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Nonlocal PT-symmetric integrable equations and related Riemann–Hilbert problems☆

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ABSTRACT

We aim to discuss about how to construct and classify nonlocal PT-symmetric integrable equations via nonlocal group reductions of matrix spectral problems. The nonlocalities considered are reverse-space, reverse-time and reverse-spacetime, each of which can involve either the transpose or the Hermitian transpose. The associated spectral problems are used to formulate a kind of Riemann–Hilbert problems and thus inverse scattering transforms. Soliton solutions are generated from specific Riemann–Hilbert problems with the identity jump matrix. We focus on two expository examples: nonlocal PT-symmetric matrix nonlinear Schrödinger and modified Korteweg–de Vries equations.

1. Introduction

Zero curvature equations associated with matrix Lie algebras generate Liouville integrable equations[1–4]. The Liouville integrability means that there exist infinitely many symmetries and conservation laws. Basic tools to explore such Liouville integrability are the trace identity[5] and the variational identity[6], corresponding to semi-simple Lie algebras or non-semi-simple Lie algebras, respectively. Among the well-known hierarchies of integrable equations are the KdV hierarchy, the AKNS hierarchy and the Kaup–Newell hierarchy[7,8], and various hierarchies of their associated integrable couplings[9,10].

A generating scheme[11] starts from two matrix spectral problems:

$$-i\phi_x = U\phi,\ -i\phi_t = V\phi, \tag{1.1}$$

where i is the unit imaginary number, $U = U(u,\lambda)$ and $V = V(u,\lambda)$ are a Lax pair of given matrices belonging to a loop algebra (called a spectral matrix and a Lax matrix, respectively), and ϕ is a matrix eigenfunction associated with an eigenvalue λ. The zero curvature equation

$$U_t - V_x + i[U,V] = 0, \tag{1.2}$$

which is the compatibility condition of the two spectral problems, determines an integrable equation[5]. Its bi-Hamiltonian structure[12], which yields a series of commuting symmetries and conservation laws[13,14], can often be furnished through applying the trace identity or the variational identity. Conservation laws and recursion operators can also be generated by performing symbolic computations with computer algebra systems (see, e.g.,[15,16]). Generally, conservation laws correspond to pairs of symmetries and adjoint symmetries[17,18].

A PT-symmetry is a fundamental symmetry of physical laws under a parity transformation: either $x \to -x$ or $x \to -x,\ y \to -y,\ z \to -z$ (but not $x \to -x,\ y \to -y$, which is a rotation); time reversal: $t \to -t$; and charge conjugation: $i \to -i$. It is a symmetry of nature at the fundamental level[19,20]. An equation is called to be PT-symmetric, if it is invariant under a PT-symmetry.

In this article, we would like to discuss about how to generate and classify nonlocal integrable group reductions of matrix spectral problems, which lead to nonlocal PT-symmetric integrable equations. The nonlocalities considered include reverse-space, reverse-time and reverse-spacetime, each of which involve either the transpose or the Hermitian transpose. Many illustrative examples of nonlocal integrable equations were recently given in soliton theory (see, e.g.,[21–32]). Based on special linear Lie algebras, we would like to consider the matrix AKNS spectral problems. The associated spectral problems are used to establish a kind of Riemann–Hilbert problems and thus inverse scattering transforms. Soliton solutions are obtained from solving specific Riemann–Hilbert problems with the identity jump matrix. We focus on two expository examples: nonlocal PT-symmetric matrix nonlinear Schrödinger and modified Korteweg–de Vries equations. A few concluding remarks are given in the last section.

☆ In celebration of the retirement of Professor Chaudry Masood Khalique.

