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A polynomial conjecture connected with rogue waves in the KdV equation

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ABSTRACT

A polynomial conjecture, associated with rational solutions including rogue wave solutions of the KdV equation, is presented. The conjecture can be used to show that for the bilinear KdV equation, an arbitrary linear combination of two Wronskian polynomial solutions with a difference two between the Wronskian orders will again be a solution.

1. Introduction

Integrable equations possess abundant exact solutions which exhibit various nonlinear phenomena in complex systems.[1–3] Particularly, optical solitons and rogue waves are of current importance and general interest within the physical and engineering sciences.[4–6] There are plenty of studies on such nonlinear dispersive waves,[7–9] and their corresponding equations and even hierarchies of equations can be solved through Wronskian or Casoratian formulations (see, e.g., Refs. 10, 11) and Riemann–Hilbert problems (see, e.g., Ref. 12).

It is known[10] that for the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.1}$$

a Wronskian $f = W(\phi_0, \phi_1, \dots, \phi_{N-1})$ presents a solution $u = -2(\ln f)_{xx}$, if the entries satisfy that

$$-\phi_{i,xx} = \sum_{j=1}^{N-1} \lambda_{ij}(t)\phi_j,\ 0 \le i \le N-1, \tag{1.2}$$

and

$$\phi_{i,t} = -4\phi_{i,xxx} + \xi(t)\phi_i,\ 0 \le i \le N-1, \tag{1.3}$$

where λ_{ij} and ξ are arbitrary functions of t. Rational solutions correspond to the case of zero eigenvalues of the coefficient matrix $\Lambda = (\lambda_{ij})_{0\le i,j\le N-1}$.[10] One example of such rational solutions is $u = \frac{2}{x^2}$ (see Ref. 13 for more examples). Through the x-translational invariance: $\tilde{u}(x,t) = u(x+c,t)$ and the Galilean invariance: $\tilde{u}(x,t) = u(x+6ct,t)+c$, where c is an arbitrary constant, we can generate a rogue wave solution to the KdV equation:

$$\tilde{u}(x,t) = \frac{2}{(x+6ct+ai)^2} + c, \tag{1.4}$$

where $a \neq 0$ and c are arbitrary real constants. A special case of it with $a = 1/2$ and $c = -1$ gives the rogue wave solution presented in Ref. 14:

$$\tilde{u}(x,t) = \frac{8}{(2x-12t+i)^2} - 1. \tag{1.5}$$

In this paper, we would like to present a polynomial conjecture connected with rational solutions to the KdV equation within the Wronskian formulation. Such rational solutions can lead to rogue wave solutions to the KdV equation through the x-translational (or t-translational) invariance and the Galilean invariance. In Section 2, we will analyze the conjecture and present two illustrative examples. Concluding remarks will be given in the last section.

2. A polynomial conjecture

Following Ref. 10, for a sequence of smooth functions of x: $\phi_i = \phi_i(x)$, $i \ge 0$, let us define

$$\phi_i^{(j)} = \frac{d^j\phi_i}{dx^j},\ \Phi_i^{(j)} = (\phi_0^{(j)}, \phi_1^{(j)}, \dots, \phi_{i-1}^{(j)})^T,\ i,j \ge 0. \tag{2.1}$$

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For $m, n \geq 1$, we denote a Wronskian of order m by

$$(\widehat{m-1}) = W(\phi_0, \phi_1, \ldots, \phi_{m-1}) = \det(\boldsymbol{\Phi}_m^{(0)}, \boldsymbol{\Phi}_m^{(1)}, \ldots, \boldsymbol{\Phi}_m^{(m-1)}), \tag{2.2}$$

and a generalized Wronskian of order $m+n$, by

$$(\widehat{m-1}, i_1, i_2, \ldots, i_n) = \det(\boldsymbol{\Phi}_{m+n}^{(0)}, \boldsymbol{\Phi}_{m+n}^{(1)}, \ldots, \boldsymbol{\Phi}_{m+n}^{(m-1)}, \boldsymbol{\Phi}_{m+n}^{(i_1)}, \boldsymbol{\Phi}_{m+n}^{(i_2)}, \ldots, \boldsymbol{\Phi}_{m+n}^{(i_n)}), \tag{2.3}$$

where $m \leq i_1 < i_2 < \cdots < i_n$. In a Wronskian of order m, we have a square matrix of size m, and in a generalized Wronskian of order $m+n$, we have a square matrix of size $m+n$.

