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Soliton solutions by means of Hirota bilinear forms

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ABSTRACT

The paper aims to provide a brief overview of soliton solutions obtained through the Hirota direct method. A bilinear formulation of soliton solutions in both (1+1)-dimensions and (2+1)-dimensions is discussed, together with applications to various integrable equations. The Hirota conditions for N-soliton solutions are analyzed and a few open questions regarding higher-dimensional cases and generalized bilinear equations are presented.

1. Introduction

Integrable equations possess a kind of exact multiple wave solutions, called N-soliton solutions. Among basic approaches to soliton solutions are the inverse scattering transform,[1,2] the Riemann–Hilbert technique,[3] the Darboux transformation,[4] and the Hirota direct method.[5] Significant solutions in mathematical physics, such as breather, complexion, lump and rogue wave solutions, are particular reductions of N-soliton solutions in different situations. Solitons superimposed in fibers are applied to optical communications, which enable to produce faster, richer, more secure, and more flexible communication systems.[6]

The Hirota direct method provides a standard and powerful approach to N-soliton solutions, indeed.[5] The innovative bilinear derivatives play a crucial role in generating soliton solutions[7] as well as lump solutions,[8] and Hirota bilinear forms are the key in related basic theories.[5] It is a characteristic feature that integrable equations can be transformed into Hirota bilinear forms under dependent variable transformations. This is also reflected by the Bell polynomial theory, which tells when a nonlinear equation can be expressed as a bilinear equation, either Hirota bilinear or generalized bilinear, through an exponential function relation.[9,10]

In this paper, we would like to provide a brief survey on soliton solutions via the Hirota direct method. In Section 2, we will introduce both Hirota bilinear derivatives and Hirota bilinear forms. In Section 3, we will formulate N-soliton solutions via Hirota bilinear forms, and analyze the Hirota conditions for N-soliton solutions. In Section 4, we will present a few basic examples of applications, including some novel examples. Finally in Section 5, we will give a summary, and discuss higher-dimensional cases and generalized bilinear equations and their related soliton problems.

2. Hirota bilinear derivatives and forms

2.1. Basic definitions

It is known that Hirota bilinear derivatives with respect to x and t are defined as follows[7]:

$$D_x^m D_t^n f\cdot g = (\partial_x-\partial_{x'})^m(\partial_t-\partial_{t'})^n f(x,t)g(x',t')|_{x'=x,t'=t}, \tag{2.1}$$

where $m,n\geq 0$ and $m+n\geq 1$. Particularly, we have

$$D_x f\cdot g = f_xg-fg_x,\ \ D_x^2 f\cdot g = f_{xx}g-2f_xg_x+fg_{xx}. \tag{2.2}$$

When $f=g$, we get Hirota bilinear expressions:

$$D_x^m D_t^n f\cdot f = (\partial_x-\partial_{x'})^m(\partial_t-\partial_{t'})^n f(x,t)f(x',t')|_{x'=x,t'=t}, \tag{2.3}$$

where $m,n\geq 0$ and $m+n\geq 1$. The first two of such expressions read

$$D_x f\cdot f = 0,\ \ D_x^2 f\cdot f = 2(f_{xx}f-f_x^2), \tag{2.4}$$

when $n=0$. Hirota bilinear derivatives with respect to x,y and t can be defined completely similarly.

Since we can see that Hirota bilinear expressions of odd degree are all zero,[8] we assume that $P(x,t)$ (or $P(x,y,t)$) is an even polynomial in x,t (or x,y,t). To guarantee the existence of polynomial solutions,

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particularly in the case of lump solutions, we also assume that P has no constant term, i.e., $P(\mathbf{0}) = 0$.

A Hirota bilinear equation in (1 + 1)- or (2 + 1)-dimensions is

$$P(D_x, D_t)f \cdot f = 0 \text{ or } P(D_x, D_y, D_t)f \cdot f = 0. \tag{2.5}$$

If a nonlinear partial differential equation (PDE) can be transformed into a Hirota bilinear equation under a dependent variable transformation, we say it possesses a Hirota bilinear form. The Bell polynomial theory[10] shows what nonlinear equations can possess Hirota bilinear forms.

