Physics Letters A 379 (2015) 1975–1978

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Lump solutions to the Kadomtsev–Petviashvili equation

Wen-Xiu Ma [1]

College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, PR China

ARTICLE INFO

Article history:
Received 31 March 2015
Received in revised form 18 June 2015
Accepted 30 June 2015
Available online 2 July 2015
Communicated by R. Wu

Keywords:
Lump solution
Hirota bilinear form
Kadomtsev–Petviashvili equation

ABSTRACT

Through symbolic computation with Maple, a class of lump solutions, rationally localized in all directions in the space, to the $(2+1)$-dimensional Kadomtsev–Petviashvili (KP) equation is presented, making use of its Hirota bilinear form. The resulting lump solutions contain six free parameters, two of which are due to the translation invariance of the KP equation and the other four of which satisfy a non-zero determinant condition guaranteeing analyticity and rational localization of the solutions. Three contour plots with different determinant values are sequentially made to show that the corresponding lump solution tends to zero when the determinant approaches zero. Two particular lump solutions with specific values of the involved parameters are plotted, as illustrative examples.

1. Introduction

In mathematical physics, the Kadomtsev–Petviashvili (KP) equation, named after Boris B. Kadomtsev and Vladimir I. Petviashvili, is a partial differential equation that describes nonlinear wave motion [1]. The KP equation is usually written as

$$P_{KP}(u) := (u_t + 6uu_x + u_{xxx})_x - \sigma u_{yy} = 0, \ \sigma = \pm 1, \tag{1.1}$$

which is classified as the KPI equation when $\sigma = 1$ and the KPII equation when $\sigma = -1$. The equation can be applied to physics as a way to model water waves of long wavelengths with weakly non-linear restoring forces and frequency dispersion. It is a two-dimensional generalization of the one-dimensional Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.2}$$

Like the KdV equation, the KP equation is completely integrable, i.e., the solution to its Cauchy problem can be determined entirely by quadrature [2]. The equation can also be solved using the inverse scattering transform much like the nonlinear Schrödinger equation [3].

It is known that all integrable equations possess soliton solutions – exponentially localized solutions in certain directions. Hirota bilinear forms [4] play an important role in presenting soliton solutions, though some intelligent guesswork is often necessary [5]. In contrast to soliton solutions, lump solutions are a kind of rational function solutions, localized in all directions in the space. General rational function solutions were presented for the KdV equation, the Boussinesq equation and the Toda lattice equation (see, e.g., [6–8]), systematically through the Wronskian and Casoratian determinant techniques for integrable equations [9,10]. The idea of adopting rational functions was also used to establish a powerful approach to exact traveling wave solutions [11]. The Adomian decomposition method was adopted to derive exact series solutions converging to rational solutions for the KdV equation [12], and bilinear Bäcklund transformations were used to construct rational solutions to $(3+1)$-dimensional generalized KP equations (see, e.g., [13]). Particular examples of lump solutions are found for many integrable equations such as the KPI equation [14–16], the three-dimensional three-wave resonant interaction [17], the B-KP equation [18], the Davey–Stewartson-II equation [16] and the Ishimori-I equation [19]. It is natural and interesting to search for lump solutions to nonlinear partial differential equations, taking advantage of Hirota bilinear forms.

In this paper, we would like to focus on the $(2+1)$-dimensional KPI equation and present a general class of lump solutions by symbolic computation with Maple. The $(2+1)$-dimensional KPI equation has a Hirota bilinear form, and so, we will do a search for positive quadratic function solutions to the corresponding $(2+1)$-dimensional bilinear KPI equation. The obtained quadratic function solutions contain a set of six free parameters, and taking special choices of the involved parameters covers a particular class of lump solutions generated from computing long wave limits of

E-mail address: mawx@cas.usf.edu.

[1] Permanent address: Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA.

http://dx.doi.org/10.1016/j.physleta.2015.06.061

soliton solutions. Finally, a few concluding remarks will be given at the end of the paper.