Conjecture. *Let $N \geq 3$ be an arbitrary natural number. If*

$$\phi_{0,xx} = 0, \ \phi_{i+1,xx} = \phi_i, \ i \geq 0, \tag{2.4}$$

then we have an identity for generalized Wronskians:

$$\begin{aligned} &(\widehat{N-1})(\widehat{N-1}, N+2, N+3) + (\widehat{N-3}, N, N+1)(\widehat{N+1}) \\ &-(\widehat{N-2}, N)(\widehat{N-1}, N+1, N+3) - (\widehat{N-3}, N-1, N+1)(\widehat{N}, N+2) \\ &+(\widehat{N-2}, N+1)(\widehat{N-1}, N+1, N+2) \\ &+(\widehat{N-3}, N-1, N)(\widehat{N}, N+3) = 0. \end{aligned} \tag{2.5}$$

Those conditions in (2.4) define a sequence of polynomials ϕ_i, $i \geq 0$, of the variable x. Therefore, the above conjecture is about the polynomials of x which satisfy (2.4).

More symmetrically, the identity (2.5) can be written as

$$\begin{aligned} &(\widehat{N-3}, N-2, N-1)(\widehat{N-1}, N+2, N+3) \\ &+(\widehat{N-3}, N, N+1)(\widehat{N-1}, N, N+1) \\ &-(\widehat{N-3}, N-2, N)(\widehat{N-1}, N+1, N+3) \\ &-(\widehat{N-3}, N-1, N+1)(\widehat{N-1}, N, N+2) \\ &+(\widehat{N-3}, N-2, N+1)(\widehat{N-1}, N+1, N+2) \\ &+(\widehat{N-3}, N-1, N)(\widehat{N-1}, N, N+3) \\ &= 0. \end{aligned} \tag{2.6}$$

This can be further expressed as a sum:

$$\sum_{\substack{a+b+c+d=2 \\ -2\leq a<b\leq 1,\ 0\leq c<d\leq 3}} (-1)^{a+b+1}(\widehat{N-3}, N+a, N+b)(\widehat{N-1}, N+c, N+d) = 0. \tag{2.7}$$

This sum exhibits a characteristic property of determinants. The question for us is what it really reflects.

The identity (2.6) looks like the simplest case of the Plücker relations,[7] and it might be helpful in proving it to apply the Laplace expansion for determinants and Jacobi's identity for Wronskians:

$$\begin{aligned} &(W(g_1, g_2, \ldots, g_{m-1}, h))_x W(g_1, g_2, \ldots, g_m) \\ &- W(g_1, g_2, \ldots, g_{m-1}, h)(W(g_1, g_2, \ldots, g_m))_x \\ &= W(g_1, g_2, \ldots, g_{m-1}) W(g_1, g_2, \ldots, g_m, h), \end{aligned}$$

where $g_1, g_2, \ldots, g_m, h$ are sufficiently differentiable functions and $W(g_1, \ldots, g_{m-1}) = 1$ when $m = 1$.

Let us now illustrate the identity (2.5) (i.e., the identity (2.6)). We take

$$\psi(\eta) = \sinh(\eta x) = \sum_{i=0}^{\infty} \phi_i \eta^{2i+1}, \tag{2.8}$$

where η is a free parameter. This analytic function satisfies $\psi_{xx} = \eta^2 \psi$ and thus the resulting sequence of polynomials ϕ_i, $i \geq 0$, satisfies the conditions in (2.4), the first few of which read

$$\phi_0 = x, \ \phi_1 = \frac{1}{6}x^3, \ \phi_2 = \frac{1}{120}x^5, \ \phi_3 = \frac{1}{5040}x^7, \ \phi_4 = \frac{1}{362880}x^9, \ \phi_5 = \frac{1}{39916800}x^{11}. \tag{2.9}$$

Then, it directly follows that when $N = 3$, we have

$$\begin{aligned} &(\widehat{N-3}, N-2, N-1) = \frac{1}{45}x^6, \ (\widehat{N-3}, N, N+1) = x^2, \\ &(\widehat{N-3}, N-2, N) = \frac{2}{15}x^5, \ (\widehat{N-3}, N-1, N+1) = x^3, \\ &(\widehat{N-3}, N-2, N+1) = \frac{1}{3}x^4, \ (\widehat{N-3}, N-1, N) = \frac{1}{3}x^4, \end{aligned}$$

and

$$\begin{aligned} &(\widehat{N-1}, N+2, N+3) = \frac{2}{2835}x^{11}, \ (\widehat{N-1}, N, N+1) = \frac{1}{4465125}x^{15}, \\ &(\widehat{N-1}, N+1, N+3) = \frac{1}{4725}x^{12}, \ (\widehat{N-1}, N, N+2) = \frac{1}{297675}x^{14}, \\ &(\widehat{N-1}, N+1, N+2) = \frac{1}{42525}x^{13}, \ (\widehat{N-1}, N, N+3) = \frac{1}{42525}x^{13}; \end{aligned}$$

