equation for the variable v above is linear with respect to v, when the variable u is fixed. An integrable coupling

$$u_t = K(u),\ \ v_t = S(u,v), \tag{1.3}$$

is called to be nonlinear, when S is nonlinear with respect to v. An integrable coupling of the form

$$u_t = K(u),\ \ v_t = S(u,v),\ \ w_t = T(u,v,w), \tag{1.4}$$

is referred to as a bi-integrable coupling (see, e.g., [13,14]).

Linear integrable couplings involve extensions of symmetry equations [1,15,16] and play a crucial role in classifying integrable equations. However, nonlinear couplings exhibit much richer structures. There are several systematic ways to construct linear integrable cou-

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https://doi.org/10.1016/j.physleta.2024.129783
Received 21 June 2024; Received in revised form 5 August 2024; Accepted 7 August 2024
Available online 12 August 2024
0375-9601/

plings, often beginning with perturbed spectral matrices, defined as before, and the enlarged spectral matrices:

$$\hat{U}=\begin{bmatrix} U(u) & 0 \\ U_{1,a}(v) & 0 \end{bmatrix} \text{ and } \hat{U}=\begin{bmatrix} U(u) & U_{2,a}(v) \\ 0 & 0 \end{bmatrix}, \tag{1.5}$$

where either $U_{1,a}$ or $U_{2,a}$ may not be a square matrix. We have a feasible method for constructing nonlinear integrable couplings from a selection of spectral matrices of the following forms [17]:

$$\hat{U}=\begin{bmatrix} U(u) & 0 \\ U_a(v) & U(u)+U_a(v) \end{bmatrix} \text{ and } \hat{U}=\begin{bmatrix} U(u) & U_a(v) \\ 0 & U(u)+U_a(v) \end{bmatrix}. \tag{1.6}$$

In this paper, we aim to investigate the relationship between integrable couplings and three-dimensional unital algebras. We demonstrate that expansions within these algebras encompass various types of bi-integrable couplings, including nonlinear ones. To illustrate, we apply this general procedure to the KdV equation. Our findings contribute supplementary insights to the existing body of research on integrable couplings.

2. Integrable couplings by three-dimensional algebras

There are five unital associative algebras of dimension three over the complex number field (see [18]). Let us denote the identity element by $\mathbf{1}$. Each of these five algebras consists of linear combinations of three basis elements: the identity element $\mathbf{1}$ and two additional elements denoted as a and b. Based on the definition of the identity element, we have

$$\mathbf{1}\cdot\mathbf{1}=\mathbf{1},\ \mathbf{1}\cdot a=a\cdot\mathbf{1}=a,\ \mathbf{1}\cdot b=b\cdot\mathbf{1}=b. \tag{2.1}$$

It remains to specify

$$\text{(I)}\qquad aa=0,\ bb=0,\ ab=ba=0, \tag{2.2}$$

$$\text{(II)}\qquad aa=b,\ bb=0,\ ab=ba=0, \tag{2.3}$$

$$\text{(III)}\qquad aa=a,\ bb=0,\ ab=ba=0, \tag{2.4}$$

$$\text{(IV)}\qquad aa=a,\ bb=b,\ ab=ba=0, \tag{2.5}$$

$$\text{(V)}\qquad aa=a,\ bb=0,\ ab=b,\ ba=0, \tag{2.6}$$

for five algebras, where we denote $c\cdot d$ by cd for convenience. The fifth one could be generated by the three elements $\mathbf{1}, a', b$ satisfying

$$a'a'=\mathbf{1},\ a'b=-ba'=b. \tag{2.7}$$

Such an element a' could be taken as $a'=2a-1$.

Let us assume that we have an integrable equation

$$u_t=K(u,u_x,\cdots,u^{(n)}), \tag{2.8}$$

where u denotes a column vector of dependent variables and $u^{(n)}$ stands for the n-th derivative. Moreover, we assume that the integrable equation possesses a Lax pair of matrix spectral problems:

$$\phi_x=U(u,\lambda)\phi,\ \phi_t=V(u,\lambda)\phi, \tag{2.9}$$

i.e., it is generated from the zero curvature equation

$$U_t-V_x+[U,V]=0, \tag{2.10}$$

and there is an operator Φ satisfying the recursion operator property and the hereditary property:

$$L_K\Phi=0 \text{ and } L_{\Phi X}\Phi=\Phi L_X\Phi, \tag{2.11}$$

where X is an arbitrary vector field and L_X is the Lie derivative along X:

$$(L_X\Phi)Y=\Phi[\![X,Y]\!]+[\![X,\Phi Y]\!], \tag{2.12}$$

with Y being another arbitrary vector field and $[\![X,Y]\!]$ denoting the commutator of X and Y. The hereditary property [19] allows for the definition of a bi-differential calculus [20], which yields commuting hierarchies of symmetries and conserved quantities.

