Pramana – J. Phys. (2024) 98:68
https://doi.org/10.1007/s12043-024-02765-8

Solving a non-local linear differential equation model of the Newtonian-type

WEN-XIU MA[1,2,3,4]

[1]Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China
[2]Department of Mathematics, King Abdulaziz University, 21589 Jeddah, Saudi Arabia
[3]Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
[4]Material Science Innovation and Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa
E-mail: mawx@cas.usf.edu

MS received 10 November 2023; revised 16 February 2024; accepted 19 February 2024

Abstract. Motivated by recent studies on non-local integrable models, we consider a non-local inhomogeneous linear differential equation model of Newtonian type:

$x''(t) = \lambda x(t) + \mu x(-t) + f(t),\ t \in \mathbb{R},$

where λ and μ are real constants and f is continuous. Through decomposing functions into their even and odd parts, we transform the non-local model into a local model, and then with the classical ODE technique, solve the resulting local model under the even and odd constraints. The general solution involving two arbitrary constants is presented in nine cases of the coefficients.

Keywords. Non-local differential equation model; inhomogeneous equation model; general solution.

PACS Nos **02.30.Hq; 02.30.Ik; 02.30.Ks**

1. Introduction

Non-local differential equations have various applications in physical sciences and engineering [1,2]. One popular application of non-local dynamics is pantograph modelling [3]. It has a long history in pantograph mechanics and pantograph transport [4]. In particular, the eidograph was invented in 1821 to improve upon the practical utility of the pantograph [5]. Non-local equation models contain delay differential equation models [6,7]. Such models have been introduced to analyse ultradian oscillations of insulin and glucose [8,9], and to describe physiological control systems regarding dynamical respiratory and hematopoietic diseases [10]. Two concrete examples are discrete delay equation models (see, e.g., [11]),

$$x''(t) = F(t, x(t), x(t-a)),\ a > 0 \tag{1}$$

and pantograph equation models (see, e.g., [12]),

$$x''(t) = F(t, x(t), x(\lambda t)),\ 0 < \lambda < 1, \tag{2}$$

where F is a continuous function.

Recently, there have been abundant studies on non-local integrable equations (see, e.g., [13,14]). This motivates us to consider a class of non-local differential equation models, involving the value of the unknown function at the inverse point of time t. The following equations are among the examples of such equation models of the Newtonian type:

$$x''(t) = F(t, x(t), x(-t)),\ t \in \mathbb{R} \tag{3}$$

and

$$x''(t) = F(t, x(t), x(t^{-1})),\ t > 0, \tag{4}$$

where F is again a continuous function.

In this paper, we would like to solve the non-local inhomogeneous linear differential equation model of the Newtonian type:

$$x''(t) = \lambda x(t) + \mu x(-t) + f(t),\ t \in \mathbb{R}, \tag{5}$$

where λ and μ are arbitrary real constants and f is a given continuous function. The left-hand side of the model is the acceleration, while the right-hand side is

Published online: 03 May 2024

a force acting on entangled particles. This is the simplest example of non-local linear models of the above Newtonian-type. Obviously, this non-local model possesses a linear superposition principle, which exhibits its solution structure. We will determine a general solution, which contains two arbitrary constants, to model (5) in nine cases of the coefficients λ and μ. The results also show that the dimension of the solution space of the corresponding homogeneous counterpart non-local model is two. The conclusion is given in the last section.

2. General solution to the non-local model

We solve the non-local inhomogeneous differential equation model of the Newtonian-type:

$$x''(t) = \lambda x(t) + \mu x(-t) + f(t),\ t \in \mathbb{R}, \tag{6}$$

where λ and μ are arbitrary real constants and f is a continuous function on $\mathbb{R}$. Let us make the even and odd function decompositions

$$x(t) = y(t) + z(t), \qquad f(t) = g(t) + h(t), \tag{7}$$

where y and g are even functions, and z and h are odd functions. Actually, we have

$$\begin{cases} y(t) = \frac{1}{2}(x(t) + x(-t)),\ z(t) = \frac{1}{2}(x(t) - x(-t)), \\ g(t)=\frac{1}{2}(f(t)+f(-t)),\ h(t)=\frac{1}{2}(f(t)-f(-t)). \end{cases} \tag{8}$$

Then, by balancing the even and odd functions, the inhomogeneous non-local model (6) becomes

$$y''(t) = (\lambda + \mu)y(t) + g(t) \tag{9}$$

and

$$z''(t) = (\lambda - \mu)z(t) + h(t). \tag{10}$$

This is a local and decoupled model, which can be solved by the traditional approach. The decomposition into even and odd function parts makes it possible to solve the non-local model (6).