2. Lump solutions to the KP equation

The $(2+1)$-dimensional KP equation (1.1) is a member of the KP soliton hierarchy [20]. Under the link between f and u:

$$u = 2(\ln f)_{xx}, \tag{2.1}$$

it becomes the following Hirota bilinear equation:

$$\begin{aligned} B_{KP}(f) &:= (D_x D_t + D_x^4 - \sigma D_y^2) f \cdot f \\ &= 2[f_{xt} f - f_t f_x + f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2 - \sigma (f_{yy} f - f_y^2)], \\ \sigma &= \pm 1, \end{aligned} \tag{2.2}$$

which is identified as the bilinear KPI equation when $\sigma = 1$ and the bilinear KPII equation when $\sigma = -1$. The transformation (2.1) is also a characteristic one in establishing Bell polynomial theories of soliton equations [21,22], and the actual relation between the KP equation and the bilinear KP equation reads

$$P_{KP}(u) = \left[\frac{B_{KP}(f)}{f^2} \right]_{xx}. \tag{2.3}$$

Therefore, if f solves the bilinear KP equation (2.2), then $u = 2(\ln f)_{xx}$ will solve the KP equation (1.1).

To search for quadratic function solutions to the $(2+1)$-dimensional bilinear KPI equation in (2.2), we begin with

$$\begin{aligned} &f = g^2 + h^2 + a_9, \ g = a_1 x + a_2 y + a_3 t + a_4, \\ &h = a_5 x + a_6 y + a_7 t + a_8, \end{aligned} \tag{2.4}$$

where a_i, $1 \le i \le 9$, are real parameters to be determined. A simpler form $g^2 + a_5$ does not generate analytic solutions, which are rationally localized in all directions in the space; and so, we start with (2.4). A direct Maple symbolic computation with f above generates the following set of constraining equations for the parameters:

$$\begin{aligned} &\Big\{ a_1 = a_1, \ a_2 = a_2, \ a_3 = \frac{a_1 {a_2}^2 - a_1 {a_6}^2 + 2 a_2 a_5 a_6}{{a_1}^2 + {a_5}^2}, \ a_4 = a_4, \\ &a_5 = a_5, \ a_6 = a_6, \ a_7 = \frac{2 a_1 a_2 a_6 - {a_2}^2 a_5 + a_5 {a_6}^2}{{a_1}^2 + {a_5}^2}, \ a_8 = a_8, \\ &a_9 = \frac{3({a_1}^2 + {a_5}^2)^3}{(a_1 a_6 - a_2 a_5)^2} \Big\}, \end{aligned} \tag{2.5}$$

which needs to satisfy a determinant condition

$$\Delta := a_1 a_6 - a_2 a_5 = \begin{vmatrix} a_1 & a_2 \\ a_5 & a_6 \end{vmatrix} \ne 0. \tag{2.6}$$

This set leads to a class of positive quadratic function solutions to the bilinear KPI equation in (2.2):

$$\begin{aligned} f &= \left(a_1 x + a_2 y + \frac{a_1 {a_2}^2 - a_1 {a_6}^2 + 2 a_2 a_5 a_6}{{a_1}^2 + {a_5}^2} t + a_4 \right)^2 \\ &\quad + \left(a_5 x + a_6 y + \frac{2 a_1 a_2 a_6 - {a_2}^2 a_5 + a_5 {a_6}^2}{{a_1}^2 + {a_5}^2} t + a_8 \right)^2 \\ &\quad + \frac{3({a_1}^2 + {a_5}^2)^3}{(a_1 a_6 - a_2 a_5)^2}, \end{aligned} \tag{2.7}$$

and the resulting class of quadratic function solutions, in turn, yields a class of lump solutions to the $(2+1)$-dimensional KPI equation in (1.1) through the transformation (2.1):

$$u = \frac{4(a_1^2 + a_5^2) f - 8(a_1 g + a_5 h)^2}{f^2}, \tag{2.8}$$

where the function f is defined by (2.7), and the functions of g and h are given as follows:

$$g = a_1 x + a_2 y + \frac{a_1 {a_2}^2 - a_1 {a_6}^2 + 2 a_2 a_5 a_6}{{a_1}^2 + {a_5}^2} t + a_4, \tag{2.9}$$

$$h = a_5 x + a_6 y + \frac{2 a_1 a_2 a_6 - {a_2}^2 a_5 + a_5 {a_6}^2}{{a_1}^2 + {a_5}^2} t + a_8. \tag{2.10}$$

