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A multi-component integrable hierarchy and its integrable reductions

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ABSTRACT

We present a multi-component integrable hierarchy via the zero curvature formulation. The trace identity is applied to construction of its Hamiltonian structure, and two integrable reductions are generated under similarity transformations. The adopted matrix spectral problem is associated with a special Lie sub-algebra of the general linear algebra.

1. Introduction

Integrable equations are based on Lax pairs or matrix spectral problems associated with matrix Lie algebras [1,2]. The most commonly used is the special linear algebra [3–5], but special orthogonal algebras also generate integrable hierarchies with Hamiltonian structures (see, e.g., [6]). The key point is to choose a pseudoregular element in a matrix loop algebra to form a spectral matrix, and then the corresponding zero curvature equations will naturally produce an integrable hierarchy. Hamiltonian structures could be constructed via the trace identity [7] if the associated Lie algebra is semisimple, and the variational identity [8] if the associated Lie algebra is non-semisimple. This usually guarantees the Liouville integrability.

Integrable reductions could be made under similarity transformations, which keep the original zero curvature equations to be invariant. Local and nonlocal reduced integrable equations have been generated from matrix spectral problems associated with the Ablowitz-Kaup-Newell-Segur (AKNS) spectral problems (see, e.g., [9–11] and [12–17] for local and nonlocal reductions, respectively). It has been shown that through pairs of group reductions, we can generate novel kinds of nonlocal reduced integrable equations [18–20]. All such studies provide great supplements to the traditional theories of partial differential equations.

This letter aims to present an application of the zero curvature formulation to a matrix spectral problem with multiple components. In Section 2, we present a special matrix Lie sub-algebra of the general linear algebra and formulate a matrix spectral problem. Via zero curvature equations, we generate an integrable hierarchy with multiple components, and using the trace identity, we construct its Hamiltonian structure. In Section 3, we propose two group reductions of the spectral matrix, which lead to two reduced integrable hierarchies. A typical example in the first reduced hierarchy reads

$$p_{t_3} = p_{xxx} + 3p_x\alpha\beta p^T p + 3p\alpha\beta p^T p_x - p_x\alpha p^T p\beta - 2p\alpha p_x^T p\beta,$$

where α and β are two constant commuting symmetric and orthogonal matrices and p^T denotes the matrix transpose of the potential p. Two typical examples in the second reduced hierarchy are

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$$ip_{t_2}=p_{xx}+2p\beta p^{\dagger}p-p\alpha p^{T}p^{*}\beta\alpha,$$

and

$$p_{t_3}=p_{xxx}+3p_x\beta p^{\dagger}p+3p\beta p^{\dagger}p_x-p_x\alpha p^{T}p^{*}\beta\alpha-2p\alpha p_x^{T}p^{*}\beta\alpha,$$

where α and β are two real commuting symmetric and orthogonal matrices, and $p^{\dagger}$ and p^{*} denote the Hermitian transpose and the complex conjugate of the potential p, respectively. In the last section, we give a conclusion and some concluding remarks.

2. A multi-component integrable hierarchy

2.1. A matrix Lie algebra

Let n be a given natural number, and α be a given symmetric and orthogonal matrix of order n:

$$\alpha^{T}=\alpha,\ \alpha^{T}\alpha=I_n, \tag{2.1}$$

where I_n is the identity matrix of order n.

Consider a class of square matrices

$$A=\begin{bmatrix}-a & b & 0\\ c & d & \alpha^{T}b^{T}\\ 0 & c^{T}\alpha^{T} & a\end{bmatrix}, \tag{2.2}$$

where a is a scalar, b and c^{T} are row vectors of dimension n and d is a square matrix which satisfies

$$(\alpha d)^{T}=-\alpha d. \tag{2.3}$$

It is direct to check that such matrices form a matrix Lie algebra, under the matrix commutator: $[A,B]=AB-BA$. All matrices in (2.2) are determined by the property that

$$\begin{bmatrix}0 & 0 & 1\\ 0 & -\alpha & 0\\ 1 & 0 & 0\end{bmatrix}A \tag{2.4}$$

is skew-symmetric.