2.2. Illustrative examples

In the case of (1 + 1)-dimensions, the bilinear Korteweg–de Vries (KdV) equation

$$(D_x^4 + D_x D_t)f \cdot f = 2(f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 + f_{xt}f - f_x f_t) = 0 \tag{2.6}$$

generates the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.7}$$

under the dependent variable transformation $u = 2(\ln f)_{xx}$. Actually, we have

$$u_t + 6uu_x + u_{xxx} = \Big[\frac{(D_x^4 + D_x D_t)f \cdot f}{f^2}\Big]_x. \tag{2.8}$$

The bilinear Boussinesq equation reads

$$(D_x^4 + D_t^2)f \cdot f = 2(f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 + f_{tt}f - f_t^2) = 0, \tag{2.9}$$

which gives the Boussinesq equation

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0, \tag{2.10}$$

if we take $u = 6(\ln f)_{xx}$. Precisely, we can show that

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = \Big[\frac{3(D_x^4 + D_t^2)f \cdot f}{f^2}\Big]_{xx}. \tag{2.11}$$

In the case of (2 + 1)-dimensions, the bilinear Kadomtsev–Petviashvili (KP) equation is

$$(D_x^4 + D_x D_t - D_y^2)f \cdot f = 0, \tag{2.12}$$

i.e.,

$$2(f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 + f_{xt}f - f_x f_t - f_{yy}f + f_y^2) = 0. \tag{2.13}$$

This is equivalent to the KP equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0, \tag{2.14}$$

under the dependent variable transformation $u = 2(\ln f)_{xx}$. Similarly, we have

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = \Big[\frac{(D_x^4 + D_x D_t - D_y^2)f \cdot f}{f^2}\Big]_{xx}. \tag{2.15}$$

The bilinear B-type Kadomtsev–Petviashvili (BKP) equation reads

$$\begin{aligned} B(f) &:= (D_x^6 - 5D_x^3 D_y + D_x D_t - 5D_y^2)f \cdot f \\ &= 2(f_{6x}f - 6f_{5x}f_x + 15f_{4x}f_{xx} - 10f_{xxx}^2 \\ &\quad - 5f_{xxxy}f + 15f_{xxy}f_x - 15f_{xy}f_{xx} + 5f_y f_{xxx} \\ &\quad + f_{xt}f - f_x f_t - 5f_{yy}f + 5f_y^2) = 0, \end{aligned} \tag{2.16}$$

which engenders the BKP equation

$$N(u) := (u_t + 15uu_{xxx} + 15u_x^3 - 15u_x u_y + u_{5x})_x - 5u_{xxxy} - 5u_{yy} = 0, \tag{2.17}$$

under the dependent variable transformation $u = 2(\ln f)_x$. In fact, we can show that

$$N(u) = (B(f)/f^2)_x. \tag{2.18}$$

Based on Hirota bilinear forms, symbolic computation can be applied, in search of exact multiple wave solutions, including lump solutions as long wave limits (see, e.g., Refs. 11, 12).

3. Hirota bilinear formulation of solitons

3.1. N-Soliton solutions

We focus on two cases of (1 + 1)-dimensions and (2 + 1)-dimensions. Assume that the wave vectors are given by

$$\mathbf{k}_i = (k_i, -\omega_i) \text{ or } (k_i, l_i, -\omega_i),\ 1 \le i \le N. \tag{3.1}$$

An N-soliton solution is defined as follows:

$$f = \sum_{\mu=0,1} \exp\Big(\sum_{i=1}^N \mu_i \eta_i + \sum_{i<j} a_{ij}\mu_i\mu_j\Big), \tag{3.2}$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_N)$, $\mu = 0, 1$ means that each μ_i takes either 0 or 1,

$$\eta_i = k_i x\,(+l_i y) - \omega_i t + \eta_{j,0},\ \ \eta_{j,0} = \text{const.},\ \ 1 \le i \le N, \tag{3.3}$$

and

$$e^{a_{ij}} = A_{ij} = -\frac{P(\mathbf{k}_i - \mathbf{k}_j)}{P(\mathbf{k}_i + \mathbf{k}_j)},\ \ 1 \le i < j \le N. \tag{3.4}$$

