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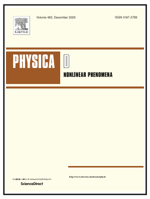

Matrix integrable hierarchies connected with the symplectic Lie algebras sp(2m) and their bi-Hamiltonian structures and Darboux transformations

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ABSTRACT

This study introduces a framework of matrix spectral problems associated with the general symplectic Lie algebras sp(2m), and establishes their corresponding integrable hierarchies through the zero-curvature formulation. The trace identity is employed to establish the bi-Hamiltonian structures, while the associated Lax pairs ensure the existence of Darboux transformations. Furthermore, the N-fold Darboux transformation is systematically formulated through iterations of first-order Darboux transformations, and an explicit single-step application is also presented.

1. Introduction

In soliton theory, Lax pairs play a central role in the study of integrable systems. The concept of a Lax pair [1] involves formulating a linear eigenvalue problem associated with a given matrix Lie algebra. By constructing appropriate solutions to the corresponding stationary zero-curvature equation, one can generate a commuting hierarchy of model equations with remarkable integrable properties, including infinitely many symmetries and conserved Hamiltonian functionals [2,3].

The associated Lax pairs enable the explicit construction of general soliton solutions, which are closely related to the inverse scattering transform [4–6], the Darboux transformation (see, e.g., [7–10]), the Riemann-Hilbert problem [11,12], the dressing procedure [13,14], vertex operators [15–18], and τ-functions (see, e.g., [19,20]). These constructions are often framed in terms of affine (also called Kac-Moody or Euclidean) Lie algebras (see, e.g., [3,17]), specifically within Hermitian symmetric spaces (see, e.g., [2,21–23]).

In the formulation of Lax pairs, we typically denote the potential (or the dependent variable) by u, and the spectral parameter by λ. A Lax pair consists of a set of linear eigenvalue problems:

$$\big(\partial_x - U(u,\lambda)\big)\phi = 0,\ \big(\partial_t - V\big(u,\lambda\big)\big)\phi = 0, \tag{1.1}$$

or equivalently,

$$\phi_x = U(u,\lambda)\phi,\ \phi_t = V(u,\lambda)\phi, \tag{1.2}$$

where ϕ is the eigenfunction, and U and V, referred to as the spatial and temporal spectral matrices, respectively, belong to the loop algebra $\tilde{g}$ associated with a Lie algebra g:

$$\tilde{g} = \Big\{ \sum_{l\le k} g_l\lambda^l \mid g_l \in g,\ k \in \mathbb{Z} \Big\}. \tag{1.3}$$

The loop algebra $\tilde{g}$ can be decomposed into a direct sum:

$$\tilde{g} = \tilde{g}_+ \oplus \tilde{g}_-, \tag{1.4}$$

where the two sub-Lie algebras are defined as

$$\tilde{g}_+ = \Big\{ \sum_{0\le l\le k} g_l\lambda^l \mid g_l \in g,\ k \ge 0 \Big\},\ \tilde{g}_- = \Big\{ \sum_{l\le 0} g_l\lambda^l \mid g_l \in g \Big\}. \tag{1.5}$$

The way the potential and the spectral parameter enter the Lax pair is essential for characterizing the associated integrable system. The compatibility condition, also known as the zero-curvature condition, is expressed as

$$[\partial_x - U, \partial_t - V] = 0,\ \text{ or equivalently, }\ U_t - V_x + [U,V] = 0, \tag{1.6}$$

where $[U,V]$ denotes the Lie bracket of U and V. This condition ensures that the eigenfunction ϕ evolves consistently in both the spatial and temporal directions, thereby leading to the integrable system.

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To illustrate, let us consider the Ablowitz-Kaup-Newell-Segur (AKNS) system, a classical framework for generating scalar integrable equations. The AKNS spectral problems [24] are defined through the Lax pair matrices in $\widetilde{\mathrm{sl}}(2,\mathbb{C})$:

$$U(u,\lambda)=\begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix},\ V(u,\lambda)=\begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \tag{1.7}$$

where p and q are scalar potentials, and a, b, and c are polynomials in the spectral parameter λ with differential polynomial coefficients in u. The explicit forms of these entries depend on the particular integrable equation being modeled. Imposing the zero-curvature condition (1.6) produces nonlinear evolution equations for u. A prototypical example is the nonlinear Schrödinger (NLS) system:

$$p_t = p_{xx} - 2p^2 q,\ \ q_t = -q_{xx} + 2pq^2, \tag{1.8}$$

which arises from the specific choice:

$$a = 2\lambda^2 - pq,\ \ b = 2\lambda p + p_x,\ \ c = 2\lambda q - q_x. \tag{1.9}$$

The integrable structure of such systems is characterized by their associated spectral problems, which give rise to soliton solutions, and infinite hierarchies of symmetries and conserved Hamiltonian quantities.

By appropriately selecting U and V, one can systematically derive many classical integrable systems, including the sine-Gordon and Korteweg-de Vries equations. These systems not only possess remarkable integrable properties but also admit powerful analytical approaches, including the inverse scattering transform and the Darboux transformation. More generally, the Lax pair (or equivalently, the zero-curvature) formulation provides a unifying principle for constructing Liouville-integrable Hamiltonian hierarchies associated with various Lie algebras: special linear algebras (see, e.g., [24–32]), special orthogonal algebras (see, e.g., [33–35]), and non-semisimple Lie algebras (see, e.g., [36–39]). Such hierarchies play a central role in modern integrable systems theory, offering a systematic framework for revealing soliton dynamics, Hamiltonian structures, and underlying conservation laws and symmetries.

