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Reducing Lax pairs to obtain integrable matrix modified Korteweg–de Vries models

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Abstract. This paper explores the matrix modified Korteweg–de Vries (mKdV) integrable models using similarity transformations. The study employs the Lax pair formulation as a foundation, proposing pairs of similarity transformations to reduce the Lax pairs of the Ablowitz–Kaup–Newell–Segur matrix spectral problems, thereby deriving integrable matrix mKdV models. Four illustrative scenarios are discussed to present specific examples of these reduced integrable models.

Keywords. Lax pair; Ablowitz–Kaup–Newell–Segur matrix spectral problem; zero-curvature equation; similarity transformation.

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1. Introduction

It starts with formulating Lax pairs [1] to generate integrable models [2], where the spectral matrices originate from matrix Lie algebras [3]. Infinitely many commuting symmetries and conservation laws can be derived from the associated Lax pairs, which are linked to bi-Hamiltonian structures [4]. The inverse scattering transform can then be applied to their Cauchy problems [5,6].

The matrix Ablowitz–Kaup–Newell-Segur (AKNS) spectral problems provide a universal framework for typical integrable models, such as the nonlinear Schrödinger (NLS) equation and the modified Korteweg–de Vries (mKdV) equation. A single similarity transformation can be used to reduce Lax pairs and get the corresponding reduced integrable models [7]. Applying a pair of similarity transformations can produce a variety of reduced integrable models [8]. The challenge lies in balancing the reductions applied to the potentials generated by the two similarity transformations, ensuring the invariance of the associated zero-curvature equations [9].

Similarity transformations have also been widely applied in the formulation of non-local integrable models involving reflection points [10]. A complete classification of lower-order non-local integrable models associated with the matrix AKNS spectral problems has identified three types of non-local NLS equations and two types of non-local modified Korteweg–de Vries mKdV equations [11]. Additionally, various efficient approaches have been developed to study reduced integrable models, particularly for constructing their soliton solutions.

The classical inverse scattering transform remains a powerful technique for solving Cauchy problems of non-local integrable models [12,13]. Moreover, techniques, such as the Hirota bilinear method, Bäcklund transforms, Darboux transformation and Riemann–Hilbert method have proven to be effective. Also, several innovative mathematical frameworks have been proposed to investigate non-local reduced integrable models (see, e.g., [11,14–19]).

In this paper, we aim to present the reduced integrable mKdV models through a pair of similarity transformations, based on the matrix AKNS spectral problems. The

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crucial step involves formulating two similarity transformations that are consistent with each other. In §2, we lay the foundation for the subsequent analysis by revisiting the AKNS matrix spectral problems, their associated integrable mKdV models and the general framework for conducting pairs of similarity transformations. In §3, we analyse four application scenarios within the generating scheme, each using distinct sets of block matrices to construct the pair of similarity transformations. These examples of integrable mKdV models highlight the richness of the reduced matrix AKNS integrable models. The final section provides a summary of our results, along with some concluding remarks.

2. Matrix-integrable mKdV models via similarity transformations

2.1 *Revisiting the matrix AKNS integrable hierarchies*

Let m and n be two natural numbers. As usual, we define two matrix potentials, p and q, as follows:

$$p = p(x,t) = (p_{jk})_{m\times n},\quad q = q(x,t) = (q_{kj})_{n\times m}, \tag{1}$$

and denote the dependent variable by $u = u(p,q)$, which is a vector-valued function of p and q. For each $r \geq 0$, we associate a pair of standard matrix AKNS spectral problems:

$$-i\phi_x = U\phi,\quad -i\phi_t = V^{[r]}\phi, \tag{2}$$

where the Lax pairs are defined as follows:

$$U = U(u,\lambda) = \lambda\Lambda + P \tag{3}$$

and

$$V^{[r]} = V^{[r]}(u,\lambda) = \lambda^r\Omega + Q^{[r]}, \tag{4}$$

where

$$\begin{cases} \Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix}, \\ P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \\ \Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix}, \\ Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}. \end{cases} \tag{5}$$

In these Lax pairs, I_k is the identity matrix of size k, λ denotes the spectral parameter, α_1, α_2 and β_1, β_2 are two pairs of arbitrary distinct constants and $Q^{[0]}$ is the $(m+n)$th-order zero matrix. Additionally, starting with the initial condition $W^{[0]} = \Omega$, we define the following Laurent series:

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]} = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}. \tag{6}$$

This series represents the unique Laurent series solution to the stationary zero-curvature equation

$$W_x = i[U, W]. \tag{7}$$

Such a series solution is a crucial object for generating hierarchies of integrable models (see [20,21]).

