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A MULTI-COMPONENT LAX INTEGRABLE HIERARCHY WITH HAMILTONIAN STRUCTURE

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Abstract

A Lax integrable multi-component hierarchy is generated from a matrix spectral problem involving two arbitrary matrices, within the the framework of zero curvature equations. Its Hamiltonian structure is established be means of the trace variational identity. An example with five components is presented, together with the first two Hamiltonian systems in the resulting soliton hierarchy.

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1. Introduction

It is important to explore integrability [1] for nonlinear differential and/or difference equations. One of the problems contributing to the theory is to search for completely integrable differential and/or difference equations and to accumulate characteristics of integrability. To the end, we would here like to present a Lax integrable multi-component hierarchy from a matrix spectral problem involving two arbitrary matrices, within the the framework of zero curvature equations.

It is known that the multiplicity of a given system of differential and/or difference equations causes difficulties in determining its integrable properties. Our result will provide a useful supplement to existing integrable theories of scalar differential and/or equations, of which a few criteria such as the inverse scattering transform, the Painlevé test, Bäcklund transformation are available for testing integrability [2, 3].

The paper is structured as follows. In Section 2., starting from a new higher-order matrix spectral problem, the associated Lax integrable equations of multi-components are furnished. In Section 3., one representative hierarchy of the resulting Lax integrable equations is presented and shown to possess Hamiltonian structure based on the trace variational identity. Moreover, a concrete example with five components is computed, together with two first Hamiltonian systems in the resulting Lax integrable hierarchy. A few concluding remarks are given in Section 4..

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2. Lax Pairs and Associated Zero Curvature Equations

We consider a higher-order matrix spectral problem involving two arbitrary matrices:

$$\phi_x = U\phi = U(u,\lambda)\phi,\ U = \left[\begin{array}{cc} \lambda J_2 & q \\ -q^T & r \end{array}\right],\ J_2 = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right], \tag{2.1}$$

where λ is a spectral parameter, r is a skew-symmetric (i.e., $r^T = -r$) matrix and $u = p(q,r)$ is a vector potential. Here p is an arrangement of variables in two arbitrary matrices q and r into a vector.

To derive associated Lax integrable equations, let us first solve the stationary zero curvature equation

$$V_x = [U, V] \tag{2.2}$$

of the spectral problem (2.1). We try a solution V of the following type:

$$V = \left[\begin{array}{cc} aJ_2 & b \\ -b^T & c \end{array}\right], \tag{2.3}$$

where a is scalar and $c^T = -c$. Therefore, based on

$$[U, V] = \left[\begin{array}{cc} -qb^T + bq^T & \lambda J_2 b - aJ_2 q + qc - br \\ \lambda b^T J_2 - aq^T J_2 - rb^T + cq^T & -q^T b + b^T q + [r, c] \end{array}\right],$$

the stationary zero curvature equation (2.2) becomes

$$\left\{\begin{array}{l} a_x J_2 = bq^T - qb^T, \\ b_x = \lambda J_2 b - aJ_2 q + qc - br, \\ c_x = b^T q - q^T b + [r, c]. \end{array}\right. \tag{2.4}$$

As usual, we seek and specify a formal solution as follows:

$$V = \left[\begin{array}{cc} aJ_2 & b \\ -b^T & c \end{array}\right] = \sum_{k=0}^{\infty} V_k \lambda^{-k},\ V_k = \left[\begin{array}{cc} a_k J_2 & b_k \\ -(b_k)^T & c_k \end{array}\right], \tag{2.5}$$

where $(c_k)^T = -c_k$, $k \geq 0$. Then, the three equations in (2.4) recursively define all $a_k,\ b_k$ and c_k, $k \geq 0$, while the initial values need to satisfy

$$b_0 = 0,\ a_{0,x} = 0,\ c_{0,x} = [r, c_0].$$

From now on, for any integer $n \geq 1$, we take

$$V^{(n)} = (\lambda^n V)_+ + \Delta_n = \sum_{j=0}^{n} V_j \lambda^{n-j} + \Delta_n,\ \Delta_n = \left[\begin{array}{cc} 0 & 0 \\ 0 & \delta_n \end{array}\right], \tag{2.6}$$