2.2. A multi-component integrable hierarchy

As usual, let λ denote the spectral parameter, and assume that the potential vector reads:

$$u=u(x,t)=(p,q^{T})^{T},\ p=p(x,t)=(p_1,\cdots,p_n),\ q=q(x,t)=(q_1,\cdots,q_n)^{T}. \tag{2.5}$$

We consider the spatial matrix spectral problem as follows:

$$-i\phi_x=U\phi=U(u,\lambda)\phi,\ U=\begin{bmatrix}-\lambda & p & 0\\ q & 0 & \alpha^{T}p^{T}\\ 0 & q^{T}\alpha^{T} & \lambda\end{bmatrix}, \tag{2.6}$$

which provides a counterpart of the AKNS spectral problem [3].

We first solve the stationary zero curvature equation

$$W_x=i[U,W], \tag{2.7}$$

by looking for a Laurent series solution:

$$W=\begin{bmatrix}-a & b & 0\\ c & d & \alpha^{T}b^{T}\\ 0 & c^{T}\alpha^{T} & a\end{bmatrix}=\sum_{s\geq 0}\lambda^{-s}W^{[s]},\ W^{[s]}=\begin{bmatrix}-a^{[s]} & b^{[s]} & 0\\ c^{[s]} & d^{[s]} & \alpha^{T}b^{[s]T}\\ 0 & c^{[s]T}\alpha^{T} & a^{[s]}\end{bmatrix}. \tag{2.8}$$

Note that we have

$$[U,W]=\begin{bmatrix}pc-bq & -\lambda b+pd+ap & 0\\ \lambda c-dq-qa & [U,W]_{22} & -\lambda\alpha^{T}b^{T}-d\alpha^{T}p^{T}+\alpha^{T}p^{T}a\\ 0 & \lambda c^{T}\alpha^{T}-aq^{T}\alpha^{T}+q^{T}\alpha^{T}d & q^{T}b^{T}-c^{T}p^{T}\end{bmatrix},$$

where

$$[U,W]_{22}=qb-cp-\alpha^T b^T q^T \alpha^T+\alpha^T p^T c^T \alpha^T.$$

Thus, the stationary zero curvature equation gives the recursion relation:

$$\begin{cases} a_x^{[0]}=0,\ b^{[0]}=0,\ c^{[0]}=0,\ d_x^{[0]}=0,\\ b^{[s+1]}=ib_x^{[s]}+pd^{[s]}+a^{[s]}p,\\ c^{[s+1]}=-ic_x^{[s]}+d^{[s]}q+qa^{[s]},\\ a_x^{[s+1]}=i(b^{[s+1]}q-pc^{[s+1]})=-b_x^{[s]}q-pc_x^{[s]},\\ d_x^{[s+1]}=i(qb^{[s+1]}-c^{[s+1]}p-\alpha^T b^{[s+1]T}q^T\alpha^T+\alpha^T p^T c^{[s+1]T}\alpha^T), \end{cases} \tag{2.9}$$

