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Soliton hierarchies and soliton solutions of type $(-\lambda^*, -\lambda)$ reduced nonlocal nonlinear Schrödinger equations of arbitrary even order

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ABSTRACT

We present mixed-type reduced soliton hierarchies of nonlocal integrable nonlinear Schrödinger equations of arbitrary even order by conducting two nonlocal group reductions for the Ablowitz–Kaup–Newell–Segur matrix spectral problems. Based on specific distributions of eigenvalues and adjoint eigenvalues, we construct soliton solutions by solving the corresponding reflectionless generalized Riemann–Hilbert problems, where eigenvalues could equal adjoint eigenvalues.

1. Introduction

Nonlinear integrable equations are often generated via zero curvature equations and their Hamiltonian structures can be presented by the trace identity[1] or the variational identity,[2] which produce infinitely many symmetries and conservation laws. Based on matrix spectral problems, with which zero curvature equations are associated, the inverse scattering transform solves Cauchy problems of integrable equations. By conducting group reductions for matrix spectral problems, which keep the zero curvature equations invariant, one can obtain both local and nonlocal reduced integrable equations.

Nonlocal integrable equations have formed a new research area, supplementing the classical theory of partial differential equations. By taking one nonlocal group reduction, three kinds of nonlocal nonlinear Schrödinger (NLS) equations and two kinds of nonlocal modified Korteweg–de Vries (mKdV) equations can be generated from the Ablowitz–Kaup–Newell–Segur (AKNS) matrix spectral problems.[3,4] The inverse scattering transform has been successfully applied to analysis of soliton solutions to nonlocal integrable equations (see, e.g., Refs. 5–8).

Integrable equations can also be solved by other efficient approaches, which include Darboux transformation, the Hirota bilinear method and Riemann–Hilbert problems, and their soliton solutions can be systematically presented, indeed (see, e.g., Refs. 9–14). Particularly, the Riemann–Hilbert technique is used to solve nonlocal integrable NLS and mKdV equations.[4,15–18] In this paper, we would like to present a kind of mixed-type reduced nonlocal integrable NLS equations of arbitrary even order by conducting two nonlocal group reductions and compute their soliton solutions through reflectionless generalized Riemann–Hilbert problems.

The rest of this paper is organized as follows. In Section 2, we recall the AKNS hierarchies of integrable equations and their matrix spectral problems to facilitate the exposition. In Section 3, we conduct two nonlocal group reductions and present type $(-\lambda^*, -\lambda)$ reduced nonlocal integrable NLS hierarchies, where λ is the spectral parameter and $*$ stands for the complex conjugate. Two scalar prototype examples of the resulting nonlocal integrable equations are

$$p_{1,t} = -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\sigma(p_1 p_1^*(-x,t) + p_1(x,-t)p_1^*(-x,-t))p_1],$$

and

$$p_{1,t} = -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\delta(p_1 p_1(x,-t) + p_1^*(-x,t)p_1^*(-x,-t))p_1],$$

where $\sigma = \pm 1$, $\delta = \pm 1$, and α and β are arbitrary real constants. Both pairs of equations are obviously PT-symmetric. In Section 4, based on the explored distribution of eigenvalues and adjoint eigenvalues, we solve the corresponding reflectionless generalized Riemann–Hilbert problems, where eigenvalues could equal adjoint eigenvalues, and compute soliton solutions to the resulting hierarchies of reduced nonlocal integrable NLS equations of arbitrary even order. In the last section, we give a conclusion and a few concluding remarks.

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2. The matrix AKNS integrable hierarchies revisited

To facilitate the subsequent exposition, let us recall the AKNS hierarchies of matrix integrable equations and their matrix spectral problems.

First, let λ denote the spectral parameter, and p and q be two matrix potentials:

$$p = p(x,t) = (p_{jk})_{m\times n},\ \ q = q(x,t) = (q_{kj})_{n\times m}, \tag{2.1}$$

where $m, n \geq 1$ are two arbitrarily given integers. The matrix AKNS spectral problems are defined as follows:

$$\begin{cases} -i\phi_x = U\phi = U(u,\lambda)\phi = (\lambda\Lambda + P)\phi, \\ -i\phi_t = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi = (\lambda^r\Omega + Q^{[r]})\phi,\ \ r \geq 0. \end{cases} \tag{2.2}$$

Here the pair of the $(m+n)$-th order square matrices, Λ and Ω, is given by

$$\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n),\ \ \Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \tag{2.3}$$

where I_s denotes the identity matrix of size s, and α_1, α_2 and β_1, β_2 are two pairs of arbitrarily given distinct real constants. The other pair of $(m+n)$-th order square matrices, P and $Q^{[r]}$, is determined by