We first solve the two resulting local equation models of the Newtonian-type, eqs (9) and (10). The construction process requires us to pay attention to the fact that y is even and z is odd.

Depending on the three cases of

$$\lambda + \mu = 0, \quad \lambda + \mu > 0, \quad \lambda + \mu < 0, \tag{11}$$

by the ODE theory, we have the general solution to the even part local model (9):

$$\begin{cases} y(t) = c + \int_0^t \int_0^s g(r)\mathrm{d}r\mathrm{d}s, \\ y(t) = c\cosh(a_1 t) \\ \qquad +\frac{1}{a_1}\int_0^t \sinh(a_1(t-s))g(s)\mathrm{d}s, \\ y(t) = c\cos(a_2 t) + \frac{1}{a_2}\int_0^t \sin(a_2(t-s))g(s)\mathrm{d}s, \end{cases} \tag{12}$$

respectively, where c is an arbitrary constant and

$$a_1 = \sqrt{\lambda + \mu}, \qquad a_2 = \sqrt{-\lambda - \mu}. \tag{13}$$

Similarly, depending on the three cases of

$$\lambda - \mu = 0, \qquad \lambda - \mu > 0, \qquad \lambda - \mu < 0, \tag{14}$$

by the ODE theory, we have the general solution to the odd part local model (10):

$$\begin{cases} z(t) = dt + \int_0^t \int_0^s h(r)\mathrm{d}r\mathrm{d}s, \\ z(t) = d\cosh(b_1 t) \\ \qquad +\frac{1}{b_1}\int_0^t \sinh(b_1(t-s))h(s)\mathrm{d}s, \\ z(t) = d\cos(b_2 t) + \frac{1}{b_2}\int_0^t \sin(b_2(t-s))h(s)\mathrm{d}s, \end{cases} \tag{15}$$

respectively, where d is an arbitrary constant and

$$b_1 = \sqrt{\lambda - \mu}, \qquad b_2 = \sqrt{\mu - \lambda}. \tag{16}$$

Now, we can present the general solution to the inhomogeneous non-local model (6) of the Newtonian-type in the following nine cases of the coefficients.

Case 1.1. $\lambda + \mu = 0,\ \lambda - \mu = 0$: This means that

$$\lambda = \mu = 0. \tag{17}$$

The general solution reads as

$$\begin{aligned} x(t) &= c + \int_0^t \int_0^s g(r)\mathrm{d}r\mathrm{d}s + dt + \int_0^t \int_0^s h(r)\mathrm{d}r\mathrm{d}s \\ &= c + dt + \int_0^t \int_0^s f(r)\mathrm{d}r\mathrm{d}s, \end{aligned} \tag{18}$$

where c and d are arbitrary constants.

Case 1.2. $\lambda + \mu = 0,\ \lambda - \mu > 0$: This exactly tells that

$$\lambda = -\mu,\ \mu < 0. \tag{19}$$

The general solution is given by

$$\begin{aligned} x(t) = c &+ \int_0^t \int_0^s g(r)\mathrm{d}r\mathrm{d}s + d\sinh(b_1 t) \\ &+\frac{1}{b_1}\int_0^t \sinh(b_1(t-s))h(s)\mathrm{d}s, \end{aligned} \tag{20}$$

where $b_1 = \sqrt{\lambda - \mu} = \sqrt{2\lambda}$, and c and d are arbitrary constants.

Case 1.3. $\lambda + \mu = 0,\ \lambda < \mu$: This means that

$$\lambda = -\mu,\ \mu > 0. \tag{21}$$

The general solution reads as

$$x(t) = c + \int_0^t \int_0^s g(r)\mathrm{d}r\mathrm{d}s + d\sin(b_2 t) + \frac{1}{b_2}\int_0^t \sin(b_2(t-s))h(s)\mathrm{d}s, \tag{22}$$

where $b_2 = \sqrt{\mu - \lambda} = \sqrt{2\mu}$, and c and d are arbitrary constants.