where $s\ge 0$. Upon choosing

$$a^{[0]}=1,\ d^{[0]}=0, \tag{2.10}$$

and taking the constant of integration to be zero,

$$a^{[s]}|_{u=0}=0,\ d^{[s]}|_{u=0}=0,\ s\ge 1, \tag{2.11}$$

we can work out

$$\begin{cases} b^{[1]}=p,\ c^{[1]}=q,\ a^{[1]}=0,\ d^{[1]}=0;\\ b^{[2]}=ip_x,\ c^{[2]}=-iq_x,\ a^{[2]}=-pq,\ d^{[2]}=-qp+\alpha^T p^T q^T \alpha^T;\\ b^{[3]}=-p_{xx}-2pqp+p\alpha^T p^T q^T \alpha^T,\\ c^{[3]}=-q_{xx}-2qpq+\alpha^T p^T q^T \alpha^T q,\\ a^{[3]}=-i(p_x q-pq_x),\\ d^{[3]}=i[-(qp_x-q_x p)+\alpha^T(p_x^T q^T-p^T q_x^T)\alpha^T];\\ b^{[4]}=-ip_{xxx}-3ip_x qp-3ipqp_x+ip_x\alpha^T p^T q^T\alpha^T+2ip\alpha^T p_x^T q^T \alpha^T,\\ c^{[4]}=iq_{xxx}+3iq_x pq+3iqpq_x-2i\alpha^T p^T q_x^T \alpha^T q-i\alpha^T p^T q^T \alpha^T q_x. \end{cases} \tag{2.12}$$

Then, we introduce the temporal matrix spectral problems:

$$-i\phi_t=V^{[r]}\phi=V^{[r]}(u,\lambda)\phi,\ V^{[r]}=(\lambda^r W)_+=\sum_{s=0}^{r}\lambda^s W^{[r-s]},\ r\ge 0. \tag{2.13}$$

Now it is direct to see that the compatibility conditions of the two matrix spectral problems in (2.6) and (2.13), i.e., the zero curvature equations:

$$U_{t_r}-V_x^{[r]}+i[U,V^{[r]}]=0,\ r\ge 0, \tag{2.14}$$

generate the multi-component integrable hierarchy:

$$u_{t_r}=K^{[r]}=(ib^{[r+1]},-ic^{[r+1]T})^T,\ \text{i.e., } p_{t_r}=ib^{[r+1]},\ q_{t_r}=-ic^{[r+1]},\ r\ge 0. \tag{2.15}$$

The first two nonlinear integrable equations are the generalized nonlinear Schrödinger equations

$$\begin{cases} ip_{t_2}=p_{xx}+2pqp-p\alpha p^T q^T \alpha,\\ iq_{t_2}=-q_{xx}-2qpq+\alpha p^T q^T \alpha q, \end{cases} \tag{2.16}$$

and the generalized modified Korteweg-de Vries equations

$$\begin{cases} p_{t_3}=p_{xxx}+3p_x qp+3pqp_x-p_x\alpha p^T q^T\alpha-2p\alpha p_x^T q^T \alpha,\\ q_{t_3}=q_{xxx}+3q_x pq+3qpq_x-2\alpha p^T q_x^T \alpha q-\alpha p^T q^T \alpha q_x. \end{cases} \tag{2.17}$$

2.3. Hamiltonian structure

To construct a Hamiltonian structure for the integrable hierarchy (2.15), we apply the trace identity

$$\frac{\delta}{\delta u}\int \operatorname{tr}\Big(W\frac{\partial U}{\partial\lambda}\Big)\,dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\operatorname{tr}\Big(W\frac{\partial U}{\partial u}\Big), \tag{2.18}$$

where γ is a constant. Obviously, we have

$$\mathrm{tr}\big(W\frac{\partial U}{\partial \lambda}\big)=2a,\ \mathrm{tr}\big(W\frac{\partial U}{\partial u}\big)=2(c^T,b)^T, \tag{2.19}$$

and thus, we obtain

$$\frac{\delta}{\delta u}\int a^{[s+1]}\,dx=\lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}(c^{[s]T},b^{[s]})^T,\ s\ge 0. \tag{2.20}$$

Taking $s=2$, we find $\gamma=0$, and then we have

$$\frac{\delta}{\delta u}\int H^{[s]}\,dx=(c^{[s]T},b^{[s]})^T,\ \mathcal{H}^{[s]}=-\int \frac{a^{[s+1]}}{s}\,dx,\ s\ge 1. \tag{2.21}$$