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.4}$$

which is called the potential matrix, and

$$Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \tag{2.5}$$

where $a^{[s]}, b^{[s]}, c^{[s]}$ and $d^{[s]}$ are defined recursively by

$$b^{[0]} = 0,\ \ c^{[0]} = 0,\ \ a^{[0]} = \beta_1 I_m,\ \ d^{[0]} = \beta_2 I_n, \tag{2.6a}$$

$$b^{[s+1]} = \frac{1}{\alpha}(-ib^{[s]}_x - pd^{[s]} + a^{[s]}p),\ \ s \geq 0, \tag{2.6b}$$

$$c^{[s+1]} = \frac{1}{\alpha}(ic^{[s]}_x + qa^{[s]} - d^{[s]}q),\ \ s \geq 0, \tag{2.6c}$$

$$a^{[s]}_x = i(pc^{[s]} - b^{[s]}q),\ \ d^{[s]}_x = i(qb^{[s]} - c^{[s]}p),\ \ s \geq 1, \tag{2.6d}$$

with zero constants of integration being taken. Particularly, we can work out

$$Q^{[1]} = \frac{\beta}{\alpha}P,\ \ Q^{[2]} = \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2}I_{m,n}(P^2 + iP_x),$$

and

$$Q^{[3]} = \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P,P_x] + P_{xx} + 2P^3),$$

where $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ and $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. Based on the recursive relations in (2.6), we can also see that

$$W = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{2.7}$$

presents a Laurent series solution to the stationary zero curvature equation

$$W_x = i[U, W]. \tag{2.8}$$

The compatibility conditions of the two matrix spectral problems in (2.2), i.e., the zero curvature equations:

$$U_t - V^{[r]}_x + i[U, V^{[r]}] = 0,\ \ r \geq 0, \tag{2.9}$$

yield one matrix AKNS integrable hierarchy (see, e.g., Ref. 19 for more details):

$$p_t = i\alpha b^{[r+1]},\ \ q_t = -i\alpha c^{[r+1]},\ \ r \geq 0. \tag{2.10}$$

By a Lax operator algebra theory[20,21] and the trace identity,[1] we can directly show that the hierarchy (2.10) defines a hierarchy of commuting flows, each of which possesses a bi-Hamiltonian structure and thus infinitely many commuting conservation laws. The first nonlinear (i.e., $r = 2$) integrable system in the hierarchy gives us the AKNS matrix NLS equations:

$$p_t = -\frac{\beta}{\alpha^2}i(p_{xx} + 2pqp),\ \ q_t = \frac{\beta}{\alpha^2}i(q_{xx} + 2qpq), \tag{2.11}$$

where p and q are the two matrix potentials defined by (2.1).

3. Type $(-\lambda^*, -\lambda)$ reduced nonlocal NLS hierarchies

Let Σ_1 and Σ_2 be a pair of constant invertible Hermitian matrices of sizes m and n, respectively, and Δ_1 and Δ_2, another pair of constant invertible symmetric matrices of sizes m and n, respectively. We consider a pair of nonlocal group reductions for the spectral matrix U:

$$U^\dagger(-x,t,-\lambda^*) = (U(-x,t,-\lambda^*))^\dagger = -\Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{3.12}$$

and

$$U^T(x,-t,-\lambda) = (U(x,-t,-\lambda))^T = -\Delta U(x,t,\lambda)\Delta^{-1}, \tag{3.13}$$

where $\dagger$ and T denote the Hermitian transpose and the matrix transpose, respectively, and Σ and Δ are the two constant invertible matrices defined by

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},\ \ \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{3.14}$$

Equivalently, these two group reductions require

$$P^\dagger(-x,t) = -\Sigma P(x,t)\Sigma^{-1}, \tag{3.15}$$

and

$$P^T(x,-t) = -\Delta P(x,t)\Delta^{-1}, \tag{3.16}$$

respectively. More precisely, they need the following reductions on the matrix potentials p and q:

$$q(x,t) = -\Sigma_2^{-1}p^\dagger(-x,t)\Sigma_1, \tag{3.17}$$

and

$$q(x,t) = -\Delta_2^{-1}p^T(x,-t)\Delta_1, \tag{3.18}$$

respectively. It therefore follows that the matrix potential p must satisfy

$$\Sigma_2^{-1}p^\dagger(-x,t)\Sigma_1 = \Delta_2^{-1}p^T(x,-t)\Delta_1, \tag{3.19}$$

or the matrix potential q must satisfy

$$\Sigma_1^{-1}q^\dagger(-x,t)\Sigma_2 = \Delta_1^{-1}q^T(x,-t)\Delta_2, \tag{3.20}$$

to guarantee that both group reductions in (3.12) and (3.13) are compatible.