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2. Matrix NLS equations and mKdV equations

Let $m, n \in \mathbb{N}$ be arbitrarily given. Suppose that $u = u(p, q)$, where p and q are potential matrices:

$$p = (p_{\mu\nu})_{1\leq\mu\leq m,\ 1\leq\nu\leq n},\ \ q = (q_{\nu\mu})_{1\leq\nu\leq n,\ 1\leq\mu\leq m}, \tag{2.1}$$

and set

$$\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n),\ \ \Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \tag{2.2}$$

where α_i and β_i, $i = 1, 2$, are two pairs of different numbers. For a fixed $r \in \mathbb{N}$, we consider a Lax pair in the AKNS matrix spectral problems as follows:

$$U = \lambda\Lambda + P(u),\ \ V^{[r]} = \lambda^r\Omega + Q^{[r]}(u, \lambda), \tag{2.3}$$

where P and $Q^{[r]}$ are traceless, and the degree of $Q^{[r]}$ with respect to λ is less than r.

The matrix nonlinear Schrödinger (NLS) equations has the potential matrix:

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}. \tag{2.4}$$

and the $Q^{[2]}$ matrix:

$$Q^{[2]} = \frac{\beta}{\alpha}\lambda \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2} \begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix}, \tag{2.5}$$

where $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$. The corresponding zero curvature equation

$$U_t - V^{[2]}_x + i[U, V^{[2]}] = 0$$

gives the matrix NLS equations:

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} + 2pqp),\ \ q_t = \frac{\beta}{\alpha^2} i(q_{xx} + 2qpq). \tag{2.6}$$

Consider the following zero curvature equation

$$U_t - V^{[3]}_x + i[U, V^{[3]}] = 0,$$

where $V^{[3]} = \lambda^3\Lambda + Q^{[3]}$ with

$$Q^{[3]} = \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3), \tag{2.7}$$

in which $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. This zero curvature equation presents the matrix modified Korteweg–de Vries (mKdV) equations:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_x qp),\ \ q_t = -\frac{\beta}{\alpha^3} i(q_{xxx} + 3q_x pq + 3qpq_x). \tag{2.8}$$

The above matrix NLS equations and mKdV equations possess bi-Hamiltonian structures, leading to infinitely many symmetries and conservation laws (see, e.g.,[33,34]), and it is easy to see that they are all PT-symmetric.

3. Nonlocal integrable equations

Let us consider the cases of replacing eigenvalues: $\lambda \to -\lambda^*, -\lambda, \lambda$. We will see those replacements produce nonlocal integrable reductions (see, e.g.,[29–32]).

The other case of $\lambda \to \lambda^*$ only produces local integrable reductions (see, e.g.,[35]).

To obtain reduced integrable equations, we can consider the following PT-symmetric nonlocal group reductions:

$$U^\dagger(\tilde{x}, \tilde{t}, -\lambda^*) = -CU(x, t, \lambda)C^{-1}, \tag{3.1}$$

$$U^T(\tilde{x}, \tilde{t}, -\lambda) = -CU(x, t, \lambda)C^{-1}, \tag{3.2}$$

$$U^T(\tilde{x}, \tilde{t}, \lambda) = CU(x, t, \lambda)C^{-1}, \tag{3.3}$$

where $(\tilde{x}, \tilde{t}) = (-x, t), (x, -t)$ or $(-x, -t)$, and

$$C = \begin{bmatrix} \Sigma_1 0 & \\ 0 & \Sigma_2 \end{bmatrix},\ \ \Sigma_j^\dagger = \Sigma_j \text{ or } \Sigma_j^T = \Sigma_j,\ \ j = 1, 2. \tag{3.4}$$

3.1. Nonlocal PT-symmetric NLS reductions

In the case of NLS equations, it is easy to see that all admissible nonlocal PT-symmetric group reductions can be divided into three categories of reverse-space, reverse-time and reverse-spacetime reductions:

$$U^\dagger(-x, t, -\lambda^*) = -CU(x, t, \lambda)C^{-1}, \tag{3.5}$$

$$U^T(x, -t, -\lambda) = -CU(x, t, \lambda)C^{-1}, \tag{3.6}$$

$$U^T(-x, -t, \lambda) = CU(x, t, \lambda)C^{-1}. \tag{3.7}$$

They lead to the potential reductions:

$$q(x, t) = -\Sigma_2^{-1} p^\dagger(-x, t)\Sigma_1, \tag{3.8}$$

$$q(x, t) = -\Sigma_2^{-1} p^T(x, -t)\Sigma_1, \tag{3.9}$$

$$q(x, t) = \Sigma_2^{-1} p^T(-x, -t)\Sigma_1, \tag{3.10}$$

respectively, and the associated Lax matrices satisfy

$$(V^{[2]})^\dagger(-x, t, -\lambda^*) = CV^{[2]}(x, t, \lambda)C^{-1}, \tag{3.11}$$

$$(V^{[2]})^T(x, -t, -\lambda) = CV^{[2]}(x, t, \lambda)C^{-1}, \tag{3.12}$$

$$(V^{[2]})^T(-x, -t, \lambda) = CV^{[2]}(x, t, \lambda)C^{-1}, \tag{3.13}$$

respectively. It is then direct to see that the corresponding nonlocal integrable reverse-space, reverse-time and reverse-spacetime NLS equations are given as follows:

$$p_t = -\frac{\beta}{\alpha^2} i[p_{xx} - 2p\Sigma_2^{-1} p^\dagger(-x, t)\Sigma_1 p],\ \ \Sigma_i^\dagger = \Sigma_i,\ \ i = 1, 2, \tag{3.14}$$

$$p_t = -\frac{\beta}{\alpha^2} i[p_{xx} - 2p\Sigma_2^{-1} p^T(x, -t)\Sigma_1 p],\ \ \Sigma_i^T = \Sigma_i,\ \ i = 1, 2, \tag{3.15}$$

$$p_t = -\frac{\beta}{\alpha^2} i[p_{xx} + 2p\Sigma_2^{-1} p^T(-x, -t)\Sigma_1 p],\ \ \Sigma_i^T = \Sigma_i,\ \ i = 1, 2. \tag{3.16}$$

Obviously, those three nonlocal matrix NLS equations are also PT-symmetric.

3.2. Nonlocal PT-symmetric mKdV reductions

In the case of mKdV equations, we can similarly see that all admissible PT-symmetric group reductions are composed of the following complex and real reverse-spacetime reductions:

$$U^\dagger(-x, -t, -\lambda^*) = -CU(x, t, \lambda)C^{-1}, \tag{3.17}$$

$$U^T(-x, -t, \lambda) = CU(x, t, \lambda)C^{-1}. \tag{3.18}$$

They generate the potential reductions:

$$q(x, t) = -\Sigma_2^{-1} p^\dagger(-x, -t)\Sigma_1, \tag{3.19}$$

$$q(x, t) = \Sigma_2^{-1} p^T(-x, -t)\Sigma_1, \tag{3.20}$$

respectively, and the associated Lax matrices satisfy

$$(V^{[3]})^\dagger(-x, -t, -\lambda^*) = -CV^{[3]}(x, t, \lambda)C^{-1}, \tag{3.21}$$

$$(V^{[3]})^T(-x, -t, \lambda) = CV^{[3]}(x, t, \lambda)C^{-1}, \tag{3.22}$$

respectively. It is now obvious to see that the corresponding nonlocal integrable complex and real reverse-spacetime mKdV equations read

$$p_t = -\frac{\beta}{\alpha^3}[p_{xxx} - 3p\Sigma_2^{-1} p^\dagger(-x, -t)\Sigma_1 p_x - 3p_x\Sigma_2^{-1} p^\dagger(-x, -t)\Sigma_1 p],$$

$$\Sigma_i^\dagger = \Sigma_i,\ i=1,2, \tag{3.23}$$

and

$$p_t = -\frac{\beta}{\alpha^3}[p_{xxx} + 3p\Sigma_2^{-1}p^T(-x,-t)\Sigma_1 p_x + 3p_x\Sigma_2^{-1}p^T(-x,-t)\Sigma_1 p],$$
$$\Sigma_i^T = \Sigma_i,\ i=1,2. \tag{3.24}$$

It is also easy to find that those two nonlocal matrix mkdV equations are PT-symmetric.