This enables us to furnish the Hamiltonian structure for the integrable hierarchy (2.15):

$$u_{t_r}=K_r=J\frac{\delta \mathcal{H}^{[r]}}{\delta u},\ J=\begin{bmatrix} 0 & iI_m \\ -iI_m & 0 \end{bmatrix},\ r\ge 1, \tag{2.22}$$

where J is a Hamiltonian operator and $\mathcal{H}^{[r]}$ is defined by (2.21). This Hamiltonian structure establishes a relation between symmetries and conserved quantities. Infinitely many symmetries, which are commuting

$$[\![K_{s_1},K_{s_2}]\!]=K'_{s_1}(u)[K_{s_2}]-K'_{s_2}(u)[K_{s_1}]=0,\ s_1,s_2\ge 0, \tag{2.23}$$

are guaranteed by a Lax operator:

$$[\![V^{[s_1]},V^{[s_2]}]\!]=V^{[s_1]\prime}(u)[K^{[s_2]}]-V^{[s_2]\prime}(u)[K^{[s_1]}]+[V^{[s_1]},V^{[s_2]}]=0,\ s_1,s_2\ge 0, \tag{2.24}$$

which can be proved directly. It then follows from the Hamiltonian structure that

$$\{\mathcal{H}_{s_1},\mathcal{H}_{s_2}\}_J=\int \big(\frac{\delta \mathcal{H}^{[s_1]}}{\delta u}\big)^T J\frac{\delta \mathcal{H}^{[s_2]}}{\delta u}\,dx=0,\ s_1,s_2\ge 0. \tag{2.25}$$

A bi-Hamiltonian structure [21] can be presented as well by combining J with a recursion relation for K_s from (2.9) [22].

3. Integrable reductions

Let β be another constant square matrix of order n which satisfies

$$\beta^T=\beta,\ \beta^T\beta=I_n,\ \alpha\beta=\beta\alpha, \tag{3.1}$$

where α is the square matrix involved in the spectral matrix U by (2.6).

Case 1: First, we take a reduction for the spectral matrix U:

$$CU(\lambda)C^{-1}=U(-\lambda),\ C=\begin{bmatrix} 0 & 0 & 1 \\ 0 & \beta & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{3.2}$$

Noting that

$$CU(\lambda)C^{-1}=\begin{bmatrix} \lambda & q^T\alpha^T\beta^T & 0 \\ \beta\alpha^T p^T & 0 & \beta q \\ 0 & p\beta^T & -\lambda \end{bmatrix},$$

the above reduction equivalently requires

$$q=\alpha\beta p^T \text{ or } p=q^T\alpha\beta. \tag{3.3}$$

It is direct to see that

$$CW(\lambda)C^{-1}=-W(-\lambda), \tag{3.4}$$

since both Laurent series $CW(\lambda)C^{-1}$ and $W(-\lambda)$ of λ solve the stationary zero curvature equation (2.7) with $U(-\lambda)$ and possess the opposite initial values at $\lambda=\infty$. Therefore, by virtue of $V^{[r]}=(\lambda^r W)_+$, we further have

$$CV^{[r]}(\lambda)C^{-1}=(-1)^{r+1}V^{[r]}(-\lambda),\ r\ge 0. \tag{3.5}$$

This implies that

$$\begin{aligned} &C\big(U_{t_{2s+1}}(\lambda)-V_x^{[2s+1]}(\lambda)+i[U(\lambda),V^{[2s+1]}(\lambda)]\big)C^{-1}\\ &=U_{t_{2s+1}}(-\lambda)-V_x^{[2s+1]}(-\lambda)+i[U(-\lambda),V^{[2s+1]}(-\lambda)],\ s\ge 0, \end{aligned} \tag{3.6}$$

and thus, we obtain a reduced integrable hierarchy

$$p_{t_{2s+1}} = ib^{[2s+2]}|_{q=\alpha\beta p^T},\ s \geq 0, \tag{3.7}$$

each of which possesses infinitely many symmetries and conservation laws inherited from the original ones under the potential reduction (3.3). The first nonlinear reduced integrable equation is

$$p_{t_3} = p_{xxx} + 3p_x\alpha\beta p^T p + 3p\alpha\beta p^T p_x - p_x\alpha p^T p\beta - 2p\alpha p_x^T p\beta. \tag{3.8}$$