Hamiltonain formulation:

Hamiltonian structures are fundamental in the study of integrable systems, providing a rigorous framework for analyzing the integrability and conservation properties of the associated models. One of the most effective approaches for generating Hamiltonian structures is based on the trace identity (see [25,26]) for semi-simple Lie algebras and its generalization, the variational identity (see [27]), for non-semi-simple Lie algebras. In particular, the trace identity has proven to be a powerful and elegant tool for constructing bi-Hamiltonian formulations and establishing the integrability of nonlinear evolution equations.

Let $W \in \tilde{g}_-$ be a solution to the stationary zero-curvature equation

$$[\partial_x - U, W] = 0,\quad \text{i.e.,}\quad W_x - [U, W] = 0. \tag{1.10}$$

Geometrically, the left hand side can be interpreted as

$$\mathrm{ad}^*_W(\partial_x - U) = 0, \tag{1.11}$$

via the coadjoint action ad^* of $\tilde{g}$ on the dual space of a central extension algebra $\hat{g}$ of $\tilde{g}$.

The trace identity provides a fundamental link between the Lax representation and the Hamiltonian structure of an integrable hierarchy (see [25] for details):

$$\frac{\delta}{\delta u}\int \mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\,\mathrm{tr}\Big(W\frac{\partial U}{\partial u}\Big), \tag{1.12}$$

where $\frac{\delta}{\delta u}$ denotes the variational derivative with respect to u and tr represents the trace of a matrix, and γ is a constant independent of the spectral parameter λ. In particular, γ can be determined by

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln \mathrm{tr}\,(W^2). \tag{1.13}$$

This identity establishes a direct correspondence between the spectral problem and the Hamiltonian formulation of the hierarchy, elucidating how the variational derivative of a trace functional involving the spectral parameter encapsulates the integrable structure of the hierarchy.

Darboux transformation:

A Darboux transformation consists of a gauge transformation

$$\phi' = D\phi \tag{1.14}$$

accompanied by a transformed potential $u' = u'(u)$, where $D = D(u, \lambda)$ is a matrix function depending on the potential u and the spectral parameter λ. The new eigenfunction ϕ' is required to satisfy the same type of matrix spectral problems:

$$\phi'_x = U'\phi' = U(u',\lambda)\phi',\ \ \phi'_t = V'\phi' = V(u',\lambda)\phi', \tag{1.15}$$

where U' and V' preserve the structure form of the original Lax pair matrices (see, e.g., [40,41]). The matrix D, referred to as the Darboux matrix, must satisfy the spatial and temporal compatibility conditions:

$$U'D = DU + D_x,\ \ V'D = DV + D_t, \tag{1.16}$$

which provide a constructive scheme for generating new Lax pairs. The Darboux transformation can also be derived from the invariant conditions under the adjoint action of the corresponding Lie group $\hat{G}$ on the centrally extended algebra $\hat{g}$:

$$D(\partial_x - U)D^{-1} = \partial_x - U',\ \ D(\partial_t - V)D^{-1} = \partial_t - V'. \tag{1.17}$$

Assume the matrices U and V are of order n. A commonly used first-order Darboux matrix takes the form:

$$D(\lambda) = \lambda I_n - S, \tag{1.18}$$

where I_n denotes the $n \times n$ identity matrix, and S is an $n \times n$ matrix independent of the spectral parameter λ.

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be distinct eigenvalues with corresponding eigenfunctions $\phi^{[i]}$ satisfying:

$$\phi^{[i]}_x U(u,\lambda_i)\phi^{[i]},\ \ \phi^{[i]}_t = V(u,\lambda_i)\phi^{[i]},\ \ 1 \le i \le n, \tag{1.19}$$

where u is a known solution of (1.6). Define the matrices

$$H = (\phi^{[1]}, \cdots, \phi^{[n]}),\ \ A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n), \tag{1.20}$$

and then the matrix S can be expressed as:

$$S = HAH^{-1}, \tag{1.21}$$

which is equivalent to the relations

$$D(\lambda_i)\phi^{[i]} = 0,\ \ 1 \le i \le n. \tag{1.22}$$

This means that the Darboux matrix $D(\lambda)$ annihilates all the chosen eigenfunctions. The transformed potential $u' = u'(u)$ thereby defines a new solution to the integrable model (1.6). This Darboux framework is applicable to general AKNS-type flows in both lower- and higher-dimensional cases [40–43].

This paper introduces a class of matrix spectral problems associated with the symplectic Lie algebras $\mathrm{sp}(2m)$ and constructs corresponding matrix Liouville integrable hierarchies via the zero-curvature formulation. The resulting hierarchies are demonstrated to possess bi-Hamiltonian structures by means of the trace identity. Darboux transformations are developed using the annihilating generating procedure, and a concrete application of the first-order Darboux transformation is presented. The paper concludes with a summary of the main results and final remarks.

2. Matrix sp(2*m*)-integrable hierarchies

Let m be a given natural number. We consider the symplectic Lie algebra:

$$\mathrm{sp}(2m) = \mathrm{sp}(2m,\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}_{2m\times 2m} \middle|\, B^T = B,\ C^T = C \right\}, \tag{2.1}$$

and denote by $\widetilde{\mathrm{so}}(2m)$ its loop algebra, defined as the algebra of Laurent series in the spectral parameter λ with coefficients in $\mathrm{sp}(2m)$, as introduced in (1.3).