The zero-curvature equations:

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0,\quad r \geq 0 \tag{8}$$

ensure the consistency of the two matrix spectral problems in (2). Based on the specific forms in (3) and (4), these generate the matrix AKNS hierarchy of integrable models:

$$p_t = i\alpha b^{[r+1]},\quad q_t = -i\alpha c^{[r+1]},\quad r \geq 0, \tag{9}$$

where $\alpha = \alpha_1 - \alpha_2$. The simplest case with $m = n = 1$ yields the AKNS integrable hierarchy with scalar potentials, p and q [22]. Each system within the matrix AKNS integrable hierarchy possesses a bi-Hamiltonian structure, along with infinitely many symmetries and conserved quantities (see [23–25]).

When $r = 2s+1$, $s \geq 1$, the matrix AKNS integrable hierarchy (9) reduces to the matrix mKdV integrable hierarchy. Furthermore, when $s = 1$, we obtain the first nonlinear integrable model – the integrable matrix mKdV equations:

$$\begin{cases} p_t = -\dfrac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp), \\ q_t = -\dfrac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \end{cases} \tag{10}$$

where $\beta = \beta_1 - \beta_2$. The corresponding Lax matrix $V^{[3]}$ is given by

$$V^{[3]} = \lambda^3\Omega + \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3), \tag{11}$$

where $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. We note that many other significant examples of higher-order matrix AKNS integrable models can similarly be generated (see [26]).

2.2 *Pairs of similarity transformations*

To introduce a pair of similarity transformations, we start by taking two constant, invertible, symmetric square matrices of order m, denoted by Σ_1 and Δ_1, and two constant, invertible, symmetric square matrices of order n, denoted by Σ_2 and Δ_2. We then define two invertible constant square matrices of order $m + n$ as follows, as done in [9,27,28]:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{12}$$

Clearly, both Σ and Δ satisfy the following similarity properties:

$$\begin{aligned} &\Sigma \Lambda \Sigma^{-1} = \Delta \Lambda \Delta^{-1} = \Lambda, \\ &\Sigma \Omega \Sigma^{-1} = \Delta \Omega \Delta^{-1} = \Omega, \end{aligned} \tag{13}$$

where Λ and Ω are defined as in (5). Assuming that A^T stands for the matrix transpose of a matrix A, we propose the following pair of similarity transformations:

$$\begin{aligned} &\Sigma U(\lambda) \Sigma^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T, \\ &\Delta U(\lambda) \Delta^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T, \end{aligned} \tag{14}$$

whose constant terms correspond to the identities in (13). It will be proved later that the original zero-curvature equations of the mKdV equations remain invariant under each of these similarity transformations.

Obviously, the two similarity transformations lead to the following relations for the potential matrix P:

$$\Sigma P \Sigma^{-1} = -P^T, \quad \Delta P \Delta^{-1} = -P^T. \tag{15}$$

These transformations give rise to the following pairs of constraints for the two matrix potentials p and q:

$$p^T = -\Sigma_2 q \Sigma_1^{-1}, \quad q^T = -\Sigma_1 p \Sigma_2^{-1} \tag{16}$$

and

$$p^T = -\Delta_2 q \Delta_1^{-1}, \quad q^T = -\Delta_1 p \Delta_2^{-1}. \tag{17}$$

Since both Σ and Δ are symmetric, the two constraints in (16) and (17) are compatible. Under each of the following two equivalent conditions:

$$\Sigma_1 p \Sigma_2^{-1} = \Delta_1 p \Delta_2^{-1} \tag{18}$$

or

$$\Sigma_2 q \Sigma_1^{-1} = \Delta_2 q \Delta_1^{-1}, \tag{19}$$

the two sets of constraints in (16) and (17) imply each other.

Therefore, under the condition given either in (18) or (19), the two similarity transformations in (14) generate the reduced AKNS matrix spectral problems:

$$-i\phi_x = U\phi, \ U = \begin{bmatrix} \alpha_1 \lambda I_m & p \\ -\Sigma_2^{-1} p^T \Sigma_1 & \alpha_2 \lambda I_n \end{bmatrix}, \tag{20}$$

where p must satisfy (18), or the other reduced AKNS matrix spectral problems:

$$-i\phi_x = U\phi, \ U = \begin{bmatrix} \alpha_1 \lambda I_m & -\Sigma_1^{-1} q^T \Sigma_2 \\ q & \alpha_2 \lambda I_n \end{bmatrix}, \tag{21}$$

where q must satisfy (19).