When such a function f solves a Hirota bilinear equation, we need a set of requirements, called the Hirota N-soliton condition (see Refs. 13, 14 for details). Observe that

$$\begin{aligned} &P(D_{x_1}, D_{x_2}, \dots, D_{x_M})f \cdot f \\ &= \tilde{P}(\{H(\mathbf{k}_{i_1}, \dots, \mathbf{k}_{i_n}) \,|\, 1 \le i_1 < i_2 < \cdots < i_n \le N\}), \end{aligned} \tag{3.5}$$

where $\tilde{P}$ is another polynomial satisfying

$$\tilde{P}(\mathbf{0}) = 0,$$

and where

$$H(\mathbf{k}_{i_1}, \dots, \mathbf{k}_{i_n}),\ \ 1 \le i_1 < i_2 < \cdots < i_n \le N,$$

are called Hirota functions of the wave vectors.[13,14] Those functions are polynomials in the wave vectors and will be defined later. Therefore, a Hirota bilinear equation possesses an N-soliton solution if and only if those Hirota functions are all zero.

The first example of integrable equations possessing N-soliton solutions is the KdV equation.[15]

3.2. Hirota N-soliton conditions

Let f be defined as before, and $\hat{\xi}$ mean that no ξ is involved. Then[13,14]

$$\begin{aligned} &P(D_{x_1}, \dots, D_{x_M})f \cdot f \\ &= (-1)^{\frac{1}{2}N(N-1)} \frac{H(\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_N)}{\prod_{1\le i<j\le N} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\eta_1+\eta_2+\cdots+\eta_N} \\ &+ \sum_{n=1}^{N-1} (-1)^{\frac{1}{2}(N-n)(N-n-1)} \sum_{1\le i_1<\cdots<i_n\le N} \frac{H(\mathbf{k}_1, \dots, \hat{\mathbf{k}}_{i_1}, \dots, \hat{\mathbf{k}}_{i_n}, \dots, \mathbf{k}_N)}{\prod_{\substack{1\le i<j\le N \\ i,j\notin\{i_1,\dots,i_n\}}} P(\mathbf{k}_i + \mathbf{k}_j)} \\ &\quad \times e^{\eta_1+\cdots+\hat{\eta}_{i_1}+\cdots+\hat{\eta}_{i_n}+\cdots+\eta_N} \\ &+ \sum_{n=1}^{N-1} \sum_{1\le i_1<\cdots<i_n\le N} e^{2(\eta_{i_1}+\cdots+\eta_{i_n}+\sum_{1\le r<s\le n} a_{i_r i_s})} P(D_{x_1}, \dots, D_{x_M})\tilde{f} \cdot \tilde{f}, \end{aligned} \tag{3.6}$$

where

$$\tilde{f} = \tilde{f}_{i_1\cdots i_n} = \sum_{\tilde{\mu}_{i_1\cdots i_n}=0,1} \exp\Big(\sum_{\substack{1\le i\le N \\ i\notin\{i_1,\dots,i_n\}}} \mu_i\tilde{\eta}_i + \sum_{\substack{1\le i<j\le N \\ i,j\notin\{i_1,\dots,i_n\}}} a_{ij}\mu_i\mu_j\Big), \tag{3.7}$$

$$\tilde{\eta}_i = \eta_i + \sum_{r=1}^n a_{ii_r}, \tag{3.8}$$

in which $\tilde{\mu}_{i_1\cdots i_n} = (\mu_1, \dots, \hat{\mu}_{i_1} \cdots, \hat{\mu}_{i_n}, \dots, \mu_N)$ and $\tilde{\mu}_{i_1\cdots i_n} = 0, 1$ means that each μ_i in $\tilde{\mu}_{i_1\cdots i_n}$ takes either 0 or 1. In the above analysis, the Hirota functions are defined by

$$H(\mathbf{k}_{i_1}, \dots, \mathbf{k}_{i_n}) :=$$

$$\sum_{\sigma=\pm1} P(\sum_{r=1}^{n}\sigma_r \mathbf{k}_{i_r}) \prod_{1\le r<s\le n} P(\sigma_r\mathbf{k}_{i_r}-\sigma_s\mathbf{k}_{i_s})\sigma_r\sigma_s,\ 1\le n\le N, \tag{3.9}$$

where $1 \le i_1 < \cdots < i_n \le N$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, and $\sigma = \pm 1$ means that each σ_r takes either 1 or –1.