4. Riemann-Hilbert problems

To formulate Riemann–Hilbert problems with respect to x, let us assume that $\alpha=\alpha_1-\alpha_2<0$ and suppose that two matrix eigenfunctions $\psi^\pm$ satisfy

$$i\psi_x^\pm = \lambda[\Lambda,\psi^\pm] + P\psi^\pm,\ \psi^\pm \to I_{m+n} \text{ when } x\to\pm\infty. \tag{4.1}$$

Then to formulate Riemann–Hilbert problems, we define two new matrices:

$$T^+ = \psi^- H_1 + \psi^+ H_2,\ T^- = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1}, \tag{4.2}$$

where

$$H_1 = \mathrm{diag}(I_m, \underbrace{0,\dots,0}_{n}),\ H_2 = \mathrm{diag}(\underbrace{0,\dots,0}_{m}, I_n). \tag{4.3}$$

Using the theory of Volterra integral equations, we can see that T^+ is analytic in $\mathbb{C}^+$ and continuous in $\bar{\mathbb{C}}^+$, and T^- is analytic in $\mathbb{C}^-$ and continuous in $\bar{\mathbb{C}}^-$, where $\mathbb{C}^+ = \{\mu\in\mathbb{C}\,|\,\mathrm{Im}\,\mu>0\}$ and $\mathbb{C}^- = \{\mu\in\mathbb{C}\,|\,\mathrm{Im}\,\mu<0\}$, and $\bar{\mathbb{C}}^\pm$ are the closures of $\mathbb{C}^\pm$, respectively.

From the definition of $\psi^\pm$, we obtain

$$\lim_{x\to\infty} T^+ = \begin{bmatrix} S_{11} & 0 \\ 0 & I_n \end{bmatrix},\ \lambda\in\bar{\mathbb{C}}^+,\ \lim_{x\to-\infty} T^- = \begin{bmatrix} \hat{S}_{11} & 0 \\ 0 & I_n \end{bmatrix},\ \lambda\in\bar{\mathbb{C}}^-, \tag{4.4}$$

and

$$\det T^+(x,\lambda) = \det S_{11}(\lambda),\ \det T^-(x,\lambda) = \det \hat{S}_{11}(\lambda), \tag{4.5}$$

where the scattering matrix S is defined by

$$\psi^- E = \psi^+ E S(\lambda),\ E = \mathrm{e}^{i\lambda\Lambda x},\ \lambda\in\mathbb{R}, \tag{4.6}$$

and we split $S(\lambda)$ and $S^{-1}(\lambda) = (S(\lambda))^{-1}$ as follows:

$$S(\lambda) = \begin{bmatrix} S_{11}(\lambda) & S_{12}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{bmatrix},\ S^{-1}(\lambda) = \begin{bmatrix} \hat{S}_{11}(\lambda) & \hat{S}_{12}(\lambda) \\ \hat{S}_{21}(\lambda) & \hat{S}_{22}(\lambda) \end{bmatrix}. \tag{4.7}$$

Finally, we can arrive at a kind of Riemann–Hilbert problems:

$$G^+(x,\lambda) = G^-(x,\lambda)G_0(x,\lambda),\ \lambda\in\mathbb{R}, \tag{4.8}$$

where

$$\begin{cases} G^+(x,\lambda) = T^+(x,\lambda)\begin{bmatrix} S_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix},\ \lambda\in\bar{\mathbb{C}}^+, \\ (G^-)^{-1}(x,\lambda) = \begin{bmatrix} \hat{S}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} T^-(x,\lambda),\ \lambda\in\bar{\mathbb{C}}^-, \end{cases} \tag{4.9}$$

and the jump matrix G_0 is

$$G_0(x,\lambda) = E\begin{bmatrix} \hat{S}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}\begin{bmatrix} I_m & \hat{S}_{12} \\ S_{21} & I_n \end{bmatrix}\begin{bmatrix} S_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}E^{-1}. \tag{4.10}$$

We have the corresponding reduction properties. For example, in the case of $\lambda\to-\lambda^*$, the eigenfunction matrix ψ satisfies

$$\psi^\dagger(-x,t,-\lambda^*) = C\psi^{-1}(x,t,\lambda)C^{-1}, \tag{4.11}$$

when $\psi\to I_{n+1},\ x,t$ or $\lambda\to\infty$. This implies that if λ is an eigenvalue, then $-\lambda^*$ is also an eigenvalue, and

$$G_0^\dagger(-x,t,-\lambda^*) = CG_0(x,t,\lambda)C^{-1}, \tag{4.12}$$

where $G_0(\lambda)$ is the jump matrix.