2.3 *Reduced integrable matrix mKdV models*

Note that we take the initial data

$$W^{[0]} = \Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix}, \tag{22}$$

as the starting term of the Laurent series solution W. Under the similarity transformations given in (14), we see from the uniqueness of solutions to the stationary zero-curvature equation that the solution W, determined by (6), satisfies

$$\begin{aligned} &\Sigma W(\lambda) \Sigma^{-1} = W^T(-\lambda) = (W(-\lambda))^T, \\ &\Delta W(\lambda) \Delta^{-1} = W^T(-\lambda) = (W(-\lambda))^T. \end{aligned} \tag{23}$$

Therefore, for all $s \geq 0$, we have

$$\begin{cases} \Sigma V^{[2s+1]}(\lambda) \Sigma^{-1} = -V^{[2s+1]T}(-\lambda) \\ \qquad = -(V^{[2s+1]}(-\lambda))^T, \\ \Delta V^{[2s+1]}(\lambda) \Delta^{-1} = -V^{[2s+1]T}(-\lambda) \\ \qquad = -(V^{[2s+1]}(-\lambda))^T, \end{cases} \tag{24}$$

and thus, we obtain

$$\begin{aligned} &\Sigma(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Sigma^{-1} \\ &\quad = -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(-\lambda))^T \end{aligned}$$

and

$$\begin{aligned} &\Delta(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Delta^{-1} \\ &\quad = -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(-\lambda))^T. \end{aligned}$$

Consequently, the matrix AKNS integrable models in (9) with $r = 2s + 1$ reduce to the following integrable mKdV models:

$$p_t = 2ib^{[2s+2]}|_{q=-\Sigma_2^{-1} p^T \Sigma_1}, \quad s \geq 0, \tag{25}$$

where p satisfies (18) or

$$q_t = -2ic^{[2s+2]}|_{p=-\Sigma_1^{-1} q^T \Sigma_2}, \quad s \geq 0, \tag{26}$$

where q satisfies (19).

The matrix spectral problems (20) and

$$-i\phi_t = V^{[2s+1]}|_{q=-\Sigma_2^{-1} p^T \Sigma_1} \phi, \quad s \geq 0, \tag{27}$$

provide Lax pairs for the reduced integrable hierarchy (25), or the matrix spectral problems (21) and

$$-i\phi_t = V^{[2s+1]}|_{p=-\Sigma_1^{-1} q^T \Sigma_2} \phi, \quad s \geq 0 \tag{28}$$

provide Lax pairs for the reduced integrable hierarchy (26).

As a consequence of the Lax operator algebras (see [29]), the resulting reduced integrable models possess infinitely many commuting symmetries. It is important to note that since Σ_1, Σ_2, Δ_1 and Δ_2 are arbitrary, selecting specific forms for these matrices allows for the construction of a wide variety of integrable mKdV models. These models serve as concrete examples of the broader class of reduced integrable matrix AKNS models. However, when $r = 2s,\ s \geq 0$, the similarity properties observed in (24) do not hold.

3. Four implementation scenarios

In this section, we explore four distinct scenarios by selecting four sets of pairs of similarity transformations. Each scenario presents an illustrative example of a reduced matrix AKNS spectral problem and its corresponding integrable mKdV equations. We focus on the case where $m = 2$ and $n = 3$, with the spectral matrix given by

$$U = U(u,\lambda) = \begin{bmatrix} \alpha_2\lambda I_2 & p \\ q & \alpha_2\lambda I_3 \end{bmatrix}, \tag{29}$$

where p satisfy(18) and q is determined by the second equation in either (16) or (17). Alternatively, q may satisfy (19), in which case p is given by the first equation in either (16) or (17).

Example 1. Let us begin by introducing a pair of similarity transformations. We consider the following specific pairs of matrices:

$$\begin{aligned} &\Sigma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix},\ \Sigma_2 = \begin{bmatrix} 0 & 0 & \delta_1 \\ 0 & \delta_2 & 0 \\ \delta_1 & 0 & 0 \end{bmatrix}; \\ &\Delta_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix},\ \Delta_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_1 \end{bmatrix}, \end{aligned} \tag{30}$$

where σ_1, δ_1 and δ_2 are arbitrary non-zero constants. In this manner, the similarity transformations (14) generate the expressions for p and q:

$$p = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_3 & p_2 & p_1 \end{bmatrix},\ q = \begin{bmatrix} -\frac{\sigma_1}{\delta_1}p_1 & -\frac{\sigma_1}{\delta_1}p_3 \\ -\frac{\sigma_1}{\delta_2}p_2 & -\frac{\sigma_1}{\delta_2}p_2 \\ -\frac{\sigma_1}{\delta_1}p_3 & -\frac{\sigma_1}{\delta_1}p_1 \end{bmatrix}. \tag{31}$$