Case 2: Second, we take another reduction for the spectral matrix U:

$$CU(\lambda)C^{-1} = U^\dagger(\lambda^*),\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.9}$$

where $U^\dagger(\lambda^*) = (U(\lambda^*))^\dagger$, β is again a constant square matrix of order n which satisfies (3.1), and $\dagger$ denotes the Hermitian transpose, and $*$ stands for the complex conjugate. In what follows, we assume that α and β are real.

Upon observing that

$$CU(\lambda)C^{-1} = \begin{bmatrix} -\lambda & p\beta^T & 0 \\ \beta q & 0 & \beta\alpha^T p^T \\ 0 & q^T\alpha^T\beta^T & \lambda \end{bmatrix},$$

the above reduction on the spectral matrix exactly requires

$$q^\dagger = p\beta \text{ or } p^\dagger = \beta q, \tag{3.10}$$

(note that to keep $U^\dagger(\lambda^*)$ to be in the chosen algebra, α must be real, and to impose this pair of potential reductions, β must be real). Under such a potential reduction, we have

$$CW(\lambda)C^{-1} = W^\dagger(\lambda^*), \tag{3.11}$$

which guarantees that

$$CV^{[r]}(\lambda)C^{-1} = V^{[r]\dagger}(\lambda^*),\ r \geq 0. \tag{3.12}$$

It then follows that

$$\begin{aligned} &C\big(U_{t_r}(\lambda) - V_x^{[r]}(\lambda) + i[U(\lambda), V^{[r]}(\lambda)]\big)C^{-1} \\ &= \big(U_{t_r}(\lambda^*) - V_x^{[r]}(\lambda^*) + i[U(\lambda^*), V^{[r]}(\lambda^*)]\big)^\dagger,\ r \geq 0, \end{aligned} \tag{3.13}$$

and thus, we obtain a reduced integrable hierarchy

$$p_{t_r} = ib^{r+1}|_{q=\beta p^\dagger},\ r \geq 0, \tag{3.14}$$

whose infinitely many symmetries and conservation laws are similarly inherited from the original ones under the potential reduction (3.10). The first two nonlinear reduced integrable equations read

$$ip_{t_2} = p_{xx} + 2p\beta^T p^\dagger p - p\alpha p^T p^*\beta^T\alpha, \tag{3.15}$$

and

$$p_{t_3} = p_{xxx} + 3p_x\beta p^\dagger p + 3p\beta p^\dagger p_x - p_x\alpha p^T p^*\beta\alpha - 2p\alpha p_x^T p^*\beta\alpha, \tag{3.16}$$

where $p^\dagger$ and p^* denote the Hermitian transpose and the complex conjugate of the potential p, respectively.

4. Concluding remarks

A multi-component integrable hierarchy has been presented from a newly introduced matrix spectral problem associated with a special Lie algebra of the general linear algebra. Two integrable reductions have been made, one of which leads to a reduced integrable hierarchy of generalized mKdV equations, and the other, to a reduced integrable hierarchy involving both generalized NLS equations and generalized mKdV equations.

We remark that it would be interesting to construct reduced nonlocal integrable equations from the adopted matrix spectral problem (see, e.g., [23] for the case so(3,$\mathbb{R}$)). It would also be of significant importance to explore soliton solutions of the presented integrable equations by the Darboux transformation, Riemann-Hilbert problems or the Hirota direct method, including lump solutions [24,25], complexitons [26], rogue waves [27,28] solitonless solutions and algebro-geometric solutions [29]. For lower order integrable equations in the presented hierarchies, one can directly generate multi-phase solutions through special functions including elliptic functions.

CRediT authorship contribution statement

Wen-Xiu Ma: Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

All data generated or analyzed during this study are included in this published article.

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