Based on the sub-loop algebra $\widetilde{\mathrm{sp}}(2m)_+$, we propose and study a matrix eigenvalue problem of the form:

$$\phi_x = U\phi = U(u,\lambda)\phi, \tag{2.2}$$

where λ is again the eigenvalue parameter, and the potential u consists of two symmetric $m \times m$ matrices p and q, and is expressed as

$$u = (p^T, q^T)^T = \begin{bmatrix} p \\ q \end{bmatrix}. \tag{2.3}$$

The matrix U is given by

$$U = \begin{bmatrix} \lambda I_m & p \\ q & -\lambda I_m \end{bmatrix} = \lambda\Lambda + P,\ p^T = p,\ q^T = q, \tag{2.4}$$

with

$$\Lambda = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix},\ P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}. \tag{2.5}$$

Here, I_m denotes the $m \times m$ identity matrix. Note that the matrix Λ lies in the Cartan subalgebra of $\widetilde{\mathrm{sp}}(2m)$. When $m = 1$, the spectral problem reduces to the standard scalar AKNS eigenvalue problem [24], thus providing a matrix generalization of the AKNS system. For $m > 1$, the matrix Λ defined above admits multiple eigenvalues, and the kernel of the adjoint ad_Λ (where $\mathrm{ad}_E F = [E, F]$) is noncommutative. Consequently, one cannot directly apply the Drinfeld-Sokolov scheme [2]. Typically, this generating scheme requires the matrix Λ to have distinct eigenvalues (see, e.g., [3,22]).

To establish the corresponding Liouville integrable hierarchy, we first solve the associated stationary zero-curvature Eq. (1.10) by seeking a Laurent series solution in the sub-loop algebra $\widetilde{\mathrm{sp}}(2m)_-$, which can be written as:

$$W = \begin{bmatrix} a & b \\ c & -a^T \end{bmatrix} = \sum_{l\geq 0} \lambda^{-l} W^{[l]},\ b^T = b,\ c^T = c, \tag{2.6}$$

where three fundamental components are expanded as Laurent series in the spectral parameter λ:

$$a = \sum_{l\geq 0} \lambda^{-l} a^{[l]},\ b = \sum_{l\geq 0} \lambda^{-l} b^{[l]},\ c = \sum_{l\geq 0} \lambda^{-l} c^{[l]}. \tag{2.7}$$

It is straightforward to verify that the corresponding associated stationary zero-curvature Eq. (1.10) leads to the following relations:

$$\begin{cases} a_x = pc - bq, \\ b_x = 2\lambda b - pa^T - ap, \\ c_x = qa + a^T q - 2\lambda c. \end{cases} \tag{2.8}$$

This system allows the Laurent series solution W to be determined recursively.

Furthermore, the system (2.8) implies the initial conditions:

$$a_x^{[0]} = 0,\ b^{[0]} = c^{[0]} = 0, \tag{2.9}$$

and provides the recursion relations for the coefficients of the Laurent expansion:

$$\begin{cases} b^{[l+1]} = \frac{1}{2}(b_x^{[l]} + pa^{[l]T} + a^{[l]}p), \\ c^{[l+1]} = -\frac{1}{2}(c_x^{[l]} - qa^{[l]} - a^{[l]T}q), \\ a_x^{[l+1]} = pc^{[l+1]} - b^{[l+1]}q, \end{cases} \tag{2.10}$$

where the superscript $[l]$ denotes the coefficient at order λ^{-l} for $l \geq 0$.

As usual, to specify a unique Laurent series solution, we impose the initial condition

$$a^{[0]} = I_m, \tag{2.11}$$

and take the integration constants to vanish:

$$a^{[l]}|_{u=0} = 0,\ l \geq 1. \tag{2.12}$$

Under these conditions, the sequences of $\{a^{[l]}, b^{[l]}, c^{[l]}\}$ for $l \geq 1$ can be derived recursively and uniquely. The first few terms of these sequences are given by:

$$b^{[1]} = p,\ c^{[1]} = q,\ a^{[1]} = 0;$$

$$b^{[2]} = \tfrac{1}{2}p_x,\ c^{[2]} = -\tfrac{1}{2}q_x,\ a^{[2]} = -\tfrac{1}{2}pq;$$

$$b^{[3]} = \tfrac{1}{4}(p_{xx} - 2pqp),\ c^{[3]} = \tfrac{1}{4}(q_{xx} - 2qpq),\ a^{[3]} = \tfrac{1}{4}(pq_x - p_x q);$$

$$\begin{cases} b^{[4]} = \frac{1}{8}(p_{xxx} - 3pqp_x - 3p_x qp), \\ c^{[4]} = -\frac{1}{8}(q_{xxx} - 3q_x pq - 3qpq_x), \\ a^{[4]} = -\frac{1}{8}(pq_{xx} - p_x q_x + p_{xx} q - 3pqpq); \end{cases}$$

$$\begin{cases} b^{[5]} = \frac{1}{16}\left[p_{xxxx} - 4pqp_{xx} - 2(3p_x q + pq_x)p_x - 2(2p_{xx}q + p_x q_x + pq_{xx} - 3pqpq)p\right], \\ c^{[5]} = \frac{1}{16}\left[q_{xxxx} - 4q_{xx}pq - 2q_x(3pq_x + p_x q) - 2q(2pq_{xx} + p_x q_x + p_{xx}q - 3pqpq)\right], \\ a^{[5]} = -\frac{1}{16}\left[2p(q_x p - qp_x)q - 4p_x qpq + 4pqpq_x + p_{xxx}q - pq_{xxx} + p_x q_{xx} - p_{xx}q_x\right]; \end{cases}$$

$$\begin{cases} b^{[6]} = \frac{1}{32}(p_{xxxxx} - 5p_{xxx}qp - 5pqp_{xxx} - 10p_{xx}qp_x - 10p_x qp_{xx} - 5p_{xx}q_x p - 5pq_x p_{xx} \\ \qquad -5p_x q_{xx} p - 5pq_{xx}p_x - 10p_x q_x p_x + 10p_x qpqp + 10pqp_x qp + 10pqpqp_x), \\ c^{[6]} = -\frac{1}{32}(q_{xxxxx} - 5q_{xxx}pq - 5qpq_{xxx} - 10q_{xx}pq_x - 10q_x pq_{xx} - 5q_{xx}p_x q - 5qp_x q_{xx} \\ \qquad -5q_x p_{xx} q - 5qp_{xx}q_x - 10q_x p_x q_x + 10q_x pqpq + 10qpq_x pq + 10qpqpq_x), \\ a^{[6]} = -\frac{1}{32}(p_{xxxx}q + pq_{xxxx} - p_{xxx}q_x - p_x q_{xxx} + p_{xx}q_{xx} - 5pqpq_{xx} - 5pqp_{xx}q - 5pq_{xx}pq \\ \qquad -5p_{xx}qpq - 5p_x qp_x q - 5pq_x pq_x - 5pq_x p_x q + 5p_x qpq_x + 10pqpqpq), \end{cases}$$

where p and q are symmetric $m \times m$ matrix potentials.