2.1. Generating scheme

We utilize each of the five algebras and perform expansions

$$u=p\mathbf{1}+qa+rb,\ \phi=\phi_1\mathbf{1}+\phi_2a+\phi_3b. \tag{2.13}$$

Then, by using (2.1) and each of (2.2)-(2.6), and by taking the above expansions, we obtain bi-integrable couplings:

$$\hat{u}_t=\hat{K}=(\hat{K}_1^T,\hat{K}_2^T,\hat{K}_3^T)^T, \tag{2.14}$$

where $\hat{u}=(p^T,q^T,r^T)^T$, and

$$K(p\mathbf{1}+qa+rb)=\hat{K}_1\mathbf{1}+\hat{K}_2a+\hat{K}_3b=K(p)\mathbf{1}+S(p,q)a+T(p,q,r)b. \tag{2.15}$$

More specifically, we have

$$\text{(I)}\qquad p_t=K(p),\ q_t=K'(p)[q],\ r_t=K'(p)[r], \tag{2.16}$$

$$\text{(II)}\qquad p_t=K(p),\ q_t=K'(p)[q],\ r_t=T_2(p,q,r), \tag{2.17}$$

$$\text{(III)}\qquad p_t=K(p),\ q_t=S(p,q),\ r_t=K'(p)[r], \tag{2.18}$$

$$\text{(IV)}\qquad p_t=K(p),\ q_t=S(p,q),\ r_t=S(p,r), \tag{2.19}$$

$$\text{(V)}\qquad p_t=K(p),\ q_t=S(p,q),\ r_t=T_5(p,q,r), \tag{2.20}$$

respectively corresponding to the five algebras. These integrable couplings supplement the categories of four-component integrable equations recently presented in the literature (see, e.g., [21,22]). The non-isomorphic nature of the underlying algebras ensures that they are linearly independent of each other.

Each of their corresponding associated Lax pairs could be formulated as follows:

$$\hat{\phi}_x=\hat{U}\hat{\phi},\ \hat{\phi}_t=\hat{V}\hat{\phi}, \tag{2.21}$$

where $\hat{\phi}=(\phi_1^T,\phi_2^T,\phi_3^T)^T$, and $\hat{U}$ and $\hat{V}$ are defined by

$$\text{(I)}\quad \hat{U}=\begin{bmatrix} \hat{U}_1 & 0 & 0 \\ \hat{U}_2 & \hat{U}_1 & 0 \\ \hat{U}_3 & 0 & \hat{U}_1 \end{bmatrix},\ \hat{V}=\begin{bmatrix} \hat{V}_1 & 0 & 0 \\ \hat{V}_2 & \hat{V}_1 & 0 \\ \hat{V}_3 & 0 & \hat{V}_1 \end{bmatrix}, \tag{2.22}$$

$$\text{(II)}\quad \hat{U}=\begin{bmatrix} \hat{U}_1 & 0 & 0 \\ \hat{U}_2 & \hat{U}_1 & 0 \\ \hat{U}_3 & \hat{U}_2 & \hat{U}_1 \end{bmatrix},\ \hat{V}=\begin{bmatrix} \hat{V}_1 & 0 & 0 \\ \hat{V}_2 & \hat{V}_1 & 0 \\ \hat{V}_3 & \hat{V}_2 & \hat{V}_1 \end{bmatrix}, \tag{2.23}$$

$$\text{(III)}\quad \hat{U}=\begin{bmatrix} \hat{U}_1 & 0 & 0 \\ \hat{U}_2 & \hat{U}_1+\hat{U}_2 & 0 \\ \hat{U}_3 & 0 & \hat{U}_1 \end{bmatrix},\ \hat{V}=\begin{bmatrix} \hat{V}_1 & 0 & 0 \\ \hat{V}_2 & \hat{V}_1+\hat{V}_2 & 0 \\ \hat{V}_3 & 0 & \hat{V}_1 \end{bmatrix}, \tag{2.24}$$

$$\text{(IV)}\quad \hat{U}=\begin{bmatrix} \hat{U}_1 & 0 & 0 \\ \hat{U}_2 & \hat{U}_1+\hat{U}_2 & 0 \\ \hat{U}_3 & 0 & \hat{U}_1+\hat{U}_3 \end{bmatrix},\ \hat{V}=\begin{bmatrix} \hat{V}_1 & 0 & 0 \\ \hat{V}_2 & \hat{V}_1+\hat{V}_2 & 0 \\ \hat{V}_3 & 0 & \hat{V}_1+\hat{V}_3 \end{bmatrix}, \tag{2.25}$$