Based on (3.6)–(3.8), we can see that a Hirota bilinear equation possesses an N-soliton solution if and only if

$$H(\mathbf{k}_{i_1}, \ldots, \mathbf{k}_{i_n}) = 0,\ 1 \le i_1 < \cdots < i_n \le N,\ 1 \le n \le N. \tag{3.10}$$

This is called the Hirota N-soliton condition, or simply, the N-soliton condition. An equation possessing an N-soliton solution is often called a soliton equation. Some computational algorithms with weights have been presented for checking the Hirota N-soliton condition (see, e.g., Refs. 13, 14).

In the Hirota N-soliton condition, the case of $n = 1$ leads to the dispersion relations:

$$P(\mathbf{k}_i) = 0,\ 1 \le i \le N, \tag{3.11}$$

due to the even property of P. This is why we always assume that the dispersion relations hold, while discussing N-soliton solutions.

The 1-soliton condition is

$$P(\mathbf{k}_1) = 0, \tag{3.12}$$

which means that $f = 1 + \mathrm{e}^{\eta_1}$ is a solution. The 2-soliton condition is

$$2(P(\mathbf{k}_1 + \mathbf{k}_2)P(\mathbf{k}_1 - \mathbf{k}_2) - P(\mathbf{k}_1 - \mathbf{k}_2)P(\mathbf{k}_1 + \mathbf{k}_2)) = 0, \tag{3.13}$$

which is an identity. So there always exists the 2-soliton solution:

$$f = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + A_{12}\mathrm{e}^{\eta_1+\eta_2}. \tag{3.14}$$

Taking $N = 3$, we obtain the 3-soliton condition:

$$\sum_{\mu_1,\mu_2,\mu_3=\pm1} P(\mu_1\mathbf{k}_1 + \mu_2\mathbf{k}_2 + \mu_3\mathbf{k}_3)P(\mu_1\mathbf{k}_1 - \mu_2\mathbf{k}_2) \times P(\mu_2\mathbf{k}_2 - \mu_3\mathbf{k}_3)P(\mu_1\mathbf{k}_1 - \mu_3\mathbf{k}_3) = 0. \tag{3.15}$$

The 3-soliton solution reads

$$f = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_3} + A_{12}\mathrm{e}^{\eta_1+\eta_2} + A_{13}\mathrm{e}^{\eta_1+\eta_3} + A_{23}\mathrm{e}^{\eta_2+\eta_3} + A_{12}A_{13}A_{23}\mathrm{e}^{\eta_1+\eta_2+\eta_3}. \tag{3.16}$$

Conjecture. *Does the 3-soliton condition imply the N-soliton condition?*

There is no counterexample to this conjecture so far. If we require a sufficient Hirota N-soliton condition:

$$P(\mathbf{k}_i - \mathbf{k}_j) = 0,\ 1 \le i < j \le N, \tag{3.17}$$

which implies the Hirota N-soliton condition (3.10), we obtain the resonant N-soliton solution[16,17]:

$$f = 1 + c_1\mathrm{e}^{\eta_1} + c_2\mathrm{e}^{\eta_2} + \cdots + c_N\mathrm{e}^{\eta_N}, \tag{3.18}$$

where c_i's are arbitrary constants. Note that all wave vectors $\mathbf{k}_i$'s associated with resonant solutions form an affine space.[17]

4. Applications to integrable equations

4.1. (1 + 1)-Dimensional case

In the case of (1 + 1)-dimensions, we have various examples of integrable equations which possess N-soliton solutions (see, e.g., Refs. 18–20). The following classes of generalized integrable equations, which possess N-soliton solutions, are explored in. Ref. 13

A class of generalized KdV equations possessing N-soliton solutions is determined by a polynomial of 4th degree:

$$P(x,t) = ax^4 + bx^3t + cx^2 + dxt,\ b^2 + d^2 \neq 0, \tag{4.1}$$

where a, b, c and d arbitrary constants satisfying $b^2 + d^2 \neq 0$, which guarantees that we will have a PDE. The corresponding generalized bilinear KdV equation is