Case 2.1. $\lambda + \mu > 0,\ \lambda - \mu = 0$: This means that

$$\lambda = \mu,\ \mu > 0. \tag{23}$$

The general solution reads as

$$x(t) = c\cosh(a_1 t) + \frac{1}{a_1}\int_0^t \sinh(a_1(t-s))g(s)\mathrm{d}s + d\,t + \int_0^t \int_0^s h(r)\mathrm{d}r\mathrm{d}s, \tag{24}$$

where $a_1 = \sqrt{\lambda + \mu} = \sqrt{2\mu}$, and c and d are arbitrary constants.

Case 2.2. $\lambda + \mu > 0,\ \lambda - \mu > 0$: This exactly tells that

$$\lambda > \max(\mu, -\mu). \tag{25}$$

The general solution is given by

$$x(t) = c\cosh(a_1 t) + \frac{1}{a_1}\int_0^t \sinh(a_1(t-s))g(s)\mathrm{d}s + d\sinh(b_1 t) + \frac{1}{b_1}\int_0^t \sinh(b_1(t-s))h(s)\mathrm{d}s, \tag{26}$$

where $a_1 = \sqrt{\lambda + \mu},\ b_1 = \sqrt{\lambda - \mu}$, and c and d are arbitrary constants.

Case 2.3. $\lambda + \mu > 0,\ \lambda - \mu < 0$: This means that

$$-\mu < \lambda < \mu, \qquad \mu > 0. \tag{27}$$

The general solution reads as

$$x(t) = c\cosh(a_1 t) + \frac{1}{a_1}\int_0^t \sinh(a_1(t-s))g(s)\mathrm{d}s + d\sin(b_2 t) + \frac{1}{b_2}\int_0^t \sin(b_2(t-s))h(s)\mathrm{d}s, \tag{28}$$

where $a_1 = \sqrt{\lambda + \mu},\ b_2 = \sqrt{\mu - \lambda}$, and c and d are arbitrary constants.

Case 3.1. $\lambda + \mu < 0,\ \lambda - \mu = 0$: This means that

$$\lambda = \mu,\ \lambda < 0. \tag{29}$$

The general solution reads as

$$x(t) = c\cos(a_2 t) + \frac{1}{a_2}\int_0^t \sin(a_2(t-s))g(s)\mathrm{d}s + d\,t + \int_0^t \int_0^s h(r)\mathrm{d}r\mathrm{d}s, \tag{30}$$

where $a_2 = \sqrt{-\lambda - \mu} = \sqrt{-2\lambda}$, and c and d are arbitrary constants.

Case 3.2. $\lambda + \mu < 0,\ \lambda - \mu > 0$: This exactly tells that

$$\mu < \lambda < -\mu,\ \mu < 0. \tag{31}$$

The general solution is given by

$$x(t) = c\cos(a_2 t) + \frac{1}{a_2}\int_0^t \sin(a_2(t-s))g(s)\mathrm{d}s + d\sinh(b_1 t) + \frac{1}{b_1}\int_0^t \sinh(b_1(t-s))h(s)\mathrm{d}s, \tag{32}$$

where $a_2 = \sqrt{-\lambda - \mu},\ b_1 = \sqrt{\lambda - \mu}$, and c and d are arbitrary constants.

Case 3.3. $\lambda + \mu < 0,\ \lambda - \mu < 0$: This means that

$$\lambda < \min(\mu, -\mu). \tag{33}$$

The general solution reads as

$$x(t) = c\cos(a_2 t) + \frac{1}{a_2}\int_0^t \sin(a_2(t-s))g(s)\mathrm{d}s + d\sin(b_2 t) + +\frac{1}{b_2}\int_0^t \sin(b_2(t-s))h(s)\mathrm{d}s, \tag{34}$$

where $a_2 = \sqrt{-\lambda - \mu}$, $b_2 = \sqrt{\mu - \lambda}$, and c and d are arbitrary constants.