It is now straightforward to observe that the corresponding reduced integrable matrix mKdV equations with $u = (p_1, p_2, p_3)^T$ are given by

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3}p_{1,xxx} + \frac{3\beta\sigma_1}{\alpha^3\delta_1\delta_2} \\ \quad \times\{[\delta_1 p_2^2 + 2\delta_2(p_1^2+p_3^2)]p_{1,x} \\ \quad + \delta_1 p_2(p_1+p_3)p_{2,x} + (\delta_1 p_2^2 + 4\delta_2 p_1 p_3)p_{3,x}\}, \\ p_{2,t} = -\frac{\beta}{\alpha^3}p_{2,xxx} + \frac{3\beta\sigma_1}{\alpha^3\delta_1\delta_2} \\ \quad \times\{[4\delta_1 p_2^2 + \delta_2(p_1+p_3)^2]p_{2,x} \\ \quad + \delta_2 p_2(p_1+p_3)(p_{1,x}+p_{3,x})\}, \\ p_{3,t} = -\frac{\beta}{\alpha^3}p_{3,xxx} + \frac{3\beta\sigma_1}{\alpha^3\delta_1\delta_2} \\ \quad \times\{(\delta_1 p_2^2 + 4\delta_2 p_1 p_3)p_{1,x} + \delta_1 p_2(p_1+p_3)p_{2,x} \\ \quad + [\delta_1 p_2^2 + 2\delta_2(p_1^2+p_3^2)]p_{3,x}\}, \end{cases} \tag{32}$$

where α, β, σ_1, δ_1 and δ_2 are arbitrary but non-zero constants.

When taking

$$\alpha = -\sigma_1 = 1,\ \beta = -\delta_1 = -\delta_2 = -1, \tag{33}$$

the equations further reduce to the following integrable mKdV equations:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 3[(2p_1^2 + p_2^2 + 2p_3^2)p_{1,x} \\ \quad + p_2(p_1+p_3)p_{2,x} + (4p_1p_3 + p_2^2)p_{3,x}], \\ p_{2,t} = p_{2,xxx} + 3\{p_2(p_1+p_3)(p_{1,x}+p_{3,x}) \\ \quad + [4p_2^2 + (p_1+p_3)^2]p_{2,x}\}, \\ p_{3,t} = p_{3,xxx} + 3[(4p_1p_3 + p_2^2)p_{1,x} \\ \quad + p_2(p_1+p_3)p_{2,x} + (2p_1^2 + p_2^2 + 2p_3^2)p_{3,x}]. \end{cases} \tag{34}$$

Example 2. Next, we explore the second scenario by choosing the following specific pairs of matrices:

$$\begin{aligned} &\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix},\ \Sigma_2 = \begin{bmatrix} 0 & 0 & \delta_1 \\ 0 & \delta_2 & 0 \\ \delta_1 & 0 & 0 \end{bmatrix}; \\ &\Delta_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix},\ \Delta_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_1 \end{bmatrix}, \end{aligned} \tag{35}$$

where σ_1, δ_1 and δ_2 are arbitrary non-zero constants. Note that we have swapped the selections for Σ_1 and Δ_1 in Example 1. Under these choices, the similarity transformations (14) result in the expressions for p

and q:

$$p=\begin{bmatrix} p_1 & p_2 & p_3 \\ p_3 & p_2 & p_1 \end{bmatrix},\ q=\begin{bmatrix} -\frac{\sigma_1}{\delta_1}p_3 & -\frac{\sigma_1}{\delta_1}p_1 \\ -\frac{\sigma_1}{\delta_2}p_2 & -\frac{\sigma_1}{\delta_2}p_2 \\ -\frac{\sigma_1}{\delta_1}p_1 & -\frac{\sigma_1}{\delta_1}p_3 \end{bmatrix}. \tag{36}$$