On the basis of the above computations, we set $\Delta_k = 0$ $(k \geq 0)$ to introduce the temporal matrix eigenvalue problems:

$$\phi_{t_k} = V^{[k]}\phi = V^{[k]}(u,\lambda)\phi,\ V^{[k]} = (\lambda^k W)_+ = \sum_{l=0}^{k} \lambda^l W^{[k-l]},\ k \geq 0. \tag{2.13}$$

These present the temporal parts of the Lax pairs associated with the spatial problem (2.4). The compatibility (zero-curvature) conditions ensuring the solvability of the spatial and temporal eigenvalue problems are given by

$$U_{t_k} - V_x^{[k]} + [U, V^{[k]}] = 0,\ k \geq 0. \tag{2.14}$$

These zero-curvature equations generate a hierarchy of integrable systems involving two matrix-valued potentials,

$$u_{t_k} = X^{[k]}(u) = \begin{bmatrix} 2b^{[k+1]} \\ -2c^{[k+1]} \end{bmatrix}, \tag{2.15}$$

or equivalently,

$$p_{t_k} = 2b^{[k+1]},\ q_{t_k} = -2c^{[k+1]},\ k \geq 0, \tag{2.16}$$

where p and q are symmetric $m \times m$ matrix-valued functions.

As particular examples, this integrable hierarchy includes coupled systems of integrable NLS equations and modified Korteweg-de Vries (mKdV) equations, as well as their higher-order generalizations:

$$p_{t_2} = \frac{1}{2}(p_{xx} - 2pqp),\ q_{t_2} = -\frac{1}{2}(q_{xx} - 2qpq); \tag{2.17}$$

$$p_{t_3} = \frac{1}{4}(p_{xxx} - 3pqp_x - 3p_x qp),\ q_{t_3} = \frac{1}{4}(q_{xxx} - 3q_x pq - 3qpq_x); \tag{2.18}$$

$$\begin{cases} p_{t_4} = \frac{1}{8}[p_{xxxx} - 2(2p_{xx}qp + pq_{xx}p + 2pqp_{xx}) - 2(p_x q_x p + 3p_x qp_x + pq_x p_x) + 6pqpqp], \\ q_{t_4} = -\frac{1}{8}[q_{xxxx} - 2(2q_{xx}pq + qp_{xx}q + 2qpq_{xx}) - 2(q_x p_x q + 3q_x pq_x + qp_x q_x) + 6qpqpq], \end{cases} \tag{2.19}$$

where $p^T = p$ and $q^T = q$. These models are typical examples of coupled integrable systems, further extending the class of multi-component integrable NLS and mKdV-type systems (see, e.g., [44–46]).

3. Bi-Hamiltonian structures

The bi-Hamiltonian structures of the integrable hierarchy (2.16) can be systematically constructed by applying the classical trace identity (1.12) to the spatial matrix eigenvalue problem (2.4).

3.1. Application of the trace identity to Hamiltonian structures

The trace identity involves the solution W defined in (2.6), based on which the Hamiltonian densities of the hierarchy (2.16) can be systematically derived. Concretely, we have

$$\mathrm{tr}\Big(W\frac{\partial U}{\partial\lambda}\Big) = 2\,\mathrm{tr}\,a,\ \mathrm{tr}\Big(W\frac{\partial U}{\partial u}\Big) = (c,b)^T = \left[\begin{array}{c} c^T \\ b^T \end{array}\right], \tag{3.1}$$

and consequently, the classical trace identity (1.12) in this case yields

$$\frac{\delta}{\delta u}\int \lambda^{-(l+1)}(2\,\mathrm{tr}\,a^{[l+1]})\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma-l}(c^{[l]},b^{[l]})^T,\ l\geq 0.$$

When evaluated at $l=2$, this relation yields $\gamma=0$, and therefore, we obtain

$$\frac{\delta}{\delta u}\mathcal{H}^{[k]} = (c^{[k+1]},b^{[k+1]})^T,\ k\geq 0, \tag{3.2}$$

where the Hamiltonian functionals are computed as

$$\mathcal{H}^{[k]} = -\int \frac{2}{k+1}(\mathrm{tr}\,a^{[k+2]})\,dx,\ k\geq 0. \tag{3.3}$$

This allows us to establish the Hamiltonian structures for the integrable hierarchy (2.16):

$$u_{t_k} = X^{[k]} = J_1\frac{\delta\mathcal{H}^{[k]}}{\delta u},\ J_1 = \left[\begin{array}{cc} 0 & 2I_m \\ -2I_m & 0 \end{array}\right],\ k\geq 0, \tag{3.4}$$

where J_1 is clearly skew-symmetric and therefore Hamiltonian, and $\mathcal{H}^{[k]}$ are the functionals defined above. An essential property of Hamiltonian structures is the interrelation $S = J_1\frac{\delta\mathcal{H}}{\delta u}$ between a conserved functional $\mathcal{H}$ and a symmetry S within the same nonlinear model.