In this class of lump solutions, all six involved parameters of a_1, a_2, a_4, a_5, a_6, a_8 are arbitrary provided that the solutions are well defined, i.e., if the determinant condition (2.6) is satisfied. That determinant condition precisely means that two directions (a_1, a_2) and (a_5, a_6) in the xy-plane are not parallel.

Note that the solutions in (2.8) are analytic in the xy-plane if and only if the parameter $a_9 > 0$. The analyticity of the solutions in (2.8) is guaranteed if (2.6) holds. The condition (2.6) also leads to $a_1^2 + a_5^2 \ne 0$, and so $a_9 > 0$. It is readily observed that at any given time t, all the above lump solutions $u \to 0$ if and only if the corresponding sum of squares $g^2 + h^2 \to \infty$, or equivalently, $x^2 + y^2 \to \infty$ due to (2.6). Therefore, the condition (2.6) guarantees both analyticity and localization of the solutions in (2.8). Actually, based on the above observation, we can see that the non-zero determinant condition (2.6), the analyticity of the solutions in (2.8), and the localization of the solutions in (2.8) are equivalent to each other.

If we take a special choice for the parameters:

$$a_1 = 1, \ a_2 = a, \ a_5 = 0, \ a_6 = b, \ a_4 = c, \ a_8 = d, \tag{2.11}$$

we get the following lump solutions:

$$u = 4 \frac{-[x + ay + (a^2 - b^2)t + c]^2 + b^2 (y + 2at + d)^2 + 3/b^2}{\{[x + ay + (a^2 - b^2)t + c]^2 + b^2 (y + 2at + d)^2 + 3/b^2\}^2}, \tag{2.12}$$

which reduce to the lump solutions in [14]:

$$u = 4 \frac{-[x + ay + (a^2 - b^2)t]^2 + b^2 (y + 2at)^2 + 3/b^2}{\{[x + ay + (a^2 - b^2)t]^2 + b^2 (y + 2at)^2 + 3/b^2\}^2}, \tag{2.13}$$

under $c = d = 0$. If we further set $a = P_R$ and $b = P_I$, this exactly gives the lump solutions presented in [16]. Here $p = P_R + i P_I$ is the ratio of wave numbers on the y-axis and the x-axis. The lump solutions (2.13) are obtained from taking long wave limits of a 2-soliton solution associated with p and $\bar{p}$, $\bar{p}$ being the conjugate of p (see [14,16] for details).

Two special pairs of positive quadratic function solutions and lump solutions with specific values of the parameters are given as follows. First, a selection of the parameters:

$$a_1 = 1, \ a_2 = 2, \ a_4 = 0, \ a_5 = 1, \ a_6 = -1, \ a_8 = 0, \tag{2.14}$$

leads to

$$f = \frac{25}{2} t^2 - 8 xt + 5 yt + 2 x^2 + 2 xy + 5 y^2 + \frac{8}{3}, \tag{2.15}$$

$$u = -\frac{48 \, (21 t^2 - 48 xt - 78 yt + 12 x^2 + 12 xy - 24 y^2 - 16)}{(75 t^2 - 48 xt + 30 yt + 12 x^2 + 12 xy + 30 y^2 + 16)^2}. \tag{2.16}$$