Starting from the above established Riemann–Hilbert problems, we can also formulate associated inverse scattering transforms for solving Cauchy problems of the resulting nonlocal PT-symmetric integrable equations (see, e.g.,[22,29–32]).

5. Soliton solutions

To obtain soliton solutions to the resulting nonlocal integrable equations, we need to develop a novel formulation of solutions to Riemann–Hilbert problems with the identity jump matrix (see also[31,32]).

Let us take $G_0 = I_{m+n}$. Choose $\{\lambda_k,\hat{\lambda}_k\in\mathbb{C}\}_{k=1}^N$. Then we have the following formulation of solutions to the specific Riemann–Hilbert problems with $G_0 = I_{m+n}$:

$$G^+ = I_{m+n} - \sum_{j,l=1}^N \frac{v_j(M^{-1})_{jl}\hat{v}_l}{\lambda-\hat{\lambda}_l},\ G^- = I_{m+n} + \sum_{j,l=1}^N \frac{v_j(M^{-1})_{jl}\hat{v}_l}{\lambda-\lambda_j}, \tag{5.1}$$

where M is a new square matrix:

$$M = (m_{jl})_{N\times N},\ m_{jl} = \begin{cases} \dfrac{\hat{v}_j v_l}{\lambda_l-\hat{\lambda}_j}, & \text{if } \lambda_l\ne\hat{\lambda}_j,\ 1\le j,l\le N, \\ 0, & \text{if } \lambda_l=\hat{\lambda}_j,\ 1\le j,l\le N, \end{cases} \tag{5.2}$$

and

$$\hat{v}_j v_l = 0,\ \text{if } \lambda_l=\hat{\lambda}_j,\ 1\le j,l\le N. \tag{5.3}$$

This exactly tells that under the orthogonal condition (5.3), the two square matrices $G^\pm$ defined by (5.1) are inverse to each other. When

$$\{\lambda_k\,|\,1\le k\le N\}\cap\{\hat{\lambda}_k\,|\,1\le k\le N\} = \emptyset, \tag{5.4}$$

the above solution formulation is reduced to the existing one for local integrable equations (see, e.g.,[4,28]).

The corresponding matrix spectral problems under zero potentials generate

$$v_k(x,t) = \mathrm{e}^{i\lambda_k\Lambda x + i\lambda_k^l\Omega t}w_k,\ 1\le k\le N, \tag{5.5}$$

and

$$\hat{v}_k(x,t) = \hat{w}_k\mathrm{e}^{-i\hat{\lambda}_k\Lambda x - i\hat{\lambda}_k^l\Omega t}C,\ 1\le k\le N, \tag{5.6}$$

where w_k and $\hat{w}_k,\ 1\le k\le N$, are arbitrary column and row vectors. The orthogonal conditions for w_k and $\hat{w}_k,\ 1\le k\le N$, are

$$\hat{w}_k C w_l = 0,\ \text{if } \lambda_l=\hat{\lambda}_k,\ 1\le k,l\le N. \tag{5.7}$$

To express soliton solutions, we expand G^+ at $\lambda=\infty$ as

$$G^+(x,\lambda) = I_{m+n} + \frac{1}{\lambda}G_1^+(x) + \mathrm{O}(\frac{1}{\lambda^2}),\ \lambda\to\infty, \tag{5.8}$$

and then obtain

$$P = -[\Lambda, G_1^+] = \lim_{\lambda\to\infty}\lambda[G^+(\lambda),\Lambda]. \tag{5.9}$$

In other words,

$$p_{\mu\nu} = -\alpha(G_1^+)_{\mu,\nu+m},\ 1\le\mu\le m,\ 1\le\nu\le n, \tag{5.10}$$

where $G_1^+ = ((G_1^+)_{\mu\nu})_{(m+n)\times(m+n)}$.