Furthermore, under the group reductions in (3.12) and (3.13), we can show that

$$\begin{cases} W^\dagger(-x,t,-\lambda^*) = (W(-x,t,-\lambda^*))^\dagger = \Sigma W(x,t,\lambda)\Sigma^{-1}, \\ W^T(x,-t,-\lambda) = (W(x,-t,-\lambda))^T = \Delta W(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{3.21}$$

which implies that

$$\begin{cases} V^{[2s]\dagger}(-x,t,-\lambda^*) = (V^{[2s]}(-x,t,-\lambda^*))^\dagger = \Sigma V^{[2s]}(x,t,\lambda)\Sigma^{-1}, \\ V^{[2s]T}(x,-t,-\lambda) = (V^{[2s]}(x,-t,-\lambda))^T = \Delta V^{[2s]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{3.22}$$

and

$$\begin{cases} Q^{[2s]T}(-x,t,-\lambda^*) = (Q^{[2s]}(-x,t,-\lambda^*))^\dagger = \Sigma Q^{[2s]}(x,t,\lambda)\Sigma^{-1}, \\ Q^{[2s]T}(x,-t,-\lambda) = (Q^{[2s]}(x,-t,-\lambda))^T = \Delta Q^{[2s]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{3.23}$$

where $s \geq 0$.

Consequently, under the potential reductions (3.17) and (3.18), the integrable matrix AKNS equations in (2.10) with $r = 2s$, $s \geq 0$, are reduced to a hierarchy of nonlocal integrable NLS type equations:

$$p_t = i\alpha b^{[2s+1]}\big|_{q=-\Sigma_2^{-1}p^\dagger(-x,t)\Sigma_1 = -\Delta_2^{-1}p^T(x,-t)\Delta_1},\ \ s \geq 0, \tag{3.24}$$

where p is an $m\times n$ reduced matrix potential satisfying (3.19), Σ_1 and Σ_2 is a pair of arbitrary invertible Hermitian matrices of sizes m and

n, respectively, and Δ_1 and Δ_2 are a pair of arbitrary invertible symmetric matrices of sizes m and n, respectively. As consequences of the two group reductions, each reduced equation in the hierarchy (3.24) possesses a Lax pair of the reduced spatial and temporal matrix spectral problems in (2.2) with $r = 2s,\ s \geq 0$, and infinitely many symmetries and conservation laws reduced from those for the integrable matrix AKNS equations in (2.10) with $r = 2s, s \geq 0$.

If we fix $s = 1$, i.e., $r = 2$, then the reduced nonlocal integrable NLS type equations in (3.24) with $s = 1$ produce a kind of reduced nonlocal integrable NLS equations:

$$\begin{aligned} p_t &= -\frac{\beta}{\alpha^2} i(p_{xx} - 2p\Sigma_2^{-1} p^{\dagger}(-x,t)\Sigma_1 p) \\ &= -\frac{\beta}{\alpha^2} i(p_{xx} - 2p\Delta_2^{-1} p^{T}(x,-t)\Delta_1 p), \end{aligned} \tag{3.25}$$

where p is an $m \times n$ reduced matrix potential satisfying (3.19).

Let us now work out some examples to illustrate these reduced nonlocal integrable NLS equations, by taking different values for m, n and appropriate choices for Σ, Δ. In our subsequent construction, we will use two 2×2 matrices:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \Pi_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.26}$$

Let us first consider the case of $m = 1$ and $n = 2$. We take

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \sigma I_2,\ \Delta_1 = 1,\ \Delta_2^{-1} = \delta \Pi_2, \tag{3.27}$$

where σ and δ are real constants satisfying $\sigma^2 = \delta^2 = 1$. Then, the potential constraint (3.19) equivalently needs

$$p_2 = \sigma\delta p_1^*(-x,-t),$$

where $p = (p_1, p_2)$, and thus, the corresponding potential matrix P becomes

$$P = \begin{bmatrix} 0 & p_1 & \sigma\delta p_1^*(-x,-t) \\ -\sigma p_1^*(-x,t) & 0 & 0 \\ -\delta p_1(x,-t) & 0 & 0 \end{bmatrix}. \tag{3.28}$$

Furthermore, the corresponding reduced nonlocal integrable NLS equations become

$$p_{1,t} = -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\sigma(p_1 p_1^*(-x,t) + p_1(x,-t)p_1^*(-x,-t))p_1], \tag{3.29}$$

where $\sigma = \pm 1$ and p_1^* denotes the complex conjugate of p_1. In this pair of equations, there are three types of nonlinearities: reverse-space, reverse-time and reverse-spacetime nonlocalities.

Similarly, let us take

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \sigma \Pi_2,\ \Delta_1 = 1,\ \Delta_2^{-1} = \delta I_2, \tag{3.30}$$

where σ and δ are real constants satisfying $\sigma^2 = \delta^2 = 1$. This choice leads to the reduced potential matrix P:

$$P = \begin{bmatrix} 0 & p_1 & \sigma\delta p_1^*(-x,-t) \\ -\delta p_1(x,-t) & 0 & 0 \\ -\sigma p_1^*(-x,t) & 0 & 0 \end{bmatrix}, \tag{3.31}$$

and the reduced mixed-type nonlocal integrable NLS equations:

$$p_{1,t} = -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\delta(p_1 p_1(x,-t) + p_1^*(-x,t)p_1^*(-x,-t))p_1], \tag{3.32}$$

where $\delta = \pm 1$ and p_1^* denotes the complex conjugate of p_1 again. The mixed-type nonlocality pattern in this pair of equations is different from the one in (3.29).