Consequently, the corresponding reduced integrable matrix mKdV equations with $u=(p_1,p_2,p_3)^T$ are given by

$$\begin{cases} p_{1,t}=-\frac{\beta}{\alpha^3}p_{1,xxx}+\frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}\{(\delta_1 p_2^2+4\delta_2 p_1 p_3)p_{1,x} \\ \quad +\delta_1 p_2(p_1+p_3)p_{2,x}+[\delta_1 p_2^2+2\delta_2(p_1^2+p_3^2)]p_{3,x}\}, \\ p_{2,t}=-\frac{\beta}{\alpha^3}p_{2,xxx}+\frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}\{[4\delta_1 p_2^2+\delta_2(p_1+p_3)^2] \\ \quad \times p_{2,x}+\delta_2 p_2(p_1+p_3)(p_{1,x}+p_{3,x})\}, \\ p_{3,t}=-\frac{\beta}{\alpha^3}p_{3,xxx}+\frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}\{[\delta_1 p_2^2+2\delta_2(p_1^2+p_3^2)] \\ \quad \times p_{1,x}+\delta_1 p_2(p_1+p_3)p_{2,x}+(\delta_1 p_2^2+4\delta_2 p_1 p_3)p_{3,x}\}, \end{cases} \tag{37}$$

where α, β, σ_1, δ_1 and δ_2 are arbitrary constants but non-zero. Note that the nonlinear terms on the right-hand side of the above model coincide with those in (32), upon replacing p_1 with p_3.

By setting

$$\alpha=-\sigma_1=1,\ \beta=-\delta_1=-\delta_2=-1, \tag{38}$$

the equations simplify to the following integrable mKdV equations:

$$\begin{cases} p_{1,t}=p_{1,xxx}+3[(4p_1p_3+p_2^2)p_{1,x} \\ \quad +p_2(p_1+p_3)p_{2,x}+(2p_1^2+p_2^2+2p_3^2)p_{3,x}], \\ p_{2,t}=p_{2,xxx}+3\{p_2(p_1+p_3)(p_{1,x} \\ \quad +p_{3,x})+[4p_2^2+(p_1+p_3)^2]p_{2,x}\}, \\ p_{3,t}=p_{3,xxx}+3[(2p_1^2+p_2^2+2p_3^2)p_{1,x} \\ \quad +p_2(p_1+p_3)p_{2,x}+(4p_1p_3+p_2^2)p_{3,x}]. \end{cases} \tag{39}$$

Example 3. Now, we examine the third scenario by selecting the following specific pairs of matrices:

$$\Sigma_1=\begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix},\ \Sigma_2=\begin{bmatrix} 0 & 0 & \delta_1 \\ 0 & \delta_2 & 0 \\ \delta_1 & 0 & 0 \end{bmatrix};$$

$$\Delta_1=\begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix},\ \Delta_2=\begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_1 \end{bmatrix}, \tag{40}$$

where again σ_1, δ_1 and δ_2 are arbitrary non-zero constants. In these choices, we have used off-diagonal matrices for both Σ_1 and Δ_1. Once these matrices have been set, the similarity transformations described in (14) lead to the explicit expressions for p and q:

$$p=\begin{bmatrix} p_1 & p_2 & p_1 \\ p_3 & p_4 & p_3 \end{bmatrix},\ q=\begin{bmatrix} -\frac{\sigma_1}{\delta_1}p_3 & -\frac{\sigma_1}{\delta_1}p_1 \\ -\frac{\sigma_1}{\delta_2}p_4 & -\frac{\sigma_1}{\delta_2}p_2 \\ -\frac{\sigma_1}{\delta_1}p_3 & -\frac{\sigma_1}{\delta_1}p_1 \end{bmatrix}. \tag{41}$$

Then, it can be directly seen that the corresponding reduced integrable matrix mKdV equations with $u=(p_1,p_2,p_3,p_4)^T$ take the following form:

$$\begin{cases} p_{1,t}=-\frac{\beta}{\alpha^3}p_{1,xxx}+\frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}[(6\delta_2 p_1p_3+\delta_1 p_2p_4)p_{1,x} \\ \quad +\delta_1(p_1p_4+p_2p_3)p_{2,x}+(2\delta_2 p_1^2+\delta_1 p_2^2)p_{3,x}], \\ p_{2,t}=-\frac{\beta}{\alpha^3}p_{2,xxx}+\frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}[2\delta_2(p_1p_4+p_2p_3)p_{1,x} \\ \quad +(2\delta_2 p_1p_3+3\delta_1 p_2p_4)p_{2,x}+(2\delta_2 p_1^2+\delta_1 p_2^2)p_{4,x}], \\ p_{3,t}=-\frac{\beta}{\alpha^3}p_{3,xxx}+\frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}[(2\delta_2 p_3^2+\delta_1 p_4^2)p_{1,x} \\ \quad +(6\delta_2 p_1p_3+\delta_1 p_2p_4)p_{3,x}+\delta_1(p_1p_4+p_2p_3)p_{4,x}], \\ p_{4,t}=-\frac{\beta}{\alpha^3}p_{4,xxx}+\frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}[(2\delta_2 p_3^2+\delta_1 p_4^2)p_{2,x} \\ \quad +2\delta_2(p_1p_4+p_2p_3)p_{3,x}+(2\delta_2 p_1p_3+3\delta_1 p_2p_4)p_{4,x}], \end{cases} \tag{42}$$

where $\alpha,\beta,\sigma_1,\delta_1,\delta_2$ are arbitrary but non-zero constants.