Second, another selection of the parameters:

$$a_1 = 1, \ a_2 = -2, \ a_4 = 0, \ a_5 = -2, \ a_6 = 1, \ a_8 = 0, \tag{2.17}$$

yields

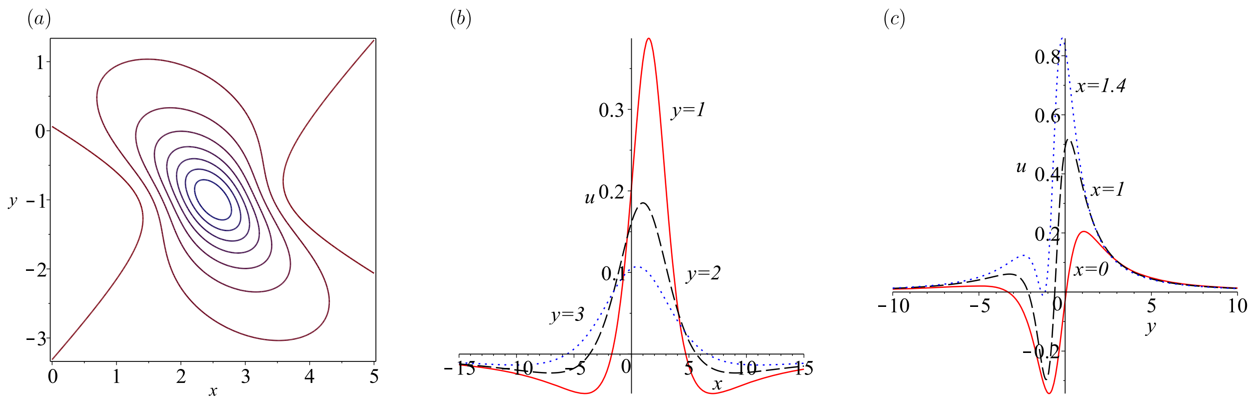

Fig. 1. Profiles of (2.16) with $t = 1$: (a) density plot, (b) x-curves, (c) y-curves.

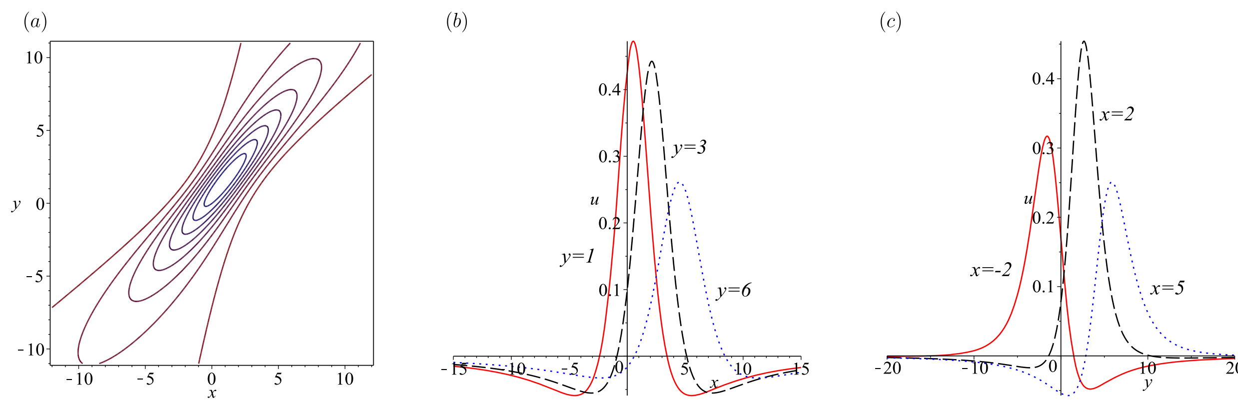

Fig. 2. Profiles of (2.19) with $t = 1$: (a) density plot, (b) x-curves, (c) y-curves.

$$f = 5\,t^2 + \frac{14\,xt}{5} - 8\,yt + 5\,x^2 - 8\,xy + 5\,y^2 + \frac{125}{3}, \tag{2.18}$$

$$u = \frac{12\,(1581\,t^2 - 1050\,xt - 1320\,yt - 1875\,x^2 + 3000\,xy - 525\,y^2 + 15\,625)}{(75\,t^2 + 42\,xt - 120\,yt + 75\,x^2 - 120\,xy + 75\,y^2 + 625)^2}. \tag{2.19}$$

Their plots when $t = 1$ are depicted in Fig. 1 and Fig. 2, respectively.