δ_n being skew-symmetric and of the same size as c, and introduce the time evolution law for the eigenfunction ϕ:

$$\phi_{t_n} = V^{(n)}\phi = V^{(n)}(u, \lambda)\phi. \tag{2.7}$$

Defining $U_1 = U|_{\lambda=0}$, we can compute that

$$((\lambda^n V)_+)_x - [U, (\lambda^n V)_+] = V_{n,x} - [U_1, V_n] = \begin{bmatrix} 0 & b_{n,x} + a_n J_2 q - q c_n + b_n r \\ * & 0 \end{bmatrix},$$

$$\Delta_{n,x} - [U, \Delta_n] = \Delta_{n,x} - [U_1, \Delta_n] = \begin{bmatrix} 0 & -q\delta_n \\ -\delta_n q^T & \delta_{n,x} - [r, \delta_n] \end{bmatrix},$$

where $n \geq 1$. Note that it is not necessary to compute the $(2,1)$-entry $*$, which actually is the negative of the transpose of the $(1,2)$-entry. It now follows that the compatibility conditions of (2.1) and (2.7) for all $n \geq 1$, i.e., the zero curvature equations

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0,\ n \geq 1,$$

associated with the Lax pairs of U and $V^{(n)}$, $n \geq 1$, engender a Lax integrable hierarchy of multi-component evolution equations

$$q_{t_n} = b_{n,x} + a_n J_2 q - q c_n + b_n r - q\delta_n,\ r_{t_n} = \delta_{n,x} - [r, \delta_n],\ n \geq 1. \tag{2.8}$$

A large number of soliton hierarchies have been presented this way (see, for example, [4]-[9]). In the following section, we would like to single out a soliton hierarchy with Hamiltonian structure among the above multi-component Lax integrable equations.

3. Hamiltonian Soliton Hierarchy

Let us take

$$\delta_n = \alpha c_n,\ n \geq 1, \tag{3.1}$$

where α is a constant. Then, due to (2.4), the Lax integrable hierarchy (2.8) becomes

$$q_{t_n} = b_{n,x} + a_n J_2 q - q c_n + b_n r - \alpha q c_n,\ r_{t_n} = \alpha(b_n^T q - q^T b_n),\ n \geq 1. \tag{3.2}$$

Furthermore, the resulting Lax integrable hierarchy can be written as

$$\begin{bmatrix} q \\ r \end{bmatrix}_{t_n} = J_c(q, r) \begin{bmatrix} b_n \\ c_n \end{bmatrix}, \tag{3.3}$$

where J_c denotes the matrix compact form of the integro-differential operator J depending on q and r:

$$u_{t_n} = (p(q, r))_{t_n} = J(q, r) p(b_{n+1}, c_{n+1}),\ n \geq 1. \tag{3.4}$$

First, it is direct to show that the operator J is Hamiltonian. Namely, J is skew-symmetric and satisfies the Jacobi identity [10]:

$$(A, JB) = -(JA, B),\ (A, J'[JB]C) + \text{cycle}(A, B, C) = 0, \tag{3.5}$$

where the inner product reads

$$(A,B)=\int \mathrm{tr}(A_1^TB_1+\frac{1}{2}A_2^TB_2)\,dx,$$

J' denotes the Gateaux derivative

$$J'[v]=\frac{\partial}{\partial\varepsilon}J(u+\varepsilon v)|_{\varepsilon=0},$$

and $A=p(A_1,A_2)$, $B=p(B_1,B_2)$ and $C=p(C_1,C_2)$ are arbitrary vectors with A_1,B_1,C_1 and A_2,B_2,C_2 are matrices of the same size as q and r, respectively,