When choosing

$$\alpha=-\sigma_1=1,\ \beta=-\delta_1=-\delta_2=-1, \tag{43}$$

we obtain the simplified integrable mKdV equations:

$$\begin{cases} p_{1,t}=p_{1,xxx}+3[(6p_1p_3+p_2p_4)p_{1,x} \\ \quad +(p_1p_4+p_2p_3)p_{2,x}+(2p_1^2+p_2^2)p_{3,x}], \\ p_{2,t}=p_{2,xxx}+3[2(p_1p_4+p_2p_3)p_{1,x} \\ \quad +(2p_1p_3+3p_2p_4)p_{2,x}+(2p_1^2+p_2^2)p_{4,x}], \\ p_{3,t}=p_{3,xxx}+3[(2p_3^2+p_4^2)p_{1,x}+(6p_1p_3 \\ \quad +p_2p_4)p_{3,x}+(p_1p_4+p_2p_3)p_{4,x}], \\ p_{4,t}=p_{4,xxx}+3[(2p_3^2+p_4^2)p_{2,x}+2(p_1p_4 \\ \quad +p_2p_3)p_{3,x}+(2p_1p_3+3p_2p_4)p_{4,x}]. \end{cases} \tag{44}$$

Example 4. Lastly, we explore the fourth scenario by defining the following specific matrix pairs:

$$\Sigma_1=\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix},\ \Sigma_2=\begin{bmatrix} 0 & 0 & \delta_1 \\ 0 & \delta_2 & 0 \\ \delta_1 & 0 & 0 \end{bmatrix};$$

$$\Delta_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}, \ \Delta_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_1 \end{bmatrix}, \tag{45}$$

where once again, we have σ_1, δ_1 and δ_2 as arbitrary non-zero constants. In these matrix choices, we have selected diagonal matrices for both Σ_1 and Δ_1. After setting these matrices, the similarity transformations outlined in (14) provide explicit expressions for p and q:

$$p = \begin{bmatrix} p_1 & p_2 & p_1 \\ p_3 & p_4 & p_3 \end{bmatrix}, \ q = \begin{bmatrix} -\frac{\sigma_1}{\delta_1} p_1 & -\frac{\sigma_1}{\delta_1} p_3 \\ -\frac{\sigma_1}{\delta_2} p_2 & -\frac{\sigma_1}{\delta_2} p_4 \\ -\frac{\sigma_1}{\delta_1} p_1 & -\frac{\sigma_1}{\delta_1} p_3 \end{bmatrix}. \tag{46}$$

Thus, it is clear that the corresponding reduced integrable matrix mKdV equations, with $u = (p_1, p_2, p_3, p_4)^T$, take the following form:

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3} p_{1,xxx} + \frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}\big[(4\delta_2 p_1^2 + \delta_1 p_2^2 + 2\delta_2 p_3^2) \\ \quad \times p_{1,x} + \delta_1(p_1p_2 + p_3p_4)p_{2,x} + (2\delta_2 p_1 p_3 \\ \quad + \delta_1 p_2 p_4)p_{3,x}\big], \\ p_{2,t} = -\frac{\beta}{\alpha^3} p_{2,xxx} + \frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}\big[2\delta_2(p_1p_2 + p_3p_4)p_{1,x} \\ \quad + (2\delta_2 p_1^2 + 2\delta_1 p_2^2 + \delta_1 p_4^2)p_{2,x} + (2\delta_2 p_1 p_3 \\ \quad + \delta_1 p_2 p_4)p_{4,x}\big], \\ p_{3,t} = -\frac{\beta}{\alpha^3} p_{3,xxx} + \frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}\big[(2\delta_2 p_1 p_3 + \delta_1 p_2 p_4)p_{1,x} \\ \quad + (2\delta_2 p_1^2 + 4\delta_2 p_3^2 + \delta_1 p_4^2)p_{3,x} + \delta_1(p_1 p_2 \\ \quad + p_3 p_4)p_{4,x}\big], \\ p_{4,t} = -\frac{\beta}{\alpha^3} p_{4,xxx} + \frac{\beta\sigma_1}{\alpha^3\delta_1\delta_2}\big[(2\delta_2 p_1 p_3 + \delta_1 p_2 p_4)p_{2,x} \\ \quad + 2\delta_2(p_1p_2 + p_3p_4)p_{3,x} + (\delta_1 p_2^2 + 2\delta_2 p_3^2 \\ \quad + 2\delta_1 p_4^2)p_{4,x}\big], \end{cases} \tag{47}$$

where $\alpha, \beta, \sigma_1, \delta_1, \delta_2$ are arbitrary non-zero constants.