The solutions (2.16) and (2.19) have a dependence on the spatial variable x in the second function h, but the solution (2.12) does not. Generally, based on (2.8), we know that when the determinant Δ in (2.6) tends to zero, the corresponding lump solution u by (2.8) tends to zero, too. Particularly, taking

$$a_1 = 1,\ a_2 = 1,\ a_4 = 0,\ a_5 = 1,\ a_6 = 1 + \varepsilon,\ a_8 = 0, \tag{2.20}$$

which leads to $\Delta = \varepsilon$, we obtain, from (2.8), the following lump solution

$$u = \frac{16\varepsilon^2 p(\varepsilon)}{[q(\varepsilon)]^2}, \tag{2.21}$$

where

$$\begin{cases} p(\varepsilon) = t^2\varepsilon^6 + 2t(2t + y)\varepsilon^5 - 4(2t^2 + 2tx + 3ty + xy + y^2)\varepsilon^3 \\ \qquad - 4(t + x + y)^2\varepsilon^2 + 48, \\ q(\varepsilon) = t^2\varepsilon^6 + 2t(2t + y)\varepsilon^5 + 2(2t + y)^2\varepsilon^4 \\ \qquad + 4(2t^2 + 2tx + 3ty + xy + y^2)\varepsilon^3 \\ \qquad + 4(t + x + y)^2\varepsilon^2 + 48. \end{cases}$$

Obviously, the limit of this solution when ε approaches zero is zero. Fig. 3 displays the contour plots of the solution (2.21) when $t = 1$ with three different values of Δ.

3. Concluding remarks

Based on the Hirota formulation and by a Maple symbolic computation, we presented a class of lump solutions to the $(2+1)$-dimensional KPI equation in (1.1), and the analyticity and localization of the resulting solutions is guaranteed by a non-zero determinant condition. A sub-class of lump solutions under special choices of the parameters involved covers the lump solutions previously presented by computing long wave limits of soliton solutions [14,16]. Three contour plots with small determinant values were sequentially made to exhibit that the corresponding lump solution tends to zero when the determinant tends to zero. Two particular lump solutions with specific values of the parameters were also plotted. For the bilinear KPII equation in (2.2), we can obtain a class of quadratic function solutions similar to (2.7):

$$\begin{aligned} f = {} & \left(a_1 x + a_2 y - \frac{a_1 a_2{}^2 - a_1 a_6{}^2 + 2\,a_2 a_5 a_6}{a_1{}^2 + a_5{}^2} t + a_4\right)^2 \\ & + \left(a_5 x + a_6 y - \frac{2\,a_1 a_2 a_6 - a_2{}^2 a_5 + a_5 a_6{}^2}{a_1{}^2 + a_5{}^2} t + a_8\right)^2 \\ & - \frac{3(a_1{}^2 + a_5{}^2)^3}{(a_1 a_6 - a_2 a_5)^2}, \end{aligned}$$

but none of them is, unfortunately, positive in the whole xy-space. Rational solutions to the KPII equation in $(3+1)$-dimensions are linked to the good Boussinesq equation by a transformation of dependent variables [23].