$$\begin{aligned} B(f) &:= (aD_x^4 + bD_x^3D_t + cD_x^2 + dD_xD_t)f\cdot f \\ &= 2[a(f_{4x}f - 4f_{3x}f_x + 3f_x^2) + b(f_{3x,t}f - 3f_{xxt}f_x + 3f_{xt}f_{xx} - f_tf_{3x}) \\ &\quad + c(f_{xx}f - f_x^2) + d(f_{xt}f - f_xf_t)] = 0 \end{aligned} \tag{4.2}$$

and its generalized KdV equation reads

$$N(u) := a(6u_xu_{xx} + u_{4x}) + b[3(u_xu_t)_x + u_{3x,t}] + cu_{xx} + du_{xt} = 0, \tag{4.3}$$

between which there is a link $N(u) = (B(f)/f^2)_x$ under the dependent variable transformation $u = 2(\ln f)_x$. The case of $b = 0$ presents the KdV equation,[15] and the case of $a = 0$, the Hirota–Satsuma equation.[18]

A class of generalized Boussinesq equations corresponds to a polynomial of 4th-degree:

$$P(x,t) = ax^4 + bx^3t + cx^2 + dxt + t^2. \tag{4.4}$$

The case of $b = d = 0$ presents the Boussinesq equation.[21,22]

A class of generalized higher-order KdV equations possessing N-soliton solutions is determined through a polynomial of 6th degree:

$$P(x,t) = ax^6 + bx^4 + cx^2 + xt. \tag{4.5}$$

The corresponding generalized higher-order bilinear KdV equation is

$$\begin{aligned} B(f) &:= (aD_x^6 + bD_x^4 + cD_x^2 + dD_xD_t)f\cdot f \\ &= 2[a(f_{6x}f - 6f_{5x}f_x + 15f_{4x}f_{xx} - 10f_{3x}^2) \\ &\quad + b(f_{4x}f - 4f_{3x}f_x + 3f_x^2) + c(f_{xx}f - f_x^2) + f_{xt}f - f_xf_t] = 0 \end{aligned} \tag{4.6}$$

and its generalized higher-order KdV equation reads

$$N(u) := a(15u_x^3 + 15u_xu_{3x} + u_{5x})_x + b(6u_xu_{xx} + u_{4x}) + cu_{xx} + du_{xt} = 0, \tag{4.7}$$

between which there is the same link $N(u) = (B(f)/f^2)_x$ under the dependent variable transformation $u = 2(\ln f)_x$. The case of $b = c = 0$ presents the Sawada–Kotera equation.[19]

A class of generalized Ramani equations corresponds to a polynomial of 6th degree:

$$P(x,t) = x^6 + ax^4 + 5x^3t + bx^2 + cxt - 5t^2. \tag{4.8}$$

The case of $a = b = c = 0$ presents the Ramani equation.[20]

4.2. (2 + 1)-Dimensional case

In the case of (2 + 1)-dimensions, we have the following few examples.

The first example is the bilinear (2 + 1)-dimensional KdV equation:

$$\begin{aligned} B(f) &:= D_y(D_t + D_x^3)f\cdot f \\ &= 2(f_{yt}f - f_yf_t + f_{xxxy}f - 3f_{xxy}f_x \\ &\quad + 3f_{xy}f_{xx} - f_yf_{xxx}) = 0, \end{aligned} \tag{4.9}$$

which is associated with

$$P(x,y,t) = yt + x^3y. \tag{4.10}$$

This is equivalent to the (2 + 1)-dimensional KdV equation[23]:

$$N(u,v) := u_t + 3(uv)_x + u_{xxx} = 0,\ u_x = v_y, \tag{4.11}$$

under the dependent variable transformation of $u = 2(\ln f)_{xy}$ and $v = 2(\ln f)_{xx}$. The link is $N(u,v) = (B(f)/f^2)_x$.

The second example is the bilinear KP equation

$$\begin{aligned} B(f) &:= (D_x^4 + D_xD_t + D_y^2)f\cdot f \\ &= 2(f_{4x}f - 4f_{3x}f_x + 3f_{xx}^2 \\ &\quad + f_{xt}f - f_xf_t + f_{yy}f - f_y^2) = 0, \end{aligned} \tag{4.12}$$

which is associated with

$$P(x,y,t)=x^4+xt+y^2. \tag{4.13}$$

It is equivalent to the KP equation

$$N(u):=(u_t+6uu_x+u_{xxx})_x+u_{yy}=0, \tag{4.14}$$

under the dependent variable transformation $u=2(\ln f)_{xx}$. The link is $N(u)=(B(f)/f^2)_{xx}$.