When choosing

$$\alpha = -\sigma_1 = 1, \\ \beta = -\delta_1 = -\delta_2 = -1, \tag{48}$$

we obtain the simplified integrable mKdV equations:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 3\big[(4p_1^2 + p_2^2 + 2p_3^2)p_{1,x} \\ \quad + (p_1p_2 + p_3p_4)p_{2,x} + (2p_1p_3 + p_2p_4)p_{3,x}\big], \\ p_{2,t} = p_{2,xxx} + 3\big[2(p_1p_2 + p_3p_4)p_{1,x} \\ \quad + (2p_1^2 + 2p_2^2 + p_4^2)p_{2,x} + (2p_1p_3 + p_2p_4)p_{4,x}\big], \\ p_{3,t} = p_{3,xxx} + 3\big[(2p_1p_3 + p_2p_4)p_{1,x} \\ \quad + (2p_1^2 + 4p_3^2 + p_4^2)p_{3,x} + (p_1p_2 + p_3p_4)p_{4,x}\big], \\ p_{4,t} = p_{4,xxx} + 3\big[(2p_1p_3 + p_2p_4)p_{2,x} \\ \quad + 2(p_1p_2 + p_3p_4)p_{3,x} + (p_2^2 + 2p_3^2 + 2p_4^2)p_{4,x}\big]. \end{cases} \tag{49}$$

We point out that it is also straightforward to compute the Lax matrix $V^{[3]}$, as defined by (11), in these four scenarios. This matrix provides the temporal part of the Lax pairs for the resulting reduced integrable mKdV models.

4. Concluding remarks

This paper investigates a pair of similarity transformations of the same form, applied to the matrix AKNS spectral problems, leading to reduced integrable matrix mKdV models. Four specific scenarios of these reduced integrable matrix mKdV models are constructed, along with their corresponding reduced matrix AKNS spectral problems. A central focus of this study is the identification of two appropriate similarity transformations that produce novel mKdV integrable models, thus extending the framework established in earlier studies (see [9,27,28]).

The examples presented demonstrate the versatility and depth of the reduced Lax pairs in constructing integrable models. By applying various similarity transformations to the zero-curvature equations, a wide range of integrable reductions can be achieved (see [30–33]). The choice of diagonal block matrices in the similarity transformations is crucial in shaping the structure of these systems. These transformations open avenues for exploring fascinating nonlinear wave phenomena, with significant potential applications in applied and engineering sciences. Moreover, they contribute to the ongoing development of integrable models linked to higher-order matrix spectral problems, as examined in [34–39].

This research explores the applications of a framework for the formulation and in-depth analysis of integrable models. Comparing these models with others could provide valuable insights in uncovering the algebraic and geometric structures inherent in various integrable models. Furthermore, studying captivating solution phenomena, such as rogue waves, lump waves and soliton waves, would be of great interest (see [40–49]). The integrable models presented in this work offer fresh perspectives on classifying multicomponent integrable models within the Lax pair framework, with the expectation that these models will contribute to promising applications in both physical and engineering sciences.