$$\text{(V)}\quad \hat{U}=\begin{bmatrix} \hat{U}_1 & 0 & 0 \\ \hat{U}_2 & \hat{U}_1+\hat{U}_2 & 0 \\ \hat{U}_3 & 0 & \hat{U}_1+\hat{U}_2 \end{bmatrix},\ \hat{V}=\begin{bmatrix} \hat{V}_1 & 0 & 0 \\ \hat{V}_2 & \hat{V}_1+\hat{V}_2 & 0 \\ \hat{V}_3 & 0 & \hat{V}_1+\hat{V}_2 \end{bmatrix}, \tag{2.26}$$

respectively, with $\hat{U}_i$ and $\hat{V}_i$, $1\le i\le 3$, being determined via

$$\begin{cases} U(p\mathbf{1}+qa+rb) = \hat{U}_1\mathbf{1}+\hat{U}_2 a+\hat{U}_3 b = U(p)\mathbf{1}+\hat{U}_1(p,q)a+\hat{U}_3(p,q,r)b, \\ V(p\mathbf{1}+qa+rb) = \hat{V}_1\mathbf{1}+\hat{V}_2 a+\hat{V}_3 b = V(p)\mathbf{1}+\hat{V}_1(p,q)a+\hat{V}_3(p,q,r)b. \end{cases} \tag{2.27}$$

Moreover, the following corresponding hereditary recursion operators could be formulated:

$$\text{(I)}\quad \hat{\Phi}(\hat{u}) = \begin{bmatrix} \hat{\Phi}_1 & 0 & 0 \\ \hat{\Phi}_2 & \hat{\Phi}_1 & 0 \\ \hat{\Phi}_3 & 0 & \hat{\Phi}_1 \end{bmatrix}, \qquad \text{(II)}\quad \hat{\Phi}(\hat{u}) = \begin{bmatrix} \hat{\Phi}_1 & 0 & 0 \\ \hat{\Phi}_2 & \hat{\Phi}_1 & 0 \\ \hat{\Phi}_3 & \hat{\Phi}_2 & \hat{\Phi}_1 \end{bmatrix}, \tag{2.28}$$

$$\text{(III)}\quad \hat{\Phi}(\hat{u}) = \begin{bmatrix} \hat{\Phi}_1 & 0 & 0 \\ \hat{\Phi}_2 & \hat{\Phi}_1+\hat{\Phi}_2 & 0 \\ \hat{\Phi}_3 & 0 & \hat{\Phi}_1 \end{bmatrix}, \tag{2.29}$$

$$\text{(IV)}\quad \hat{\Phi}(\hat{u}) = \begin{bmatrix} \hat{\Phi}_1 & 0 & 0 \\ \hat{\Phi}_2 & \hat{\Phi}_1+\hat{\Phi}_2 & 0 \\ \hat{\Phi}_3 & 0 & \hat{\Phi}_1+\hat{\Phi}_3 \end{bmatrix},$$

$$\text{(V)}\quad \hat{\Phi}(\hat{u}) = \begin{bmatrix} \hat{\Phi}_1 & 0 & 0 \\ \hat{\Phi}_2 & \hat{\Phi}_1+\hat{\Phi}_2 & 0 \\ \hat{\Phi}_3 & 0 & \hat{\Phi}_1+\hat{\Phi}_2 \end{bmatrix}, \tag{2.30}$$

where

$$\Phi(p\mathbf{1}+qa+rb) = \hat{\Phi}_1\mathbf{1}+\hat{\Phi}_2 a+\hat{\Phi}_3 b = \Phi(p)\mathbf{1}+\hat{\Phi}_2(p,q)a+\hat{\Phi}_3(p,q,r)b, \tag{2.31}$$

with $\Phi(u)$ being a hereditary recursion operator for the original equation (2.8).

The system (2.16) represents a generalization of the symmetry problem with two copies of the linearized equation, and the system (2.17) is exactly the second-order perturbation equation [1]. The first three integrable couplings in (2.16), (2.17) and (2.18) are all extensions of the first-order perturbation equation. Since the sub-vectors S, T_2, and T_5 are nonlinear, the resulting extended equations (2.17), (2.18), (2.19) and (2.20), except the first one (2.16), all present nonlinear integrable couplings of the original equation.