Observing G_1^+, we know

$$G_1^+ = -\sum_{j,l=1}^N v_j(M^{-1})_{jl}\hat{v}_l, \tag{5.11}$$

which need to satisfy

$$(G_1^+(-x,t))^\dagger = CG_1^+(x,t)C^{-1}, \tag{5.12}$$

$$(G_1^+(x,-t))^T = CG_1^+(x,t)C^{-1}, \tag{5.13}$$

or

$$(G_1^+(-x,-t))^T = -CG_1^+(x,t)C^{-1}, \tag{5.14}$$

in the three reduction cases.

Then, N-soliton solutions to the resulting nonlocal integrable equations are given as follows:

$$p_{\mu\nu} = \alpha \sum_{j,l=1}^{N} v_{j,\mu}(M^{-1})_{jl}\hat{v}_{l,m+\nu}, \tag{5.15}$$

where $v_k = (v_{k,1}, \dots, v_{k,m+n})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \dots, \hat{v}_{k,m+n})$ are assumed.

We point out that N-soliton solutions can be systematically generated by the Hirota bilinear method, and we refer the reader to[36–38] for some novel examples in (2+1)-dimensions.

6. Concluding remarks

We have constructed all nonlocal reduced integrable equations in the cases of nonlinear Schrödinger (NLS) equations and modified Korteweg–de Vries (mKdV) equations, and classified them into three categories of nonlocal PT-symmetric complex reverse-space, real reverse-time and real reverse-spacetime NLS equations and two categories of nonlocal PT-symmetric complex and real reverse-spacetime mKdV equations.

Applications to other spectral matrices could also be developed. For example, one example in the continuous case is the Kaup–Newell spectral matrix[8]:

$$U = \begin{bmatrix} -\lambda^2 & \lambda p \\ \lambda q & \lambda^2 \end{bmatrix},$$

and one example in the discrete case is the Ablowitz–Ladik spectral matrix[39]:

$$U = \begin{bmatrix} \lambda & p \\ q & \lambda^{-1} \end{bmatrix},$$

including a vector case[40]. Applications to (2+1)-dimensional cases could as well be made, for example, to a class of (2+1)-dimensional spectral problems[41]:

$$L\psi = \lambda\psi,\ L = L(\partial_x, \partial_y, u),\ u = u(x,y,t),$$

which includes the hyperbolic case:

$$L = \partial_x + \sigma_3\partial_y + u,$$

and the elliptic case:

$$L = \partial_x + i\sigma_3\partial_y + u,$$

where

$$u = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix},\ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Similarly, we can formulate other nonlocal reduced equations, not from reducing matrix spectral problems. For example, we can have nonlocal real reverse-space NLS equations:

$$p_t = -\frac{\beta}{\alpha^2} i[p_{xx} - 2p\Sigma_2^{-1}p^T(-x,t)\Sigma_1 p],\ \Sigma_i^T = \Sigma_i,\ i = 1,2,$$

nonlocal complex reverse-time NLS equations:

$$p_t = -\frac{\beta}{\alpha^2} i[p_{xx} - 2p\Sigma_2^{-1}p^\dagger(x,-t)\Sigma_1 p],\ \Sigma_i^\dagger = \Sigma_i,\ i = 1,2,$$

and nonlocal complex reverse-spacetime NLS equations:

$$p_t = -\frac{\beta}{\alpha^2} i[p_{xx} + 2p\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1 p],\ \Sigma_i^\dagger = \Sigma_i,\ i = 1,2.$$

They are all PT-symmetric, too. But how about their integrability? This is an interesting question to ask. There are other cases to generate nonlocal equations: two-step reductions, pantograph modeling, and general linear combinations of space and time variables, including reverse-space and/or reverse-time variables with shifts.