There are other studies on lump solitons of the KP equation in the literature. Evolution of lump solitons and their generation from few-cycle input pulses were numerically simulated by Minzoni and Smyth [24] and by Leblond, Kremer and Mihalache [25], respectively. Multiple lump solitons were characterized by the discrete

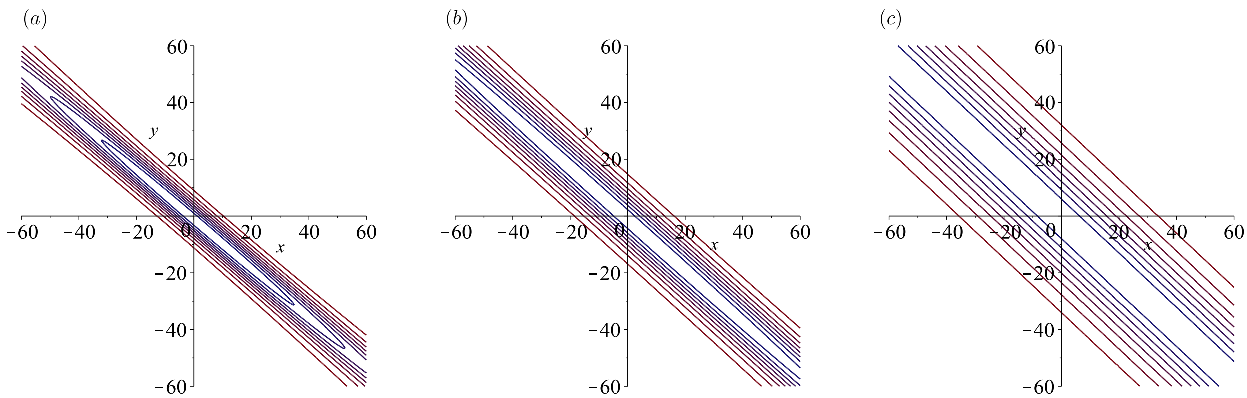

Fig. 3. Contour plots of (2.21) at $t = 1$ with (a) $\Delta = 0.3$, (b) $\Delta = 0.2$, (c) $\Delta = 0.1$.

spectrum of the nonstationary Schrödinger equation, whose related eigenfunctions have multiple poles in the spectral parameter, via the inverse scattering transform [26]; and an algorithmic construction and classification of multiple lump solitons was made through truncated Painlevé expansions by Estévez and Prada [27]. Moreover, existence of lump solitons was proved for a kind of generalized KP equations by de Bouard and Saut [28], and lump solitons of the KP equation was carried over by Dimakis and Müller-Hoissen [29] to the pseudodual chiral model as a dispersionless limit of the matrix KP equation.

We point out that resonant solutions, in terms of exponential functions, to generalized bilinear and tri-linear differential equations have been systematically analyzed [30,31]. It would be very interesting to determine when there exist positive polynomial solutions including quadratic function solutions to generalized bilinear and tri-linear equations. This kind of polynomial solutions will generate lump solutions to the corresponding nonlinear equations through $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$. Particularly, rogue wave solutions could be generated as well in terms of positive polynomial solutions. Rogue wave solutions, which draw a great deal of attention from mathematicians and physicists worldwide, are a particularly interesting class of lump-like solutions; and such solutions, usually with rational function amplitudes, could be used to describe significant nonlinear wave phenomena in both oceanography [32] and nonlinear optics [33].

For the KP equation, decreasing solutions depending rationally on the spatial variable x were identified with the Hamiltonian flow that arises in Moser's theory of the Toda system [34], and nondecreasing solutions being rational in the spatial variables x and y were linked to a generalized multi-particle Moser system [35]. There is also some direct search for rational solutions to nonlinear partial differential equations (see, e.g., [36,37] for the generalized KdV and KP equations), which can be transformed into generalized bilinear equations formulated in terms of generalized bilinear derivatives [38]. Multi-component and higher-order extensions of lump solutions exhibiting diverse soliton phenomena, particularly in $(3+1)$-dimensional cases and fully discrete cases, would be very interesting topics for future research.

Acknowledgements

The work was supported in part by NNSFC under the grants 11371326, 11271008 and 1371361, the Fundamental Research Funds for the Central Universities (2013XK03), the Natural Science Foundation of Shandong Province (Grant No. ZR2013AL016), Zhejiang Innovation Project of China (Grant No. T200905), the First-class Discipline of Universities in Shanghai and the Shanghai Univ. Leading Academic Discipline Project (No. A.13-0101-12-004), and the Distinguished Professorship at the Shanghai University of Electric Power.