The third example is the bilinear Hirota–Satsuma–Ito (HSI) equation[24]

$$\begin{aligned}B(f)&:=(D_x^3D_t+D_yD_t+D_x^2)f\cdot f\\&=2(f_{3x,t}f-3f_{xxt}f_x+3f_{xt}f_{xx}-f_tf_{xxx}\\&\quad+f_{yt}f-f_yf_t+f_{xx}f-f_x^2)=0,\end{aligned} \tag{4.15}$$

which is associated with

$$P(x,y,t)=x^3t+yt+x^2. \tag{4.16}$$

This is equivalent to the HSI equation

$$N(u):=u_{xx}+u_{yt}+3(u_xu_t)_x+u_{xxxt}=0, \tag{4.17}$$

under the dependent variable transformation $u=2(\ln f)_x$. The link is $N(u)=(B(f)/f^2)_x$.

The fourth example is the bilinear BKP equation[25]:

$$\begin{aligned}B(f)&:=(D_x^6+5D_x^3D_y+D_xD_t-5D_y^2)f\cdot f\\&=2[f_{6x}f-6f_{5x}f_x+15f_{4x}f_{xx}-10f_{3x}^3\\&\quad+5(f_{3x,y}f-3f_{xxy}f_x+3f_{xy}f_{xx}-f_yf_{3x})\\&\quad+f_{xt}f-f_xf_t-5(f_{yy}f-f_y^2)]=0,\end{aligned} \tag{4.18}$$

which is associated with

$$P(x,y,t)=x^6+5x^3y+xt-5y^2. \tag{4.19}$$

It is equivalent to the BKP equation

$$\begin{aligned}N(u)&:=(15u_x^3+15u_xu_{3x}+u_{5x})_x\\&\quad+5[u_{3x,y}+3(u_xu_y)_x]+u_{xt}-5u_{yy}=0,\end{aligned} \tag{4.20}$$

under the dependent variable transformation $u=2(\ln f)_x$. The link is $N(u)=(B(f)/f^2)_x$.

In what follows, we would like to present other two novel examples of generalized nonlinear equations possessing N-soliton solutions in the case of (2 + 1)-dimensions. The first one is a combined (2 + 1)-dimensional equation[26]:

$$\begin{aligned}B(f)&:=[a_1(D_x^4+D_xD_t)+a_2(D_x^3D_y+D_yD_t)\\&\qquad+a_3D_x^2+a_4D_xD_y+a_5D_y^2]f\cdot f=0,\end{aligned} \tag{4.21}$$

which is associated with

$$P(x,y,t)=a_1(x^4+xt)+a_2(x^3y+yt)+a_3x^2+a_4xy+a_5y^2, \tag{4.22}$$

where a_i's are arbitrary constants and satisfy $a_1^2+a_2^2\neq 0$ to guarantee the nonlinearity of the equation. This is equivalent to a nonlinear combined (2 + 1)-dimensional equation[26]:

$$\begin{aligned}N(u,v)&:=a_1(u_t+6uu_x+u_{xxx})+a_2[v_t+3(uv)_x+v_{xxx}]\\&\qquad+a_3u_x+a_4v_x+a_5v_y=0,\end{aligned} \tag{4.23}$$

where $u_y=v_x$, and the direct link is $N(u,v)=(B(f)/f^2)_x$, under the dependent variable transformation of $u=2(\ln f)_{xx}$ and $v=2(\ln f)_{xy}$.

The second one is the bilinear pKP-BKP equation[27]:

$$B(f):=(a_1D_x^6+a_2D_x^4+a_3D_x^3D_y+a_4D_x^2+a_5D_xD_t+a_6D_y^2)f\cdot f=0, \tag{4.24}$$

which is associated with

$$P(x,y,t)=a_1x^6+a_2x^4+a_3x^3y+a_4x^2+a_5xt+a_6y^2, \tag{4.25}$$

where a_i's are arbitrary constants and $a_5\neq 0$ to guarantee a PDE. This is equivalent to a nonlinear pKP-BKP equation[27]:

$$\begin{aligned}N(u)&:=a_1(15u_x^3+15u_xu_{3x}+u_{5x})_x+a_2(6u_xu_{xx}+u_{4x})\\&\quad+a_3[u_{3x,y}+3(u_xu_y)_x]+a_4u_{xx}+a_5u_{xt}+a_6u_{yy}=0,\end{aligned} \tag{4.26}$$

under the dependent variable transformation $u=2(\ln f)_x$, and the direct link is $N(u)=(B(f)/f^2)_x$. It possesses an N-soliton solution if and only if $a_3^2+5a_1a_6=0$,[27] which include the two previous BKP equations.