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References

[1] P D Lax, *Commun. Pure Appl. Math.* **21**, 467 (1968)
[2] A Das, *Integrable models* (World Scientific, Teaneck, NJ, 1989)
[3] G Z Tu, *J. Phys. A: Math. Gen.* **22**, 2375 (1989)
[4] F Magri, *J. Math. Phys.* **19**, 1156 (1978)
[5] M J Ablowitz and H Segur, *Solitons and the inverse scattering transform* (SIAM, Philadelphia, 1981)
[6] P G Drazin and R S Johnson, *Solitons: An introduction* (Cambridge University Press, Cambridge, 1983)
[7] W X Ma, *Nonlinear Anal.: Real World Appl.* **47**, 1 (2019)
[8] W X Ma, *Physica D* **446**, 133672 (2023)
[9] W X Ma, *East Asian J. Appl. Math.* **14**, 281 (2024)
[10] M J Ablowitz and Z H Musslimani, *Stud. Appl. Math.* **139**, 7 (2017)
[11] W X Ma, *Partial Differ. Equ. Appl. Math.* **4**, 100190 (2021)
[12] M J Ablowitz and Z H Musslimani, *Nonlinearity* **29**, 915 (2016)
[13] W X Ma, Y H Huang and F D Wang, *Eur. J. Appl. Math.* **33**, 1062 (2022)
[14] J L Ji and Z N Zhu, *Commun. Nonlinear Sci. Numer. Simul.* **42**, 699 (2017)
[15] C Q Song, D M Xiao and Z N Zhu, *Commun. Nonlinear Sci. Numer. Simul.* **45**, 13 (2017)
[16] M Gürses and A Pekcan, *J. Math. Phys.* **59**, 051501 (2018)
[17] M Gürses and A Pekcan, *Commun. Nonlinear Sci. Numer. Simul.* **67**, 427 (2019)
[18] C Q Song, D Y Liu and L Y Ma, *Chaos Solitons Fractals* **181**, 114707 (2024)
[19] X Wang, D L Du and H Wang, *Theor. Math. Phys.* **218**, 370 (2024)
[20] J Zhang, C P Zhang and Y N Cui, *Adv. Math. Phys.* **2017**, 9743475 (2017)
[21] Q L Zhao, Y D Zhong and X Y Li, *J. Appl. Anal. Comput.* **12**, 1353 (2022)
[22] M J Ablowitz, D J Kaup, A C Newell and H Segur, *Stud. Appl. Math.* **53**, 249 (1974)
[23] H F Wang and Y F Zhang, *Int. J. Theor. Phys.* **60**, 1797 (2021)
[24] T S Liu and T C Xia, *Nonlinear Anal. Real World Appl.* **68**, 103667 (2022)
[25] X M Zhu and J B Zhang, *Adv. Math. Phys.* **2022**, 2200092 (2022)
[26] W X Ma, *Chin. Phys. B* **32**, 020201 (2023)
[27] W X Ma, *J. Geom. Phys.* **177**, 104522 (2022)
[28] W X Ma, *Mathematics* **13**, 1438 (2025)
[29] W X Ma, *J. Phys. A: Math. Gen.* **25**, 5329 (1992)
[30] F Yu and L Li, *Nonlinear Dyn.* **87**, 2697 (2017)
[31] T A Sulaiman, U Younas, A Yusuf, M Younis, M Bilal and Shafqat-Ur-Rehman, *Mod. Phys. Lett. B* **35**, 2150109 (2021)
[32] R S Ye, Y Zhang and W X Ma, *Acta Appl. Math.* **178**, 7 (2022)
[33] U Younas, T A Sulaiman and J Ren, *Int. J. Mod. Phys. B* **37**, 2350004 (2023)
[34] W X Ma, *Indian J. Phys.* **99**, 1063 (2025)
[35] W X Ma, *Theor. Math. Phys.* **221**, 1603 (2024)
[36] X G Geng and X Zeng, *Physica D* **448**, 133738 (2023)
[37] X G Geng, M X Jia, B Xue and Y Y Zhai, *Lett. Math. Phys.* **114**, 30 (2024)
[38] W X Ma, *Int. J. Geom. Meth. Mod. Phys.* **22**, 2450293 (2025)
[39] W X Ma, *Chaos Solitons Fractals* **195**, 116309 (2025)
[40] H C Ma, Y X Bai and A P Deng, *Front. Math. China* **17**, 943 (2022)
[41] B D V Campos, *Math. Comput. Simul.* **201**, 396 (2022)
[42] J J Yang, S F Tian and Z Q Li, *J. Math. Anal. Appl.* **511**, 126103 (2022)
[43] H C Ma, S P Yue, Y D Gao and A P Deng, *Qual. Theory Dyn. Syst.* **22**, 72 (2023)
[44] M Kaplan and N Raza, *Int. J. Nonlinear Sci. Numer. Simul.* **24**, 349 (2023)
[45] W X Ma, *Rom. Rep. Phys.* **76**, 108 (2024)
[46] M Kaplan, A R Butt, H Thabet, A Akbulut, N Raza and D Kumar, *Waves Random Complex Media* **34**, 4172 (2024)
[47] D Gao, W X Ma and X Lü, *Z. Naturforsch.* **79a**, 887 (2024)
[48] J Y Chu, Y Q Liu and W X Ma, *Mod. Phys. Lett. B* **39**, 2550060 (2025)
[49] L Cheng, Y Zhang and W X Ma, *Commun. Theor. Phys.* **77**, 035002 (2025)