Let us second consider the case of $m = 1$ and $n = 4$. We take

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \mathrm{diag}(\sigma_1 I_2, \sigma_2 I_2),\ \Delta_1 = 1,\ \Delta_2^{-1} = \mathrm{diag}(\delta_1 \Pi_2, \delta_2 \Pi_2), \tag{3.33}$$

and

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \mathrm{diag}(\sigma_1 \Pi_2, \sigma_2 \Pi_2),\ \Delta_1 = 1,\ \Delta_2^{-1} = \mathrm{diag}(\delta_1 I_2, \delta_2 I_2), \tag{3.34}$$

where σ_j and δ_j are real constants satisfying $\sigma_j^2 = \delta_j^2 = 1,\ j = 1,2$. These choices can produce the reduced potential matrices:

$$P = \begin{bmatrix} 0 & p_1 & \sigma_1\delta_1 p_1^*(-x,-t) & p_3 & \sigma_2\delta_2 p_3^*(-x,-t) \\ -\sigma_1 p_1^*(-x,t) & 0 & 0 & 0 & 0 \\ -\delta_1 p_1(x,-t) & 0 & 0 & 0 & 0 \\ -\sigma_2 p_3^*(-x,t) & 0 & 0 & 0 & 0 \\ -\delta_2 p_3(x,-t) & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.35}$$

and

$$P = \begin{bmatrix} 0 & p_1 & \sigma_1\delta_1 p_1^*(-x,-t) & p_3 & \sigma_2\delta_2 p_3^*(-x,-t) \\ -\delta_1 p_1(x,-t) & 0 & 0 & 0 & 0 \\ -\sigma_1 p_1^*(-x,t) & 0 & 0 & 0 & 0 \\ -\delta_2 p_3(x,-t) & 0 & 0 & 0 & 0 \\ -\sigma_2 p_3^*(-x,t) & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.36}$$

respectively. The corresponding two classes of two-component mixed-type nonlocal integrable NLS equations read

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\sigma_1(p_1 p_1^*(-x,t) + p_1(x,-t)p_1^*(-x,-t))p_1 \\ \qquad -2\sigma_2(p_3 p_3^*(-x,t) + p_3(x,-t)p_3^*(-x,-t))p_1], \\ p_{3,t} = -\frac{\beta}{\alpha^2} i[p_{3,xx} - 2\sigma_1(p_1 p_1^*(-x,t) + p_1(x,-t)p_1^*(-x,-t))p_3 \\ \qquad -2\sigma_2(p_3 p_3^*(-x,t) + p_3(x,-t)p_3^*(-x,-t))p_3], \end{cases} \tag{3.37}$$

and

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\delta_1(p_1 p_1(x,-t) + p_1^*(-x,t)p_1^*(-x,-t))p_1 \\ \qquad -2\delta_2(p_3 p_3(x,-t) + p_3^*(-x,t)p_3^*(-x,-t))p_1], \\ p_{3,t} = -\frac{\beta}{\alpha^2} i[p_{3,xx} - 2\delta_1(p_1 p_1(x,-t) + p_1^*(-x,t)p_1^*(-x,-t))p_3 \\ \qquad -2\delta_2(p_3 p_3(x,-t) + p_3^*(-x,t)p_3^*(-x,-t))p_3], \end{cases} \tag{3.38}$$

respectively, where σ_j and δ_j are real constants satisfying $\sigma_j^2 = \delta_j^2 = 1,\ j = 1,2$. These are two-component generalizations of the previous scalar examples in (3.29) and (3.32).

Let us third consider the case of $m = 2$ and $n = 2$. We take

$$\Sigma_1 = \sigma_1 \Pi_2,\ \Sigma_2^{-1} = \sigma_2 I_2,\ \Delta_1 = \delta_1 \Pi_2,\ \Delta_2^{-1} = \delta_2 \Pi_2, \tag{3.39}$$

$$\Sigma_1 = \sigma_1 \Pi_2,\ \Sigma_2^{-1} = \sigma_2 \Pi_2,\ \Delta_1 = \delta_1 I_2,\ \Delta_2^{-1} = \delta_2 \Pi_2, \tag{3.40}$$

$$\Sigma_1 = \sigma_1 I_2,\ \Sigma_2^{-1} = \sigma_2 \Pi_2,\ \Delta_1 = \delta_1 \Pi_2,\ \Delta_2^{-1} = \delta_2 \Pi_2, \tag{3.41}$$