and when $N = 4$, we have

$$\begin{aligned} &(\widehat{N-3}, N-2, N-1) = \frac{1}{4725}x^{10}, \ (\widehat{N-3}, N, N+1) = \frac{1}{9}x^6, \\ &(\widehat{N-3}, N-2, N) = \frac{2}{945}x^9, \ (\widehat{N-3}, N-1, N+1) = \frac{17}{315}x^7, \\ &(\widehat{N-3}, N-2, N+1) = \frac{1}{105}x^8, \ (\widehat{N-3}, N-1, N) = \frac{1}{105}x^8, \end{aligned}$$

and

$$\begin{aligned} &(\widehat{N-1}, N+2, N+3) = \frac{2}{7016625}x^{17}, \\ &(\widehat{N-1}, N, N+1) = \frac{1}{46414974375}x^{21}, \\ &(\widehat{N-1}, N+1, N+3) = \frac{26}{442047375}x^{18}, \\ &(\widehat{N-1}, N, N+2) = \frac{1}{2210236875}x^{20}, \\ &(\widehat{N-1}, N+1, N+2) = \frac{2}{442047375}x^{19}, \\ &(\widehat{N-1}, N, N+3) = \frac{2}{442047375}x^{19}. \end{aligned}$$

Therefore, the identity (2.5) for generalized Wronskians with the entries in (2.9) holds when $N = 3$ and $N = 4$, indeed.

The identity (2.5) can be used to show an interesting property about rational solutions, including rogue wave solutions, to the KdV equation. The KdV equation (1.1) is transformed into a Hirota bilinear form

$$(D_x^4 + D_x D_t) f \cdot f = 2(f_{xt} f - f_t f_x + f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2) = 0, \tag{2.10}$$

under $u = -2(\ln f)_{xx}$.[10] Obviously, a polynomial solution f to the bilinear KdV equation (2.10) will lead to a rational solution to the KdV equation (1.1) by the indicated transformation. Let $N \geq 0$ be an arbitrary integer. Assume that f_N is a polynomial solution, defined by the Wronskian[10]:

$$f_N = (\widehat{N-1}) = W(\phi_0, \phi_1, \ldots, \phi_{N-1}), \tag{2.11}$$

where ϕ_i, $i \geq 0$, are polynomial functions of x and t, determined by

$$\psi(\eta) = 2\sinh(\eta x - 4\eta^3 t) = \sum_{i=0}^{\infty} \phi_i \eta^{2i+1}. \tag{2.12}$$

The corresponding rational solutions can be used to generate rogue wave solutions through the x-translational (or t-translational) invariance and the Galilean invariance of the KdV equation, as illustrated in the introduction.

Obviously, $f_1 + f_3$ and $f_2 + f_4$ still present solutions to the bilinear KdV equation (2.10). When $N \geq 3$, we can show that it is equivalent to the identity in (2.5) that $f_N + f_{N+2}$ is again a solution to the bilinear KdV equation (2.10), or more generally, an arbitrary linear combination of f_N and f_{N+2} is again a solution (see Refs. 13, 15 for more illustrative examples). This is pretty rare, since (2.10) is bilinear, not linear at all.

It is also direct to check by symbolic computation that the Boussinesq equation does not have such a property for Wronskian rational solutions (see, e.g., Ref. 16 for such solutions), and that more general linear combinations $f_N + a f_{N+1} + b f_{N+2} + c f_{N+3} + d f_{N+4}$ do not produce any other solution than $f_N + b f_{N+2}$ for the KdV equation, where a, b, c, d are constants.

3. Concluding remarks

We have discussed about a polynomial conjecture and exhibited two illustrative examples of it. The conjecture is equivalent to say that a linear combination of two Wronskian polynomial solutions with a difference two between the Wronskian orders is again a solution to the bilinear KdV equation. But there is no other solution among linear combinations $f_N + af_{N+1} + bf_{N+2} + cf_{N+3} + df_{N+4}$, where f_m is the Wronskian of order m defined by (2.11) and a, b, c, d are constants.

There exist various studies on lump solutions to nonlinear dispersive wave equations (see, e.g., Refs. 17, 18) and different nonlinear terms can join together to formulate such interesting solutions.[19,20] Recently, N-soliton solutions have been extensively studied for (2+1)-dimensional integrable equations by the Hirota bilinear method[21] and for nonlocal integrable equations by associated Riemann–Hilbert problems.[22,23] It would be extremely helpful in exploring soliton dynamics to establish clear connections between lump solutions and N-soliton solutions for both local and nonlocal integrable equations.

Declaration of competing interest

The author declares that there is no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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