There are also nonlocal PT-symmetric group reductions associated with the special orthogonal Lie algebra so(3, $\mathbb{R}$):

$$[\mathrm{e}_1, \mathrm{e}_2] = \mathrm{e}_3,\ [\mathrm{e}_2, \mathrm{e}_3] = \mathrm{e}_1,\ [\mathrm{e}_3, \mathrm{e}_1] = \mathrm{e}_2.$$

An AKNS type spectral problem associated with so(3, $\mathbb{R}$)[42] is

$$-i\phi_x = U\phi,\ U = \lambda \mathrm{e}_1 + p\mathrm{e}_2 + q\mathrm{e}_2 = \begin{bmatrix} 0 & -q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}.$$

We can have similar nonlocal group reductions to obtain a nonlocal reverse-space NLS type equation:

$$ip_t = p^*_{xx}(-x,t) - \frac{1}{2}p^2p^*(-x,t) - \frac{1}{2}(p^*(-x,t))^3,$$

a nonlocal reverse-time NLS type equation:

$$ip_t = p_{xx}(x,-t) + \frac{1}{2}p^2p(x,-t) - \frac{1}{2}(p(x,-t))^3,$$

and a nonlocal reverse-spacetime NLS type equation:

$$ip_t = -p_{xx}(-x,-t) + \frac{1}{2}p^2p(-x,-t) + \frac{1}{2}(p(-x,-t))^3.$$

They are all PT-symmetric and Liouville integrable.

Nevertheless, just based on the matrix spectral problem

$$-i\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}_x = \begin{bmatrix} 0 & -q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix},$$

we still do not know how to formulate the associated Riemann–Hilbert problems for the above nonlocal NLS type equations. The eigenfunctions of matrix spectral problems associated with so(3, $\mathbb{R}$) exhibit a very different feature of analyticity with respect to the spectral parameter λ. Further investigation is definitely needed.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

1. Lax PD. Integrals of nonlinear equations of evolution and solitary waves. *Commun Pure Appl Math*. 1968;21(5):467–490.
2. Ablowitz MA, Clarkson PA. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge: Cambridge University Press; 1991.
3. Calogero F, Degasperis A. *Solitons and Spectral Transform I*. Amsterdam: North-Holland; 1982.
4. Novikov SP, Manakov SV, Pitaevskii LP, Zakharov VE. *Theory of Solitons, the Inverse Scattering Method*. New York: Consultants Bureau; 1984.
5. Tu GZ. On Liouville integrability of zero-curvature equations and the Yang hierarchy. *J Phys A: Math Gen*. 1989;22(13):2375–2392.
6. Ma WX, Chen M. Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras. *J Phys A: Math Gen*. 2006;39(34):10787–10801.
7. Ablowitz MJ, Kaup DJ, Newell AC, Segur H. The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud Appl Math*. 1974;53(4):249–315.
8. Kaup DJ, Newell AC. An exact solution for a derivative nonlinear Schrödinger equation. *J Math Phys*. 1978;19(4):798–801.
9. Ma WX, Meng JH, Zhang HQ. Integrable couplings, variational identities and Hamiltonian formulations. *Global J Math Sci*. 2012;1(1):1–17.