and

$$\Sigma_1 = \sigma_1 \Pi_2,\ \Sigma_2^{-1} = \sigma_2 \Pi_2,\ \Delta_1 = \delta_1 \Pi_2,\ \Delta_2^{-1} = \delta_2 I_2, \tag{3.42}$$

where σ_j and δ_j are real constants satisfying $\sigma_j^2 = \delta_j^2 = 1,\ j = 1,2$. These choices can generate the corresponding reduced matrix potentials:

$$p = \begin{bmatrix} p_{11} & \sigma\delta p_{11}^*(-x,-t) \\ p_{21} & \sigma\delta p_{21}^*(-x,-t) \end{bmatrix},\ q = \begin{bmatrix} -\sigma p_{21}^*(-x,t) & -\sigma p_{11}^*(-x,t) \\ -\delta p_{21}(x,-t) & -\delta p_{11}(x,-t) \end{bmatrix}, \tag{3.43}$$

$$p = \begin{bmatrix} p_{11} & p_{12} \\ \sigma\delta p_{11}^*(-x,-t) & \sigma\delta p_{12}^*(-x,-t) \end{bmatrix},\ q = \begin{bmatrix} -\delta p_{12}(x,-t) & -\sigma p_{12}^*(-x,t) \\ -\delta p_{11}(x,-t) & -\sigma p_{11}^*(-x,t) \end{bmatrix}, \tag{3.44}$$

$$p = \begin{bmatrix} p_{11} & p_{12} \\ \sigma\delta p_{11}^*(-x,-t) & \sigma\delta p_{12}^*(-x,-t) \end{bmatrix},\ q = \begin{bmatrix} -\sigma p_{12}^*(-x,t) & -\delta p_{12}(x,-t) \\ -\sigma p_{11}^*(-x,t) & -\delta p_{11}(x,-t) \end{bmatrix}, \tag{3.45}$$

and

$$p = \begin{bmatrix} p_{11} & \sigma\delta p_{11}^*(-x,-t) \\ p_{21} & \sigma\delta p_{21}^*(-x,-t) \end{bmatrix},\ q = \begin{bmatrix} -\delta p_{21}(x,-t) & -\delta p_{11}(x,-t) \\ -\sigma p_{21}^*(-x,t) & -\sigma p_{11}^*(-x,t) \end{bmatrix}, \tag{3.46}$$

respectively, where $\sigma = \sigma_1\sigma_2$ and $\delta = \delta_1\delta_2$. Such formulations on the potential matrices enable us to obtain the following four classes of two-component mixed-type nonlocal integrable NLS equations:

$$\begin{cases} p_{11,t} = -\dfrac{\beta}{\alpha^2} i[p_{11,xx} - 2\sigma(p_{11}p_{21}^*(-x,t) + p_{11}^*(-x,-t)p_{21}(x,-t))p_{11} \\ \quad -2\sigma(p_{11}p_{11}^*(-x,t) + p_{11}^*(-x,-t)p_{11}(x,-t))p_{21}], \\ p_{21,t} = -\dfrac{\beta}{\alpha^2} i[p_{21,xx} - 2\sigma(p_{21}p_{21}^*(-x,t) + p_{21}^*(-x,-t)p_{21}(x,-t))p_{11} \\ \quad -2\sigma(p_{21}p_{11}^*(-x,t) + p_{21}^*(-x,-t)p_{11}(x,-t))p_{21}], \end{cases} \tag{3.47}$$

$$\begin{cases} p_{11,t} = -\dfrac{\beta}{\alpha^2} i[p_{11,xx} - 2\delta(p_{11}p_{12}(x,-t) + p_{12}p_{11}(x,-t))p_{11} \\ \quad -2\delta(p_{11}p_{12}^*(-x,t) + p_{12}p_{11}^*(-x,t))p_{11}^*(-x,-t)], \\ p_{12,t} = -\dfrac{\beta}{\alpha^2} i[p_{12,xx} - 2\delta(p_{11}p_{12}(x,-t) + p_{12}p_{11}(x,-t))p_{12} \\ \quad -2\delta(p_{11}p_{12}^*(-x,t) + p_{12}p_{11}^*(-x,t))p_{12}^*(-x,-t)], \end{cases} \tag{3.48}$$

$$\begin{cases} p_{11,t} = -\dfrac{\beta}{\alpha^2} i[p_{11,xx} - 2\sigma(p_{11}p_{12}^*(-x,t) + p_{12}p_{11}^*(-x,t))p_{11} \\ \quad -2\sigma(p_{11}p_{12}(x,-t) + p_{12}p_{11}(x,-t))p_{11}^*(-x,-t)], \\ p_{12,t} = -\dfrac{\beta}{\alpha^2} i[p_{12,xx} - 2\sigma(p_{11}p_{12}^*(-x,t) + p_{12}p_{11}^*(-x,t))p_{12} \\ \quad -2\sigma(p_{11}p_{12}(x,-t) + p_{12}p_{11}(x,-t))p_{12}^*(-x,-t)], \end{cases} \tag{3.49}$$