2.2. Applications to the KdV equation

Let us compute a few application examples. We consider the KdV equation

$$u_t = 6uu_x + u_{xxx}. \tag{2.32}$$

It possesses the Lax pair:

$$U(u,\lambda) = \begin{bmatrix} 0 & 1 \\ \lambda - u & 0 \end{bmatrix},\ V(u,\lambda) = \begin{bmatrix} -u_x & 4\lambda + 2u \\ -2(2\lambda+u)(u-\lambda) - u_{xx} & u_x \end{bmatrix}, \tag{2.33}$$

and the hereditary recursion operator:

$$\Phi(u) = \partial^2 + 4u + 2u_x\partial^{-1}. \tag{2.34}$$

The five resulting bi-integrable couplings are given respectively by

$$\text{(I)}\quad p_t = 6pp_x + p_{xxx},\ q_t = 6(pq)_x + q_{xxx},\ r_t = 6(pr)_x + r_{xxx}, \tag{2.35}$$

$$\text{(II)}\quad p_t = 6pp_x + p_{xxx},\ q_t = 6(pq)_x + q_{xxx},\ r_t = 6(pr)_x + 6qq_x + r_{xxx}, \tag{2.36}$$

$$\text{(III)}\quad p_t = 6pp_x + p_{xxx},\ q_t = 6(pq)_x + 6qq_x + q_{xxx},\ r_t = 6(pr)_x + r_{xxx}, \tag{2.37}$$

$$\text{(IV)}\quad p_t = 6pp_x + p_{xxx},\ q_t = 6(pq)_x + 6qq_x + q_{xxx},\ r_t = 6(pr)_x + 6rr_x + r_{xxx}, \tag{2.38}$$

$$\text{(V)}\quad p_t = 6pp_x + p_{xxx},\ q_t = 6(pq)_x + 6qq_x + q_{xxx},\ r_t = 6(pr)_x + 6qr_x + r_{xxx}, \tag{2.39}$$

the first of which is a generalization of the perturbation equation of the KdV equation and the second to the fifth of which are nonlinear integrable couplings of the KdV equation. The matrix blocks $\hat{U}_i$ and $\hat{V}_i$, $1 \le i \le 3$, in the corresponding Lax pairs (2.22)-(2.26) are determined as follows:

$$\text{(I-V)}\quad \hat{U}_1 = U(p),\ \hat{U}_2 = \begin{bmatrix} 0 & 0 \\ -q & 0 \end{bmatrix},\ \hat{U}_3 = \begin{bmatrix} 0 & 0 \\ -r & 0 \end{bmatrix},\ \hat{V}_1 = V(p),$$

and

$$\text{(I)}\quad \hat{V}_2 = \begin{bmatrix} -q_x & 2q \\ -2(\lambda+2p)q - q_{xx} & q_x \end{bmatrix},\quad \hat{V}_3 = \begin{bmatrix} -r_x & 2r \\ -2(\lambda+2p)r - r_{xx} & r_x \end{bmatrix},$$

$$\text{(II)}\quad \hat{V}_2 = \begin{bmatrix} -q_x & 2q \\ -2(\lambda+2p)q - q_{xx} & q_x \end{bmatrix},\quad \hat{V}_3 = \begin{bmatrix} -r_x & 2r \\ -2(\lambda r+2pr+q^2) - r_{xx} & r_x \end{bmatrix},$$

$$\text{(III)}\quad \hat{V}_2 = \begin{bmatrix} -q_x & 2q \\ -2(\lambda+2p+q)q - q_{xx} & q_x \end{bmatrix},\quad \hat{V}_3 = \begin{bmatrix} -r_x & 2r \\ -2(\lambda+2p)r - r_{xx} & r_x \end{bmatrix},$$

$$\text{(IV)}\quad \hat{V}_2 = \begin{bmatrix} -q_x & 2q \\ -2(\lambda+2p+q)q - q_{xx} & q_x \end{bmatrix},\quad \hat{V}_3 = \begin{bmatrix} -r_x & 2r \\ -2(\lambda+2p+r)r - r_{xx} & r_x \end{bmatrix},$$

$$\text{(V)}\quad \hat{V}_2 = \begin{bmatrix} -q_x & 2q \\ -2(\lambda+2p+q)q - q_{xx} & q_x \end{bmatrix},\quad \hat{V}_3 = \begin{bmatrix} -r_x & 2r \\ -2(\lambda+2p)r - r_{xx} & r_x \end{bmatrix}.$$