10. Ma WX. Integrable couplings and matrix loop algebras. In: Ma WX, Kaup D, eds. *Nonlinear and modern mathematical physics. AIP Conference Proceedings*. Melville, NY: American Institute of Physics; 2013:105–122. vol. 1562.
11. Ma WX. A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction. *Chin Ann Math A*. 1992;13(1):115–123.
12. Magri F. A simple model of the integrable Hamiltonian equation. *J Math Phys*. 1978;19(5):1156–1162.
13. Fuchssteiner B, Fokas AS. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Physica D*. 1981/82;4(1):47–66.
14. Olver PJ. *Applications of Lie Groups to Differential Equations*. New York: Springer-Verlag; 1986.
15. Hereman W, Adams PJ, Eklund HL, Hickman MS, Herbst BM. Direct methods and symbolic software for conservation laws of nonlinear equations. In: *Advances in Nonlinear Waves and Symbolic Computation*. New York: Nova Sci. Publ.; 2009:19–78.
16. Baldwin DE, Hereman W. A symbolic algorithm for computing recursion operators of nonlinear partial differential equations. *Int J Comput Math*. 2010;87(5):1094–1119.
17. Ibragimov NH. A new conservation theorem. *J Math Anal Appl*. 2007;333(1):311–328.
18. Ma WX. Conservation laws by symmetries and adjoint symmetries. *Discrete Contin Dyn Syst Ser S*. 2018;11(4):707–721.
19. Bender CM, Boettcher S. Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry. *Phys Rev Lett*. 1998;80(24):5243–5246.
20. Bender CM. Making sense of non-Hermitian Hamiltonians. *Rep Prog Phys*. 2007;70(6):947–1018.
21. Ablowitz MJ, Musslimani AH. Integrable nonlocal nonlinear Schrödinger equation. *Phys Rev Lett*. 2013;110(6):064105.
22. Ablowitz MJ, Musslimani ZH. Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation. *Nonlinearity*. 2016;29(3):915–946.
23. Ji JL, Zhu ZN. On a nonlocal modified Korteweg–de Vries equation: integrability, Darboux transformation and soliton solutions. *Commun Nonlinear Sci Numer Simul*. 2017;42:699–708.
24. Gerdjikov VS, Grahovski GG, Ivanov RI. On integrable wave interactions and Lax pairs on symmetric spaces. *Wave Motion*. 2017;71:53–70.
25. Song CQ, Xiao DM, Zhu ZN. Solitons and dynamics for a general integrable nonlocal coupled nonlinear Schrödinger equation. *Commun Nonlinear Sci Numer Simul*. 2017;45:13–28.
26. Zhou ZX. Darboux transformations and global solutions for a nonlocal derivative nonlinear Schrödinger equation. *Commun Nonlinear Sci Numer Simul*. 2018;62:480–488.
27. Gürses M, Pekcan A. Nonlocal nonlinear Schrödinger equations and their soliton solutions. *J Math Phys*. 2018;59(5):051501.
28. Yang J. General N-solitons and their dynamics in several nonlocal non-linear Schrödinger equations. *Phys Lett A*. 2019;383(4):328–337.
29. Ma WX. Inverse scattering for nonlocal reverse-time nonlinear Schrödinger equations. *Appl Math Lett*. 2020;102:106161.
30. Ma WX. Inverse scattering and soliton solutions of nonlocal complex reverse-spacetime mKdV equations. *J Geom Phys*. 2020;157:103845.
31. Ma WX, Huang YH, Wang FD. Inverse scattering transforms and soliton solutions of nonlocal reverse-space nonlinear Schrödinger hierarchies. *Stud Appl Math*. 2020;145(3):563–585.
32. Ma WX. Inverse scattering and soliton solutions of nonlocal reverse-spacetime nonlinear Schrödinger equations. *Proc Amer Math Soc*. 2021;149(1):251–263.
33. Ma WX, Batwa S. A binary Darboux transformation for multicomponent NLS equations and their reductions. *Anal Math Phys*. 2021;11(2):44.
34. Ma WX. Binary Darboux transformation for general matrix mKdV equations and reduced counterparts. *Chaos Solitons Fractals*. 2021;146:110824.
35. Ma WX. Riemann-Hilbert problems and soliton solutions of a multicomponent mKdV system and its reduction. *Math Methods Appl Sci*. 2019;42(4):1099–1113.
36. Ma WX. N-Soliton solution of a combined pKP-BKP equation. *J Geom Phys*. 2021;165:104191.
37. Ma WX, Yong XL, Lü X. Soliton solutions to the B-type Kadomtsev–Petviashvili equation under general dispersion relations. *Wave Motion*. 2021;103:102719.
38. Ma WX. N-Soliton solution and the Hirota condition of a (2+1)-dimensional combined equation. *Math Comput Simul*. 2021;190:270–279.
39. Abolowitz MJ, Ladik JF. Nonlinear differential-difference equations. *J Math Phys*. 1975;16(3):598–603.
40. Ma WX. An Ablowitz-Ladik integrable lattice hierarchy with multiple potentials. *Acta Math Sci*. 2020;40B(3):670–678.
41. Boiti M, Leon JJP, Martina L, Pempinelli F. Integrable nonlinear evolutions in 2+1 dimensions with nonanalytic dispersion relations. *J Phys A: Math Gen*. 1988;21(18):3611–3627.
42. Ma WX. A soliton hierarchy associated with so(3, $\mathbb{R}$). *Appl Math Comput*. 2013;220:117–122.