The standard soliton theory asserts that the vector fields $X^{[k]}$ commute:

$$[\![X^{[k_1]},X^{[k_2]}]\!] = X^{[k_1]\prime}(u)[X^{[k_2]}] - X^{[k_2]\prime}(u)[X^{[k_1]}] = 0,\ k_1,k_2\geq 0. \tag{3.5}$$

This property can be seen from an algebra of temporal spectral matrices:

$$[\![V^{[k_1]},V^{[k_2]}]\!] = V^{[k_1]\prime}(u)[X^{[k_2]}] - V^{[k_2]\prime}(u)[X^{[k_1]}] + [V^{[k_1]},V^{[k_2]}] = 0, \tag{3.6}$$

where $k_1,k_2\geq 0$ and $R'(u)[X]$ denotes the Gateaux derivative:

$$R'(u)[X] = \lim_{\varepsilon\to 0}\frac{R(u+\varepsilon X)-R(u)}{\varepsilon}.$$

This commutativity can also be verified directly by analyzing the relationship between the isospectral ($\lambda_{t_k}=0$) zero-curvature equations (see, e.g., [47] for details).

3.2. Recursion operators and bi-Hamiltonian structures

Moreover, by employing the recursion relation $X^{[k+1]} = \Phi X^{[k]}$, a straightforward yet lengthy computation results in a recursion operator $\Phi = (\Phi_{ij})_{2\times 2}$, which is established as hereditary [48], for the integrable hierarchy (2.16). This hereditary recursion operator Φ reads:

$$\Phi = \left[\begin{array}{cc} \frac{1}{2}\partial - \frac{1}{2}p\big(\partial^{-1}q(\cdot)\big) - \frac{1}{2}\big(\partial^{-1}(\cdot)q\big)p & -\frac{1}{2}p\big(\partial^{-1}(\cdot)p\big) - \frac{1}{2}\big(\partial^{-1}p(\cdot)\big)p \\ \frac{1}{2}q\big(\partial^{-1}(\cdot)q\big) + \frac{1}{2}\big(\partial^{-1}q(\cdot)\big)q & -\frac{1}{2}\partial + \frac{1}{2}q\big(\partial^{-1}p(\cdot)\big) + \frac{1}{2}\big(\partial^{-1}(\cdot)p\big)q \end{array}\right], \tag{3.7}$$

where the operators in Φ_{11} and Φ_{12} are defined as

$$\left\{\begin{array}{l} \big[p\big(\partial^{-1}q(\cdot)\big)\big]X_1 = p\big(\partial^{-1}(qX_1)\big),\ \big[\big(\partial^{-1}(\cdot)q\big)p\big]X_1 = \big(\partial^{-1}(X_1q)\big)p, \\ \big[p\big(\partial^{-1}(\cdot)p\big)\big]X_2 = p\big(\partial^{-1}(X_2p)\big),\ \big[\big(\partial^{-1}p(\cdot)\big)p\big]X_2 = \big(\partial^{-1}(pX_2)\big)p, \end{array}\right.$$

where X_1 and X_2 are $m\times m$ matrices, and the other operators in Φ_{21} and Φ_{22} are defined similarly.

Despite the nonlocality of the recursion operator defined by (3.7), the locality of the isospectral flows is maintained. The Hamiltonian formulation implies that each flow in the hierarchy preserves the integrable structure, ensuring that the derived soliton equations remain solvable by the inverse scattering transform and other analytical methods applicable to local soliton equations.

Furthermore, we can work out a second skew-symmetric operator:

$$J_2 = \Phi J_1 = \left[\begin{array}{cc} p\big(\partial^{-1}(\cdot)p\big) + \big(\partial^{-1}p(\cdot)\big)p & \partial - p\big(\partial^{-1}q(\cdot)\big) - \big(\partial^{-1}(\cdot)q\big)p \\ \partial - q\big(\partial^{-1}p(\cdot)\big) - \big(\partial^{-1}(\cdot)p\big)q & q\big(\partial^{-1}(\cdot)q\big) + \big(\partial^{-1}q(\cdot)\big)q \end{array}\right], \tag{3.8}$$

and with some detailed, albeit lengthy, analysis, we can show that J_1 and J_2 constitute a Hamiltonian pair (see [49] for a detailed proof in the case $m=1$). Therefore, the integrable hierarchy (2.16) exhibits the following bi-Hamiltonian structures [50]:

$$u_{t_k} = X^{[k]} = J_1\frac{\delta\mathcal{H}^{[k]}}{\delta u} = J_2\frac{\delta\mathcal{H}^{[k-1]}}{\delta u},\ \ k\geq 1. \tag{3.9}$$

It can then be observed that the resulting Hamiltonian functionals commute:

$$\{\mathcal{H}^{[k_1]},\mathcal{H}^{[k_2]}\}_{J_i} = 0,\ k_1,k_2\geq 0,\ i=1,2, \tag{3.10}$$

under the corresponding Poisson brackets:

$$\{\mathcal{H},\mathcal{K}\}_{J_i} = \int\Big(\frac{\delta\mathcal{H}}{\delta u}\Big)^T J_i\frac{\delta\mathcal{K}}{\delta u}\,dx,\ i=1,2. \tag{3.11}$$

The two commutativity equalities established above, (3.5) and (3.10), imply that all isospectral flows possess infinitely many conserved quantities and symmetries intrinsic to integrable systems. Moreover, leveraging the recursive and bi-Hamiltonian structures, these conserved quantities and symmetries can be systematically computed and effectively utilized. This property is fundamental to the practical analysis and application of integrable systems, as it guarantees that their solutions exhibit well-defined physical behavior and can be explored through rigorous mathematical frameworks.