Furthermore, the associated hereditary recursion operators can be expressed by the following:

$$\text{(I)}\quad \hat{\Phi}(\hat{u}) = \begin{bmatrix} \Phi(p) & 0 & 0 \\ 4q+2q_x\partial_x^{-1} & \Phi(p) & 0 \\ 4r+2r_x\partial_x^{-1} & 0 & \Phi(p) \end{bmatrix},$$

$$\text{(II)}\quad \hat{\Phi}(\hat{u}) = \begin{bmatrix} \Phi(p) & 0 & 0 \\ 4q+2q_x\partial_x^{-1} & \Phi(p) & 0 \\ 4r+2r_x\partial_x^{-1} & 4q+2q_x\partial_x^{-1} & \Phi(p) \end{bmatrix},$$

$$\text{(III)}\quad \hat{\Phi}(\hat{u}) = \begin{bmatrix} \Phi(p) & 0 & 0 \\ 4q+2q_x\partial_x^{-1} & \Phi(p+q) & 0 \\ 4r+2r_x\partial_x^{-1} & 0 & \Phi(p) \end{bmatrix},$$

$$\text{(IV)}\quad \hat{\Phi}(\hat{u}) = \begin{bmatrix} \Phi(p) & 0 & 0 \\ 4q + 2q_x\partial_x^{-1} & \Phi(p+q) & 0 \\ 4r + 2r_x\partial_x^{-1} & 0 & \Phi(p+r) \end{bmatrix},$$

$$\text{(V)}\quad \hat{\Phi}(\hat{u}) = \begin{bmatrix} \Phi(p) & 0 & 0 \\ 4q + 2q_x\partial_x^{-1} & \Phi(p+q) & 0 \\ 4r + 2r_x\partial_x^{-1} & 0 & \Phi(p+q) \end{bmatrix},$$

where $\hat{u} = (p, q, r)^T$ and $\Phi(u)$ is the hereditary recursion operator of the KdV equation, defined by (2.34).

By using these extended integrable equations, Alice-Bob systems, both local and nonlocal, can also be generated (see, e.g., [23,24]). The basic idea described above can be applied to other integrable equations, including both scalar and multi-component ones.

3. Concluding remarks

We have demonstrated that expansions over three-dimensional unital associative algebras yield bi-integrable couplings, presenting a generative idea for integrable couplings. All these algebras exhibit rich algebraic structures related to bi-integrable equations, including their associated Lax pairs and hereditary recursion operators. To illustrate our general approach, we employed the KdV equation as an example. These analyses enhance the existing body of research on integrable couplings [1,25,26].

It is evident that employing block type matrix algebras for Lax pairs enables us to generate larger classes of integrable couplings. By combining the considered form of spectral matrices with other forms found in the literature, we can create a more diverse range of integrable couplings. Furthermore, these integrable couplings may exhibit additional integrable properties, such as Hirota bilinear forms and τ-symmetry algebras [27]. Moreover, dark type integrable couplings [28] using expansions in nilpotent elements (or dark numbers [29]) can be generated [1]. The richness of dynamical structures in such nonlinear models, including soliton phenomena (see, e.g., [30–34]) and Frobenius decompositions (see, e.g., [35,36]), can be explored. On the other hand, integrable extensions, encompassing supersymmetric ones (see, e.g., [37]), bosonizations (see, e.g., [38,39]) and ren-integrable ones (see, e.g., [40]), can also be represented using expansions involving nilpotent elements, such as Grassmann numbers in supersymmetry theory. All such analyses will enrich our understanding of multi-component integrable equations (see, e.g., [21,22,41,42]) and contribute towards their classification from through an algebraic lens.

An intriguing question concerning bi-integrable couplings is whether it is possible to construct an integrable coupling that encompasses two given integrable equations. In other words, given two integrable equations $u_t = K(u)$ and $v_t = S(v)$, can we construct a larger system of the form

$$u_t = K(u),\ v_t = S(v),\ w_t = T(u, v, w), \tag{3.1}$$

while preserving overall integrability? Here, the Gateaux derivatives $T'(u)$ and $T'(v)$ are assumed to be non-zero, characterizing a special bi-integrable coupling. We hope to answer this question in the near future.

Understanding integrable couplings and their associated Lax pairs and recursion operators not only deepens our comprehension of these systems but also opens avenues for exploring new mathematical structures and solving complex physical problems. The applications of integrable equations and couplings extend across diverse fields, from fluid dynamics and plasma physics to nonlinear optics and quantum mechanics, highlighting their relevance and importance in understanding natural phenomena and engineering applications.

CRediT authorship contribution statement

Wen-Xiu Ma: Writing – review & editing, Writing – original draft, Visualization, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The author declares that there are no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The work was supported in part by National Natural Science Foundation of China (12271488, 11975145 and 11972291) and the Ministry of Science and Technology of China (G2021016032L and G2023016011L).

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