5. Concluding remarks

We have discussed the Hirota bilinear formulation of soliton solutions and presented a brief survey on illustrative examples of integrable equations, which possess N-soliton solutions. The Hirota N-soliton conditions have been given explicitly and many new examples have been discussed, which have been explored in our recent works under the help of symbolic computations.

We would like to point out that no example of Hirota bilinear equations in (3 + 1)-dimensions or higher-dimensions has been found to possess N-soliton solutions. In the case of (3 + 1)-dimensions, the Jimbo–Miwa equation[28]

$$[(D_x^3+2D_t)D_y-3D_xD_z]f\cdot f=0 \tag{5.1}$$

is the second member of soliton equations in the Sato-KP hierarchy,[29] the KP equation being the first member. But the Jimbo–Miwa equation passes the Painlevé test just for a subclass of solutions,[30] and only specific types of soliton solutions to the equation have been explored (see, e.g., Ref. 31). It would always be interesting, challenging and rewarding to look for typical examples of Hirota bilinear equations in (3 + 1)-dimensions, or even higher-dimensions, which possess N-soliton solutions.

We would also like to remark that a kind of generalized bilinear derivatives, called the $D_{p,x}$-derivatives, is defined by Ref. 32:

$$(D_{p,x}^m f\cdot g)(x)=\sum_{i=0}^{m}\binom{m}{i}\alpha_p^i(\partial_x^{n-i}f)(x)(\partial_x^i g)(x),\ m\geq 1, \tag{5.2}$$

where the powers of α_p are determined by

$$\alpha_p^i=(-1)^{r(i)},\ i=r(i)\bmod p,\ i\geq 0, \tag{5.3}$$

with $0\leq r(i)<p$. Those powers for $i=1,2,3,\ldots$ read

$$p=3:\ -,+,+,-,+,+,\ldots;$$

$$p=5:\ -,+,-,+,+,-,+,-,+,+,\ldots;$$

$$p=7:\ -,+,-,+,-,+,+,-,+,-,+,-,+,+,\ldots.$$

For example, we have

$$D_{3,x}^3f\cdot f=2f_{xxx}f,\ \ D_{3,x}^4f\cdot f=6f_{xx}^2, \tag{5.4}$$

which is different from the Hirota case (i.e., $p=2$). We can have other generalized bilinear derivatives, e.g., $D_{6,x},D_{9,x}$, associated with nonprime odd numbers.

A generalized bilinear equation reads

$$P(D_{p,x},D_{p,y},D_{p,t})f\cdot f=0, \tag{5.5}$$

and it possesses a resonant N-soliton solution

$$f=1+c_1e^{\eta_1}+c_2e^{\eta_2}+\cdots+c_Ne^{\eta_N}, \tag{5.6}$$

where c_i's are arbitrary constants, if and only if the following condition is satisfied[9,10]:

$$P(\mathbf{k}_i+\alpha_p\mathbf{k}_j)+P(\mathbf{k}_j+\alpha_p\mathbf{k}_i)=0,\ 1\leq i\leq j\leq N. \tag{5.7}$$

A generalized N-soliton condition is the condition under which a generalized bilinear equation possesses an N-soliton solution. However, what is such a generalized N-soliton condition, i.e., an N-soliton condition for a generalized bilinear equation? It is very interesting how to formulate generalized bilinear equations, for example,

$$P(D_{3,x}, D_{,t}) = 0,\ \ P(D_{3,x}, D_{3,y}, D_{3,t}) = 0, \tag{5.8}$$

which possess N-soliton solutions, in (1 + 1)-dimensions or (2 + 1)-dimensions. It is expected that some new theories could be developed in the case of generalized bilinear equations.

Declaration of competing interest

The author declares that there is no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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