Second, let us recall the trace variational identity:

$$\frac{\delta}{\delta u}\int\langle V,\frac{\partial U}{\partial\lambda}\rangle\,dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\langle V,\frac{\partial U}{\partial u}\rangle, \tag{3.6}$$

where $\langle A,B\rangle=\mathrm{tr}(AB)$ is the Killing form and γ is a constant [11]-[13]. In our case, we have

$$\langle V,\frac{\partial U}{\partial q}\rangle=-2b,\ \langle V,\frac{\partial U}{\partial r}\rangle=-2c,\ \langle V,\frac{\partial U}{\partial\lambda}\rangle=-2a.$$

It follows from the trace variational identity that

$$\begin{bmatrix}\frac{\delta}{\delta q}\\ \frac{\delta}{\delta r}\end{bmatrix}\tilde H_n=\begin{bmatrix}b_n\\ c_n\end{bmatrix},\ \tilde H_n=\int\Big(-\frac{a_{n+1}}{n}\Big)\,dx,\ n\ge1. \tag{3.7}$$

This leads to the Hamiltonian structure of the Lax integrable hierarchy (3.2):

$$\begin{bmatrix}q\\ r\end{bmatrix}_{t_n}=J_c(q,r)\begin{bmatrix}\frac{\delta}{\delta q}\\ \frac{\delta}{\delta r}\end{bmatrix}\tilde H_n,\ n\ge1. \tag{3.8}$$

When r is a 2×2 matrix, let us introduce

$$q=\begin{bmatrix}u_1 & u_3\\ u_2 & u_4\end{bmatrix},\ r=\begin{bmatrix}0 & u_5\\ -u_5 & 0\end{bmatrix}, \tag{3.9}$$

$$b_n=\begin{bmatrix}d_n & f_n\\ e_n & g_n\end{bmatrix},\ c_n=\begin{bmatrix}0 & h_n\\ -h_n & 0\end{bmatrix},\ n\ge0, \tag{3.10}$$

where $u_i,\ 1\le i\le5$, is scalar variables, and d_n,e_n,f_n,g_n and h_n are scalar functions. Then, setting $u=(u_1,u_2,u_3,u_4,u_5)^T$, the resulting Lax integrable hierarchy (3.8) can be rewritten as

$$u_{t_n}=\begin{bmatrix}d_{n,x}+u_2a_n-u_5f_n-(\alpha+1)u_3h_n\\ e_{n,x}-u_1a_n-u_5g_n-(\alpha+1)u_4h_n\\ f_{n,x}+u_4a_n+u_5d_n+(\alpha+1)u_1h_n\\ g_{n,x}-u_3a_n+u_5e_n+(\alpha+1)u_2h_n\\ \alpha h_{n,x}\end{bmatrix}=J\begin{bmatrix}d_n\\ e_n\\ f_n\\ g_n\\ h_n\end{bmatrix}=J\frac{\delta\tilde H_n}{\delta u},\ n\ge1, \tag{3.11}$$

where the Hamiltonian operator is given by

$$J = \begin{bmatrix} \partial + u_2\partial^{-1}u_2 & -u_2\partial^{-1}u_1 & u_2\partial^{-1}u_4 - u_5 & -u_2\partial^{-1}u_3 & -(\alpha+1)u_3 \\ -u_1\partial^{-1}u_2 & \partial + u_1\partial^{-1}u_1 & -u_1\partial^{-1}u_4 & u_1\partial^{-1}u_3 - u_5 & -(\alpha+1)u_4 \\ u_4\partial^{-1}u_2 + u_5 & -u_4\partial^{-1}u_1 & \partial + u_4\partial^{-1}u_4 & -u_4\partial^{-1}u_3 & (\alpha+1)u_1 \\ -u_3\partial^{-1}u_2 & u_3\partial^{-1}u_1 + u_5 & -u_3\partial^{-1}u_4 & \partial + u_3\partial^{-1}u_3 & (\alpha+1)u_2 \\ (\alpha+1)u_3 & (\alpha+1)u_4 & -(\alpha+1)u_1 & -(\alpha+1)u_2 & -\partial \end{bmatrix}. \tag{3.12}$$