and

$$\begin{cases} p_{11,t} = -\dfrac{\beta}{\alpha^2} i[p_{11,xx} - 2\delta(p_{11}p_{21}(x,-t) + p_{11}^*(-x,-t)p_{21}^*(-x,t))p_{11} \\ \quad -2\delta(p_{11}p_{11}(x,-t) + p_{11}^*(-x,-t)p_{11}^*(-x,t))p_{21}], \\ p_{21,t} = -\dfrac{\beta}{\alpha^2} i[p_{21,xx} - 2\delta(p_{21}p_{21}(x,-t) + p_{21}^*(-x,-t)p_{21}^*(-x,t))p_{11} \\ \quad -2\delta(p_{21}p_{11}(x,-t) + p_{21}^*(-x,-t)p_{11}^*(-x,t))p_{21}], \end{cases} \tag{3.50}$$

respectively, where $\sigma = \sigma_1\sigma_2 = \pm 1$ and $\delta = \delta_1\delta_2 = \pm 1$. Obviously, the nonlinearity patterns in the above four equations are different from the ones in (3.37) and (3.38).

4. Soliton solutions

4.1. Distribution of eigenvalues and adjoint eigenvalues

Under the group reduction in (3.12) (or (3.13)), we can observe that λ is an eigenvalue of the matrix spectral problems in (2.2) if and only if $\hat{\lambda} = -\lambda^*$ (or $\hat{\lambda} = -\lambda$) is an adjoint eigenvalue, namely, the adjoint matrix spectral problems hold:

$$i\tilde{\phi}_x = \tilde{\phi}U = \tilde{\phi}U(u,\hat{\lambda}),\ i\tilde{\phi}_t = \tilde{\phi}V^{[2s]} = \tilde{\phi}V^{[2s]}(u,\hat{\lambda}), \tag{4.51}$$

where $s \geq 0$. As a consequence, we can assume to have eigenvalues $\lambda:\ \mu,\ \mu^*,\ \nu$, and adjoint eigenvalues $\hat{\lambda}:\ -\mu^*,\ -\mu,\ -\nu$ where $\mu \notin \mathbb{R}$ and $\nu \in \mathbb{R}$.

Moreover, under the group reductions in (3.12) and (3.13), we can see that

$$\phi^\dagger(-x,t,-\lambda^*)\Sigma \text{ and } \phi^T(x,-t,-\lambda)\Delta, \tag{4.52}$$

will be two adjoint eigenfunctions associated with the same original eigenvalue λ, as long as $\phi(\lambda)$ is an eigenfunction of the matrix spectral problems in (2.2) associated with an eigenvalue λ.

4.2. Solitons by generalized Riemann-Hilbert problems

We would like to propose a general formulation of soliton solutions to the resulting mixed-type nonlocal integrable NLS equations by solving the corresponding reflectionless generalized Riemann–Hilbert problems (see, e.g., Refs. 19, 22, 23 for applications to local integrable equations). Let $N_1, N_2 \geq 0$ be two integers such that $N = 2N_1 + N_2 \geq 1$.

First, let us take N eigenvalues λ_k and N adjoint eigenvalues $\hat{\lambda}_k$ as follows:

$$\lambda_k,\ 1 \leq k \leq N:\ \mu_1,\ \cdots,\ \mu_{N_1},\ \mu_1^*,\ \cdots,\ \mu_{N_1}^*,\ \nu_1,\ \cdots,\ \nu_{N_2}, \tag{4.53}$$

and

$$\hat{\lambda}_k,\ 1 \leq k \leq N:\ -\mu_1^*,\ \cdots,\ -\mu_{N_1}^*,\ -\mu_1,\ \cdots,\ -\mu_{N_1},\ -\nu_1,\ \cdots,\ -\nu_{N_2}, \tag{4.54}$$

where $\mu_k \notin \mathbb{R},\ 1 \leq k \leq N_1$, and $\nu_k \in \mathbb{R},\ 1 \leq k \leq N_2$, and assume that their corresponding eigenfunctions and adjoint eigenfunctions are defined by

$$v_k,\ 1 \leq k \leq N, \text{ and } \hat{v}_k,\ 1 \leq k \leq N, \tag{4.55}$$

respectively. Obviously, in this nonlocal case, the following condition:

$$\{\lambda_k \mid 1 \leq k \leq N\} \cap \{\hat{\lambda}_k \mid 1 \leq k \leq N\} = \emptyset, \tag{4.56}$$

does not hold.