Example 2.1. Let us take

$$\lambda = 2, \qquad \mu = 1, \qquad f(t) = \mathrm{e}^t \tag{35}$$

and so the non-local model of the Newtonian-type is

$$x''(t) = 2x(t) + x(-t) + \mathrm{e}^t. \tag{36}$$

This is a special example of *Case 2.2*, for which we have

$$a_1 = \sqrt{3}, \quad b_1 = 1, \quad g(t) = \cosh t, \quad h(t) = \sinh t.$$

Then by (26), the general solution to (36) reads as

$$x(t) = c\cosh(\sqrt{3}\,t) + d\sinh t - \frac{1}{2}\mathrm{e}^t + \frac{1}{2}t\cosh t, \tag{37}$$

where c and d are arbitrary constants.

Example 2.2. Let us take

$$\lambda = 1, \quad \mu = -2, \quad f(t) = \cos t + \sin(2t), \tag{38}$$

and so the non-local model of the Newtonian-type is

$$x''(t) = x(t) - 2x(-t) + \cos t + \sin(2t). \tag{39}$$

This is a particular example of *Case 3.2*, and we have

$$a_2 = 1, \quad b_1 = \sqrt{3}, \quad g(t) = \cos t, \quad h(t) = \sin(2t).$$

Further, based on (32), we arrive at the general solution to (39):

$$x(t) = c\cos t + d\sinh(\sqrt{3}\,t) - \frac{1}{7}\sin(2t) + \frac{1}{2}t\sin t, \tag{40}$$

where c and d are again arbitrary constants.

To summarise, the general solution to the inhomogeneous non-local differential equation model (6) of the Newtonian-type is given by (18), (20), (22), (24), (26), (28), (30), (32) or (34), depending on the nine cases (17), (19), (21), (23), (25), (27), (29), (31) and (33) of the two coefficients λ and μ, respectively. The solution contains two arbitrary constants, c and d, and thus, the dimension of the solution space of the corresponding homogeneous counterpart non-local model is two.

3. Concluding remarks

We have presented the general solution to a non-local inhomogeneous linear differential equation model of the Newtonian-type in (5). The solution involves two arbitrary constants, and thus, the dimension of the solution space of the corresponding homogeneous counterpart non-local model is two. The success is to use the decomposition of functions into their even and odd parts to remove non-locality. By such an idea, one transforms the non-local model into a local and decoupled system to solve.

It is worth pointing out that there is a very different situation on the existence and uniqueness of solutions for Cauchy problems in the non-local case. Let us show this by considering a specific Cauchy problem:

$$\begin{cases} x''(t) = -\mu x(t) + \mu x(-t), \ t \in \mathbb{R}, \\ x(t_0) = x_0, \qquad x'(t_0) = x_0', \end{cases} \tag{41}$$

where $\mu > 0$ and $t_0, x_0, x_0' \in \mathbb{R}$. Based on the general solution (22), we can easily observe that upon setting $b_2 = \sqrt{2\mu}$, if $\cos(b_2 t_0) \neq 0$, then there is a unique solution:

$$x(t) = x_0 - \frac{x_0'}{b_2}\tan(b_2 t_0) + \frac{x_0'}{b_2\cos(b_2 t_0)}\sin(b_2 t), \tag{42}$$

and if $\cos(b_2 t_0) = 0$, then $x'(t_0) = 0$, and thus, there is no solution when $x_0' \neq 0$ and there are infinitely many solutions when $x_0' = 0$:

$$x(t) = x_0 - d\sin(b_2 t_0) + d\sin(b_2 t), \tag{43}$$

where d is an arbitrary constant.

There is another similar type of non-local linear differential equation model of the Newtonian-type:

$$x''(t) = \lambda x(t) + \mu x(t^{-1}) + f(t), \quad t > 0, \tag{44}$$

where λ and μ are arbitrary real constants, and f is a continuous function. The coordinate t^{-1} is the inverse of t with respect to the group operation – the multiplication, while the coordinate $-t$ in the previous model (5) is the inverse of t with respect to the other group operation – the addition. The above non-local model should be more difficult to deal with. We expect that one day, one can present some effective way to solve it.