In summary, the integrable hierarchy (2.16) admits a bi-Hamiltonian formulation, thereby exhibiting Liouville integrability. Each member of the hierarchy possesses infinitely many commuting conserved quantities $\{\mathcal{H}^{[l]}\}_{l=0}^{\infty}$ and symmetries $\{X^{[l]}\}_{l=0}^{\infty}$. The concrete examples (2.17)–(2.19) illustrate specific nonlinear coupled Liouville-integrable models endowed with bi-Hamiltonian structures, further enriching the ongoing developments in the literature (see, e.g., [51–55]).

4. Darboux transformations

We now turn to the construction of a class of Darboux transformations for the matrix integrable hierarchies associated with the symplectic Lie algebras presented above, via the annihilating generating procedure.

4.1. Compatibility conditions in the Darboux transformation

Let us consider the first-order Darboux matrix in λ:

$$D(\lambda) = \lambda I_{2m} - S, \tag{4.1}$$

where S is an auxiliary matrix to be determined. The spatial compatibility condition

$$U'D = DU + D_x,$$

with

$$U' = \lambda\Lambda + P',\ P' = \left[\begin{array}{cc} 0 & p' \\ q' & 0 \end{array}\right], p'^T = p',\ q'^T = q' \tag{4.2}$$

leads to

$$(\lambda\Lambda + P')(\lambda I_{2m} - S) = (\lambda I_{2m} - S)(\lambda\Lambda + P) - S_x.$$

By comparing the powers of λ, we obtain

$$P' = P + [\Lambda, S], \tag{4.3}$$

and

$$S_x = P'S - SP = [P + \Lambda S, S]. \tag{4.4}$$

For a fixed $k \geq 0$, the temporal compatibility condition

$$V^{[k]'} D = DV^{[k]} + D_t,$$

with

$$V^{[k]'} = (\lambda^k W')_+ = \sum_{l=0}^{k} W^{[k]'} \lambda^{k-l},\ W' = \sum_{l\geq 0} W^{[l]'} \lambda^{-l},\ W'_x = [U', W'], \tag{4.5}$$

yields

$$V^{[k]'}(\lambda I_{2m} - S) = (\lambda I_{2m} - S)V^{[k]} - S_t.$$

It then follows that

$$\begin{cases} W^{[0]'} = W^{[0]}(= \Lambda), \\ W^{[l]'} = W^{[l]} + W^{[l-1]'} S - SW^{[l-1]},\ 0 < l \leq k, \end{cases} \tag{4.6}$$

and

$$S_t = W^{[k]'} S - SW^{[k]}. \tag{4.7}$$

The expressions in (4.3) and (4.6) determine a new Lax pair of matrices, while the formulas in (4.4) and (4.7) establish the spatial and temporal compatibility conditions necessary for the existence of Darboux transformations.

4.2. *Construction of the Darboux matrix*

Following the annihilating generating procedure (see, e.g., [40]), we choose eigenfunctions $\phi^{[i]}$ satisfying the Lax pair equations:

$$\phi^{[i]}_x = U(u, \lambda_i)\phi^{[i]},\ \phi^{[i]}_t = V^{[k]}(u, \lambda_i)\phi^{[i]},\ 1 \leq i \leq 2m, \tag{4.8}$$

where $\lambda_i \in \mathbb{C},\ 1 \leq i \leq 2m$. Then the matrix S can be constructed as

$$S = HAH^{-1}, \tag{4.9}$$

where H collects the eigenfunctions as columns, and A is the diagonal matrix of their eigenvalues:

$$H = (\phi^{[1]}, \cdots, \phi^{[2m]}),\ A = \mathrm{diag}(\lambda_1, \cdots, \lambda_{2m}). \tag{4.10}$$

This construction equivalently ensures that

$$D(\lambda_i)\phi^{[i]} = 0,\ 1 \leq i \leq 2m, \tag{4.11}$$

indicating that the Darboux matrix $D(\lambda)$ annihilates the selected eigenfunctions.

To verify compatibility, we compute the derivatives of S using the identities

$$H_x = \Lambda HA + PH,\ H_t = \sum_{l=0}^{k} W^{[l]} HA^{k-l}. \tag{4.12}$$

Differentiating S with respect to x, we obtain

$$\begin{aligned} S_x &= H_x AH^{-1} - HA(H^{-1}H_x H^{-1}) \\ &= \Lambda HA^2H^{-1} + PHAH^{-1} - HAH^{-1}(\Lambda HA + PH)H^{-1} \\ &= \Lambda HA^2H^{-1} + PHAH^{-1} - HAH^{-1}\Lambda HAH^{-1} - HAH^{-1}P \\ &= [Q, S] + [\Lambda, S]S, \end{aligned}$$

which verifies the spatial compatibility condition (4.4).

Similarly, differentiating S with respect to t, we compute

$$\begin{aligned} S_t &= H_t AH^{-1} - HA(H^{-1}H_t H^{-1}) \\ &= \Big(\sum_{l=0}^{k} W^{[l]} HA^{k-l}\Big)AH^{-1} - HAH^{-1}\Big(\sum_{l=0}^{k} W^{[l]} HA^{k-l}\Big)H^{-1} \\ &= \sum_{l=0}^{k} W^{[l]} S^{k-l+1} - S\sum_{l=0}^{k} W^{[l]} S^{k-l} = \sum_{l=0}^{k} (\mathrm{ad}_{W^{[l]}} S)S^{k-l}, \end{aligned}$$

where $\mathrm{ad}_E F = [E, F]$ again. On the other hand, we can have

$$W^{[k]'} = W^{[k]} + \sum_{l=0}^{k-1} (\mathrm{ad}_{W^{[l]}} S)S^{k-l-1}, \tag{4.13}$$

from (4.6). These two formulas confirm that S satisfies the temporal compatibility condition (4.7).