Next, we introduce two matrices:

$$G^+(\lambda) = I_{m+n} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l},\ (G^-)^{-1}(\lambda) = I_{m+n} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}, \tag{4.57}$$

where M is a square matrix $M = (m_{kl})_{N\times N}$, whose entries are defined by

$$m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, & \text{if } \lambda_l \neq \hat{\lambda}_k, \\ 0, & \text{if } \lambda_l = \hat{\lambda}_k, \end{cases} \quad \text{where } 1 \leq k,l \leq N. \tag{4.58}$$

It has been shown in Ref. 16 that these two matrices $G^+(\lambda)$ and $G^-(\lambda)$ solve the corresponding reflectionless generalized Riemann–Hilbert problem, i.e., they satisfy

$$(G^-)^{-1}(\lambda)G^+(\lambda) = I_{m+n},\ \lambda \in \mathbb{R}, \tag{4.59}$$

provided that an orthogonal condition:

$$\hat{v}_k v_l = 0 \text{ if } \lambda_l = \hat{\lambda}_k,\ \text{where } 1 \leq k,l \leq N, \tag{4.60}$$

holds.

Now, let us make an asymptotic expansion

$$G^+(\lambda) = I_{m+n} + \frac{1}{\lambda}G_1^+ + \mathrm{O}(\frac{1}{\lambda^2}), \tag{4.61}$$

as $\lambda \to \infty$, to obtain

$$G_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{4.62}$$

and substituting this into the matrix spatial spectral problems in (2.2) leads to

$$P = -[\Lambda, G_1^+] = \lim_{\lambda\to\infty} [G^+(\lambda), \Lambda]. \tag{4.63}$$

Obviously, this generates soliton solutions to the matrix AKNS integrable Eqs. (2.10):

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2,\ q = -\alpha \sum_{k,l=1}^{N} v_k^2 (M^{-1})_{kl} \hat{v}_l^1, \tag{4.64}$$

where for each $1 \leq k \leq N$, we have split v_k and $\hat{v}_k$ into $v_k = ((v_k^1)^T, (v_k^2)^T)^T$ and $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2)$, where v_k^1 and $\hat{v}_k^1$ are column and row vectors of dimension m, respectively, and v_k^2 and $\hat{v}_k^2$ are column and row vectors of dimension n, respectively.

When zero matrix potentials are taken, i.e., $p = 0$ and $q = 0$ are chosen, the corresponding matrix spectral problems in (2.2) yield

$$v_k = v_k(x,t,\lambda_k) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^{2s}\Omega t} w_k,\ 1 \le k \le N, \tag{4.65}$$

where w_k, $1 \le k \le N$, are constant column vectors. According to the preceding analysis in Section 4.1, we can take the corresponding adjoint eigenfunctions as follows:

$$\hat{v}_k = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(-x,t,\lambda_k)\Sigma = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^{2s}\Omega t},\ 1 \le k \le N, \tag{4.66}$$

where

$$\hat{w}_k = w_k^\dagger \Sigma,\ 1 \le k \le N. \tag{4.67}$$

Then, the orthogonal condition (4.60) becomes

$$w_k^\dagger \Sigma w_l = 0 \text{ if } \lambda_l = \hat{\lambda}_k,\ \text{ where } 1 \le k,l \le N. \tag{4.68}$$

Finally, to present soliton solutions for the resulting nonlocal matrix integrable NLS equations (3.24), we need to check if G_1^+ defined by (4.62) satisfies the two involution properties:

$$(G_1^+)^\dagger(-x,t) = \Sigma G_1^+(x,t)\Sigma^{-1},\ (G_1^+)^T(x,-t) = \Delta G_1^+(x,t)\Delta^{-1}. \tag{4.69}$$

If so, the resulting potential matrix P given by (4.63) will satisfy the two nonlocal group reduction conditions in (3.15) and (3.16). Further, as a consequence of these conditions, we obtain the following soliton solutions:

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2, \tag{4.70}$$

for the resulting mixed-type nonlocal matrix integrable NLS equations (3.24). These solutions are reduced from the soliton solutions in (4.64) for the matrix AKNS Eqs. (2.10).

4.3. Realizing the involution properties

We would below like to build a theoretical framework for satisfying the involution properties in (4.69).

First, following the preceding analysis in Section 4.1, the adjoint eigenfunctions $\hat{v}_k$, $1 \le k \le N$, can be determined as follows:

$$\hat{v}_k = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(-x,t,\lambda_k)\Sigma = v_{N_1+k}^T(x,-t,\lambda_{N_1+k})\Delta,\ 1 \le k \le N_1, \tag{4.71}$$

$$\begin{aligned}\hat{v}_{N_1+k} &= \hat{v}_{N_1+k}(x,t,\hat{\lambda}_{N_1+k}) = v_{N_1+k}^T(-x,t,\lambda_{N_1+k})\Sigma \\ &= v_k^T(x,-t,\lambda_k)\Delta,\ 1 \le k \le N_1,\end{aligned} \tag{4.72}$$

and

$$\hat{v}_k = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(-x,t,\lambda_k)\Sigma = v_k^T(x,-t,\lambda_k)\Delta,\ 2N_1+1 \le k \le N. \tag{4.73}$$