References

[1] B.B. Kadomtsev, V.I. Petviashvili, Sov. Phys. Dokl. 15 (1970) 539.
[2] M. Mulase, Adv. Math. 54 (1984) 57.
[3] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, 1991.
[4] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, 2004.
[5] P.J. Caudrey, Philos. Trans. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. 369 (2011) 1215.
[6] W.X. Ma, Y. You, Trans. Am. Math. Soc. 357 (2005) 1753.
[7] W.X. Ma, C.X. Li, J.S. He, Nonlinear Anal. TMA 70 (2009) 4245.
[8] W.X. Ma, Y. You, Chaos Solitons Fractals 22 (2004) 395.
[9] N.C. Freeman, J.J.C. Nimmo, Phys. Lett. A 95 (1983) 1.
[10] W.X. Ma, Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal. 16 (Suppl. S1) (2009) 201.
[11] W.X. Ma, J.-H. Lee, Chaos Solitons Fractals 42 (2009) 1356.
[12] A.-M. Wazwaz, Chaos Solitons Fractals 12 (2001) 2283.
[13] W.X. Ma, A. Abdeljabbar, Appl. Math. Lett. 12 (2012) 1500.
[14] S.V. Manakov, V.E. Zakharov, L.A. Bordag, V.B. Matveev, Phys. Lett. A 63 (1977) 205.
[15] R.S. Johnson, S. Thompson, Phys. Lett. A 66 (1978) 279.
[16] J. Satsuma, M.J. Ablowitz, J. Math. Phys. 20 (1979) 1496.
[17] D.J. Kaup, J. Math. Phys. 22 (1981) 1176.
[18] C.R. Gilson, J.J.C. Nimmo, Phys. Lett. A 147 (1990) 472.
[19] K. Imai, Prog. Theor. Phys. 98 (1997) 1013.
[20] M. Sato, RIMS Kokyuroku 439 (1981) 30.
[21] C. Gilson, F. Lambert, J. Nimmo, R. Willox, Proc. R. Soc. Lond. Ser. A 452 (1996) 223.
[22] W.X. Ma, J. Phys. Conf. Ser. 411 (2013) 012021.
[23] W.X. Ma, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 2663.
[24] A.A. Minzoni, N.F. Smyth, Wave Motion 24 (1996) 291.
[25] H. Leblond, D. Kremer, D. Mihalache, Phys. Rev. A 80 (2009) 053812.
[26] J. Villarroel, M.J. Ablowitz, Commun. Math. Phys. 207 (1999) 1.
[27] P.G. Estévez, J. Prada, J. Nonlinear Math. Phys. 15 (2008) 166.
[28] A. de Bouard, J.-C. Saut, SIAM J. Math. Anal. 28 (1997) 1064.
[29] A. Dimakis, F. Müller-Hoissen, J. Gen. Lie Theory Appl. 2 (2008) 141.
[30] W.X. Ma, Rep. Math. Phys. 72 (2013) 41.
[31] W.X. Ma, Front. Math. China 8 (2013) 1139.
[32] P. Müller, C. Garrett, A. Osborne, Oceanography 18 (2005) 66.
[33] D.R. Solli, C. Ropers, P. Koonath, B. Jalali, Nature 450 (2007) 1054.
[34] I.M. Kričever, Funkc. Anal. Prilozh. 12 (1978) 76.
[35] A.P. Veselov, Usp. Mat. Nauk 35 (1980) 195.
[36] Y. Zhang, W.X. Ma, Appl. Math. Comput. 256 (2015) 252.
[37] Y.F. Zhang, W.X. Ma, Z. Naturforsch. A 70 (2015) 263.
[38] W.X. Ma, Stud. Nonlinear Sci. 2 (2011) 140.