4.3. *First-order Darboux transformation*

From the above formulation, we obtain the resulting Darboux transformation:

$$\phi' = D(\lambda)\phi,\ U' = DUD^{-1} + D_x D^{-1},\ V^{[k]'} = DV^{[k]}D^{-1} + D_t D^{-1}, \tag{4.14}$$

where $D(\lambda)$ is defined by (4.1). More specifically, we have

$$D(\lambda) = \lambda I_{2m} - S,\ U' = \lambda\Lambda + P',\ V^{[k]'} = \sum_{l=0}^{k} W^{[k]'} \lambda^{k-l}, \tag{4.15}$$

where P' and $W^{[k]'}$ are defined by (4.4) and (4.6), respectively.

Using the structures of U and U', or directly from (4.4), the new potentials are explicitly given by:

$$p' = p + 2S_{12},\ q' = q - 2S_{21}, \tag{4.16}$$

where S is defined by (4.9), and the subscript notation M_{jk} denotes the (j, k)-block component of the matrix M under the partitioning determined by the spectral matrix U, provided that S_{12} and S_{21} are symmetric.

Since $W^{[0]'} = W^{[0]}$, by the uniqueness of the stationary zero-curvature equation, the two Lax pairs produce the same integrable system. This explains why the Darboux transformation successfully generates new solutions from known ones.

4.4. *General Nth-order Darboux transformation*

We consider an Nth-order Darboux matrix of the form

$$D^{[N]}(\lambda) = \sum_{j=0}^{N} D^{[N]}_{N-j} \lambda^j, \tag{4.17}$$

with $D^{[N]}_0 = I_{2m}$, subject to the conditions

$$D^{[N]}(\lambda_i)\phi^{[i]} = 0,\ 1 \leq i \leq 2mN, \tag{4.18}$$

where $\phi^{[i]}$ are eigenfunctions corresponding to the eigenvalues λ_i, respectively. Define a matrix

$$H^{[N]} = \begin{bmatrix} \phi^{[1]} & \phi^{[2]} & \cdots & \phi^{[2mN]} \\ \lambda_1\phi^{[1]} & \lambda_2\phi^{[2]} & \cdots & \lambda_{2mN}\phi^{[2mN]} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{N-1}\phi^{[1]} & \lambda_2^{N-1}\phi^{[2]} & \cdots & \lambda_{2mN}^{N-1}\phi^{[2mN]} \end{bmatrix}. \tag{4.19}$$

When $\det H^{[N]} \neq 0$, we can solve

$$(D^{[N]}_N, D^{[N]}_{N-1}, \cdots, D^{[N]}_1)H^{[N]} = -(\lambda_1^N\phi^{[1]}, \cdots, \lambda_{2mN}^N\phi^{[2mN]}), \tag{4.20}$$

to determine $D^{[N]}_1, D^{[N]}_2, \cdots, D^{[N]}_N$ and hence the Darboux matrix $D^{[N]}(\lambda)$.

Alternatively, one can iterate the first-order Darboux transformation N times using the eigenvalues $\lambda_i,\ 1 \leq i \leq 2mN$:

$$(u; U, V^{[k]}) \to (u[1]; U[1], V^{[k]}[1]) \to (u[2]; U[2], V^{[k]}[2]) \to \cdots \to (u[N]; U[N], V^{[k]}[N]),$$

to obtain an Nth-order Darboux transformation. Due to the commutativity of Darboux transformations [7,41], we have

$$D^{[N]}(\lambda) = (\lambda I_{2m} - S_1)(\lambda I_{2m} - S_2)\cdots(\lambda I_{2m} - S_N), \tag{4.21}$$

and consequently, the Nth-order Darboux transformation can be expressed explicitly. In particular, the new potential matrix is given by

$$P' = P + [\Lambda, -D^{[N]}_1] = P + [\Lambda, S_1 + S_2 + \cdots + S_N]. \tag{4.22}$$

4.5. *Example: Explicit solutions via a single transformation*

We illustrate the procedure by considering the zero (vacuum) solution as the seed. For each eigenvalue λ_i, the associated eigenfunction

$$\phi^{[i]} = (\phi_1^{[i]T}, \phi_2^{[i]T})^T \tag{4.23}$$

is given explicitly by

$$\begin{cases} \phi_1^{[i]} = \exp(\lambda_i I_m x + \lambda_i^k I_m t)\mu_1^{[i]}, \\ \phi_2^{[i]} = \exp(-\lambda_i I_m x - \lambda_i^k I_m t)\mu_2^{[i]}, \end{cases} \quad 1 \le i \le 2m, \tag{4.24}$$

where $\mu_1^{[i]}, \mu_2^{[i]} \in \mathbb{C}^m$ are arbitrary constant column vectors. Substituting these into the Darboux framework, the new solution is explicitly given by:

$$p' = [\Lambda, S]_{12} = 2S_{12}, \ q' = [\Lambda, S]_{21} = -2S_{21}, \tag{4.25}$$

where $S = HAH^{-1}$ is constructed from the explicit eigenfunctions determined above, as defined in (4.9). The block components are taken with respect to the partitioning induced by the spectral matrix U.