These selections in (4.71), (4.72) and (4.73) generate the conditions on w_k, $1 \le k \le N$:

$$\begin{cases} w_k^T(\Sigma^*\Delta^{*-1} - \Delta\Sigma^{-1}) = 0,\ 1 \le k \le N_1, \\ w_k = \Delta^{-1}\Sigma^T w_{k-N_1}^*,\ N_1+1 \le k \le N, \\ w_k^\dagger \Sigma = w_k^T \Delta,\ 2N_1+1 \le k \le N, \end{cases} \tag{4.74}$$

where A^* denotes the complex matrix of a matrix A. Note that all these conditions aim to satisfy the reduction conditions in (3.15) and (3.16).

Next, note that when the solutions to the reflectionless generalized Riemann–Hilbert problems, defined by (4.57) and (4.58), possess the involution properties

$$(G^+)^\dagger(-\lambda^*) = \Sigma (G^-)^{-1}(\lambda)\Sigma^{-1},\ (G^+)^T(-\lambda) = \Delta (G^-)^{-1}(\lambda)\Delta^{-1}, \tag{4.75}$$

the corresponding relevant matrix G_1^+ will satisfy the involution properties in (4.69), which are consequences of the group reductions in (3.12) and (3.13). Consequently, when the conditions in (4.74) and the orthogonal condition in (4.68) are satisfied for w_k, $1 \le k \le N$, the formula (4.70), together with (4.57), (4.58), (4.65) and (4.66), presents soliton solutions to the reduced mixed-type nonlocal matrix integrable NLS equations (3.24).

Finally, for the case of $m = n/2 = N = 1$, let us compute an example of one-soliton solutions to the mixed-type scalar nonlocal integrable NLS equations. We choose $\lambda_1 = \nu$, $\hat{\lambda}_1 = -\nu$, $\nu \in \mathbb{R}$, and set $w_1 = (w_{1,1}, w_{1,2}, w_{1,3})^T$, where $w_{1,1}, w_{1,2}, w_{1,3} \in \mathbb{R}$ are arbitrary. This choice leads to a class of one-soliton solutions:

$$p_1 = \frac{2\sigma\nu(\alpha_1-\alpha_2)w_{1,1}w_{1,2}\mathrm{e}^{i(\alpha_1-\alpha_2)\nu x + i(\beta_1-\beta_2)\nu^2 t}}{\sigma(w_{1,2}^2 + w_{1,3}^2) + w_{1,1}^2 \mathrm{e}^{2i(\alpha_1-\alpha_2)\nu x}}, \tag{4.76}$$

where $\nu, w_{1,1}, w_{1,2}, w_{1,3} \in \mathbb{R}$ are arbitrary constants. It solves the mixed-type nonlocal integrable NLS equation (3.29), when the condition

$$w_{1,2}^2 - w_{1,3}^2 = 0 \tag{4.77}$$

is satisfied, and solves the mixed-type nonlocal integrable NLS equation (3.32), when the condition

$$(2\sigma\delta - 1)w_{1,2}^2 - w_{1,3}^2 = 0 \tag{4.78}$$

is satisfied. These required conditions are generated from the involution properties in (4.69). The class of solutions is analytic if and only if $w_{1,1}^2 \ne w_{1,2}^2 + w_{1,3}^2$.

5. Concluding remarks

Type $(-\lambda^*, -\lambda)$ reduced soliton hierarchies of nonlocal integrable NLS equations of even order have been presented, and their soliton solutions have been formulated through the corresponding reflectionless generalized Riemann–Hilbert problems, where eigenvalues could equal adjoint eigenvalues. The crucial step is to conduct a pair of nonlocal group reductions for the AKNS matrix spectral problems simultaneously. The resulting nonlocal integrable NLS equations are mixed-type, involving reverse-space, reverse-time and reverse-spacetime nonlocalities.

We remark that it will be of particular importance to explore soliton solutions by different approaches, including the Darboux transformation, the Hirota direct method, the Wronskian technique (see, e.g., Refs. 9, 12, 13, 24–27) and to study dynamical properties of diverse exact solutions in the nonlocal case, including lump and breather wave solutions[28–31], solitonless solutions[32] and algebro-geometric solutions,[33–35] from a perspective of Riemann–Hilbert problems. The mixed-type nonlocality involved creates big challenges for even establishing global existence of solutions. Additionally, another interesting problem is to construct reduced nonlocal integrable equations from matrix spectral problems associated with other semisimple matrix Lie algebras (see, e.g., Refs. 36, 37 for examples) and nonlocal integrable couplings associated with non-semisimple matrix Lie algebras (see, e.g., Ref. 38).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article

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