There exist various non-local integrable partial differential equations, which are formulated through conducting one group reduction (see, e.g., [14,15]) and two group reductions (see, e.g., [16,17]) of matrix spectral problems. Riemann–Hilbert problems and soliton solutions have been presented for non-local integrable nonlinear Schrödinger equations (see, e.g., [17,18]) and non-local integrable modified Korteweg–de Vries equations (see, e.g., [16,19]). Illustrative examples include

$$iu_t = u_{xx} \pm [uu(x,-t) + u(-x,t)u(-x,-t)]u, \tag{45}$$

$$iu_t = u_{xx} \pm [uu^*(-x,t) + u(-x,-t)u^*(x,-t)]u \tag{46}$$

and

$$iu_t = u_{xx} \pm [uu^*(-x,t) + u(x,-t)u^*(-x,-t)]u \tag{47}$$

and

$$u_t = u_{xxx} \pm 3[2uu^*(-x,-t)u_x + u(-x,-t)(|u|^2)_x], \tag{48}$$

$$u_t = u_{xxx} \pm 3[2uu(-x,-t)u_x + u^*(-x,-t)(|u|^2)_x], \tag{49}$$

where u^* is the complex conjugate of u. By the same idea of decomposing functions into their even and odd parts, one can study those non-local integrable equations like local equations, supplementing analytical approaches to local soliton-type solutions (see, e.g., [20–23]) and non-local soliton-type solutions (see, e.g., [24–26]).

Acknowledgements

The work was supported in part by the National Natural Science Foundation of China (12271488, 11975145 and 11972291), the Ministry of Science and Technology of China (G2021016032L and G2023016011L) and the Natural Science Foundation for Colleges and Universities in Jiangsu Province (17 KJB 110020). The author would also like to thank Arthur Danielyan and Sherwin Kouchekian for their valuable discussions.

References

[1] Y Kuang, Delay differential equations with applications in population dynamics, in: *Mathematics in Science and Engineering* (Academic Press, Boston, MA, 1993) Vol. 191
[2] R Jean-Pierre, *Automatica* **39**, 1667 (2003)
[3] A Shapira and M Tyomkyn, *Am. Math. Month.* **128**, 630 (2021)
[4] *Sci. Amer.* **73**, 87 (1895)
[5] W Wallace, *Trans. R. Soc. Edinb.* **13**, 418 (1836)
[6] R D Driver, Ordinary and delay differential equations, in: *Applied Mathematical Sciences* (Springer-Verlag, New York, 1977) Vol. 20
[7] T Erneux, Applied delay differential equations, in: *Surveys and Tutorials in the Applied Mathematical Sciences* (Springer, New York, 2009) Vol. 3
[8] J Sturis, K S Polonsky, E Mosekilde and E Van Cauter, *Am. J. Physiol. Endocrinol. Metab.* **260**, E801 (1991)
[9] A Makroglou, J X Li and Y Kuang, *Appl. Numer. Math.* **56**, 559 (2006)
[10] M C Mackey and L Glass, *Science* **197**, 287 (1977)
[11] K Engelborghs, V Lemaire, J Bélair and D Roose, *J. Math. Biol.* **42**, 361 (2001)
[12] T Kato and J B McLeod, *Bull. Am. Math. Soc.* **77**, 891 (1971)
[13] M J Ablowitz and Z H Musslimani, *Phys. Rev. Lett.* **110**, 064105 (2013)
[14] W X Ma, *Partial Differ. Equ. Appl. Math.* **4**, 100190 (2021)
[15] W X Ma, Y H Huang and F D Wang, *Eur. J. Appl. Math.* **33**, 1062 (2022)
[16] W X Ma, *Appl. Math. Lett.* **131**, 108074 (2022)
[17] W X Ma, *Int. J. Geom. Meth. Mod. Phys.* **20**, 2350098 (2023)
[18] W X Ma, *Rep. Math. Phys.* **92**, 19 (2023)
[19] W X Ma, *J. Geom. Phys.* **177**, 104522 (2022)
[20] R Hirota, The direct method in soliton theory, in: *Cambridge Tracts in Mathematics*, Series No. 155 (Cambridge University Press, New York, 2004)
[21] A Yokuş, M E Aydın, S Duran and H Durur, *Mod. Phys. Lett. B* **36**, 2250110 (2022)
[22] S Duran, A Yokus and G Kilinc, *Phys. Scr.* **98**, 125265 (2023)
[23] W X Ma, *Mathematics* **11**, 4664 (2023)
[24] J L Ji and Z N Zhu, *J. Math. Anal. Appl.* **453**, 973 (2017)
[25] M Gürses and A Pekcan, *Commun. Nonlinear Sci. Numer. Simul.* **67**, 427 (2019)
[26] W X Ma, *Chaos Solitons Fractals* **180**, 114539 (2024)