Let us consider the case $m = 2$ and determine conditions that guarantee

$$S_{12}^T = S_{12}, \ S_{21}^T = S_{21}. \tag{4.26}$$

If we take

$$\lambda_1 = \lambda_2, \ \lambda_3 = \lambda_4, \tag{4.27}$$

then, by a symbolic computation, we find that a set of two sufficient conditions is

$$\begin{cases} \mu_{1,1,1}\mu_{2,2,1} + \mu_{1,1,2}\mu_{2,2,2} - \mu_{1,2,1}\mu_{2,1,1} - \mu_{1,2,2}\mu_{2,1,2} = 0, \\ \mu_{3,1,1}\mu_{4,2,1} + \mu_{3,1,2}\mu_{4,2,2} - \mu_{3,2,1}\mu_{4,1,1} - \mu_{3,2,2}\mu_{4,1,2} = 0. \end{cases} \tag{4.28}$$

If we take

$$\lambda_1 = \lambda_2 = \lambda_3, \tag{4.29}$$

then, similarly, a set of two sufficient conditions is

$$\begin{cases} \big[(-\mu_{3,2,2}\mu_{2,1,1} + \mu_{2,2,2}\mu_{3,1,1})\mu_{1,2,1} + (\mu_{2,1,1}\mu_{3,2,1} - \mu_{2,2,1}\mu_{3,1,1})\mu_{1,2,2} \\ +(\mu_{3,2,2}\mu_{2,2,1} - \mu_{2,2,2}\mu_{3,2,1})\mu_{1,1,1}\big]\mu_{4,2,1} + \big[(-\mu_{2,1,2}\mu_{3,2,2} + \mu_{2,2,2}\mu_{3,1,2})\mu_{1,2,1} \\ +(\mu_{2,1,2}\mu_{3,2,1} - \mu_{2,2,1}\mu_{3,1,2})\mu_{1,2,2} + (\mu_{3,2,2}\mu_{2,2,1} - \mu_{2,2,2}\mu_{3,2,1})\mu_{1,1,2}\big]\mu_{4,2,2} = 0, \\ \big[(\mu_{2,1,2}\mu_{3,2,1} - \mu_{2,2,1}\mu_{3,1,2})\mu_{1,1,1} + (-\mu_{2,1,1}\mu_{3,2,1} + \mu_{2,2,1}\mu_{3,1,1})\mu_{1,1,2} \\ +(\mu_{3,1,2}\mu_{2,1,1} - \mu_{2,1,2}\mu_{3,1,1})\mu_{1,2,1}\big]\mu_{4,1,1} + \big[(\mu_{2,1,2}\mu_{3,2,2} - \mu_{2,2,2}\mu_{3,1,2})\mu_{1,1,1} \\ +(-\mu_{3,2,2}\mu_{2,1,1} + \mu_{2,2,2}\mu_{3,1,1})\mu_{1,1,2} + (\mu_{3,1,2}\mu_{2,1,1} - \mu_{2,1,2}\mu_{3,1,1})\mu_{1,2,2}\big]\mu_{4,1,2} = 0. \end{cases} \tag{4.30}$$

In the above expressions, we assume

$$\mu_{i,j,k} = \mu_{j,k}^{[i]}, \ \mu_j^{[i]} = (\mu_{j,1}^{[i]}, \cdots, \mu_{j,m}^{[i]})^T, \ 1 \le i \le 2m, \ j = 1, 2, \ 1 \le k \le m. \tag{4.31}$$

The conditions in (4.28) reflects the orthogonality between the eigenfunctions involved. For example, the first condition in (4.28) can be written as

$$\mu_1^{[1]T}\mu_2^{[2]} - \mu_2^{[1]T}\mu_1^{[2]} = 0,$$

which can also be expressed as

$$\mu^{[1]T}\Delta\mu^{[2]} = 0,$$

indicating that the orthogonality property holds:

$$\phi^{[1]T}\Delta\phi^{[2]} = 0,$$

where

$$\Delta = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}, \ \mu^{[i]} = \begin{bmatrix} \mu_1^{[i]} \\ \mu_2^{[i]} \end{bmatrix}, \ i = 1, 2.$$

Nevertheless, it is not clear what type of algebraic or geometric conditions are represented by those in (4.30).

It is remarkable that the presented Darboux transformation generates new solutions from known ones, illustrating the Bäcklund-type self-consistent closure property of the integrable systems associated with the general symplectic Lie algebras.

5. Concluding remarks

This study explores matrix integrable hierarchies arising from specific matrix eigenvalue problems formulated using the symplectic Lie algebras sp($2m$). Bi-Hamiltonian structures and Darboux transformations are established via the trace identity and the annihilating generating scheme. These bi-Hamiltonian structures are fundamental for understanding the integrable dynamics inherent in such models. In general, starting from the vacuum state, successive iterations of Darboux transformations generate N-soliton solutions.

The concrete examples presented provide specific coupled systems of nonlinear integrable equations, illustrating the practical application of the theoretical framework discussed above and highlight the integrability and rich structure exhibited by these systems. Notably, such systems can also be constructed through group reductions or similarity transformations applied to matrix spectral problems within the matrix AKNS integrable hierarchies (see, e.g., [45,56]). Moreover, group reductions enable the generation of integrable systems involving reflection points in time and space can also be generated (see, e.g., [57]).

Employing Laurent series solutions to solve the stationary zero-curvature equation is a crucial step in the construction process, as it reveals the underlying structure and integrability of the associated models. Applying the trace identity to the matrix eigenvalue problem further deepens our understanding of the bi-Hamiltonian structures inherent in these systems. Darboux transformations are developed to generate soliton solutions from the vacuum state, offering powerful analytical tools and insights into nonlinear dynamics [4,5].

Based on the eigenvalues of the associated matrix spectral problem, general soliton solutions can be classified into negatons, positons and complexitons [58]. These solutions can be further categorized into solitons, breathers, kinks, anti-kinks, lumps, rogue waves, and mixed interaction solutions, some of which arise through specific reductions of wave numbers in general soliton solutions (see, e.g., [59,60]).

Increasing the number of potentials or the order of dependence on the spectral parameter in the spatial spectral matrix naturally leads to larger and more complex integrable systems (see, e.g., [61,62]). Although the complexity of these systems grows accordingly, making their analysis more challenging, the study of such extended integrable systems continues to yield valuable insights into the fundamental principles governing nonlinear dynamics and integrability in mathematical physics [57,63].

CRediT authorship contribution statement

Wen-Xiu Ma: Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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