Taking the initial values as follows

$$a_0 = 1,\ b_0 = 0,\ h_0 = 0,$$

and setting each integration constant to zero, then the first few quantities are

$$\begin{aligned} & a_1 = 0,\ b_1 = q,\ h_1 = 0; \\ & d_2 = -u_{2x} + u_4u_5,\ e_2 = u_{1x} - u_3u_5,\ f_2 = -u_{4x} - u_2u_5,\ g_2 = u_{3x} + u_1u_5, \\ & a_2 = -\frac{1}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2),\ h_2 = u_1u_4 - u_2u_3. \end{aligned}$$

Thus, the first and second nonlinear systems of Hamiltonian equations in the Lax integrable hierarchy (3.8) with $u = (u_1, u_2, u_3, u_4, u_5)^T$ read

$$u_{t_1} = \begin{bmatrix} u_{1x} - u_3u_5, \\ u_{2x} - u_4u_5, \\ u_{3x} + u_1u_5, \\ u_{4x} + u_2u_5, \\ 0 \end{bmatrix} = J\frac{\delta\tilde{H}_1}{\delta u},$$

and

$$u_{t_2} = \begin{bmatrix} -u_{2xx} + 2u_{4x}u_5 + u_4u_{5x} - \frac{1}{2}u_2p_1 - (\alpha+1)u_1u_3u_4 \\ u_{1xx} - 2u_{3x}u_5 - u_3u_{5x} + \frac{1}{2}u_1p_2 + (\alpha+1)u_2u_3u_4 \\ -u_{4xx} - 2u_{2x}u_5 - u_2u_{5x} - \frac{1}{2}u_4p_3 - (\alpha+1)u_1u_2u_3 \\ u_{3xx} + 2u_{1x}u_5 + u_1u_{5x} + \frac{1}{2}u_3p_4 + (\alpha+1)u_1u_2u_4 \\ \alpha(u_1u_4 - u_2u_3)_x \end{bmatrix} = J\frac{\delta\tilde{H}_2}{\delta u},$$

where

$$\begin{cases} p_1 = u_1^2 + u_2^2 - (2\alpha+1)u_3^2 + u_4^2 - 2u_5^2, \\ p_2 = u_1^2 + u_2^2 + u_3^2 - (2\alpha+1)u_4^2 - 2u_5^2, \\ p_3 = -(2\alpha+1)u_1^2 + u_2^2 + u_3^2 + u_4^2 - 2u_5^2, \\ p_4 = u_1^2 - (2\alpha+1)u_2^2 + u_3^2 + u_4^2 - 2u_5^2, \end{cases}$$

the Hamiltonian operator J is defined by (3.12), and the Hamiltonian functionals $\tilde{H}_1$ and $\tilde{H}_2$ are given by

$$\tilde{H}_1 = \int \frac{1}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2)\, dx,$$
$$\tilde{H}_2 = \int [(u_1u_4 - u_2u_3)u_5 - \frac{1}{2}u_1u_{2x} + \frac{1}{2}u_{1x}u_2 - \frac{1}{2}u_3u_{4x} + \frac{1}{2}u_{3x}u_4]\, dx.$$

4. Concluding Remarks

A higher-order matrix spectral problem involving two arbitrary matrices was introduced, and an associated Lax integrable hierarchy were presented through selecting appropriate temporal parts of Lax pairs. The key in the construction is to force the associated zero curvature equations to yield evolution equations. The presented multi-component Lax integrable hierarchy was proved to possess Hamiltonian structure by means of the trace variational identity. An example with five components and its first two nonlinear Hamiltonian systems were computed.

We point out that if we take

$$\delta_n = \alpha c_{n+1},\ n \geq 1,$$

where α is a constant, the Lax pairs in Section 2. leads to another interesting Lax integrable hierarchy of multi-component Hamiltonian equations, and the resulting Hamiltonian structure is local [9]. Note that the Hamiltonian operator J defined by (3.12) is non-local. We also mention that multi-component integrable equations can be generated through other approaches such as perturbations [5, 14], loop algebra methods [15] and enlarging spectral problems based on semi-direct sums of loop algebras [16, 17]. Those approaches provide clues for classifying multi-component integrable equations, in particular, Hamiltonian equations [13, 18].

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