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Sasa–Satsuma type matrix integrable hierarchies and their Riemann–Hilbert problems and soliton solutions

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ABSTRACT

Sasa–Satsuma type matrix integrable hierarchies are generated from taking two group reductions of replacing the spectral parameter with its complex conjugate and its negative in the matrix AKNS spectral problems. Based on the Lax pairs and the adjoint lax pairs, Riemann–Hilbert problems and thus inverse scattering transforms are formulated for the resulting Sasa–Satsuma type matrix integrable hierarchies, and their soliton solutions are generated from the associated reflectionless Riemann–Hilbert problems.

1. Introduction

Integrable equations are generated from matrix spectral problems and come in hierarchies. Under specific symmetric reductions on potentials, we can obtain reduced integrable equations. Such typical examples include the nonlinear Schrödinger (NLS) equations and the modified Korteweg–de Vries (mKdV) equations. Integrable equations can often be solved by the inverse scattering transform [1,2], the Darboux transformation [3], and the Hirota bilinear method [4], and their soliton solutions can be presented explicitly [5,6]. The Riemann–Hilbert technique [7] has also become a powerful approach to integrable equations and particularly to their soliton solutions. Various integrable equations have been studied by formulating and analyzing their Riemann–Hilbert problems generated from the associated given matrix spectral problems.

We consider the (1+1)-dimensional case. Let x and t be two independent variables, and $u = u(x,t)$, a column vector of dependent variables. A Lax pair of spatial and temporal matrix spectral problems is defined by

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \quad -i\phi_t = V\phi = V(u,\lambda)\phi, \tag{1.1}$$

where i is the unit imaginary number, U and V are square matrices from loop algebras, λ is the spectral parameter and ϕ is a square matrix eigenfunction. We assume that the compatibility condition of the two matrix spectral problems generates an integrable equation

$$u_t = K(u), \tag{1.2}$$

from the zero curvature equation

$$U_t - V_x + i[U,V] = 0, \tag{1.3}$$

where $[\cdot,\cdot]$ denotes the matrix commutator. There is a kind of Lie algebraic structures underlining zero curvature equations, which guarantees the existence of infinitely many symmetries for the associated integrable equations. The adjoint Lax pair of the matrix spectral problems in (1.1) is defined as follows:

$$i\tilde{\phi}_x = \tilde{\phi}U = \tilde{\phi}U(u,\lambda), \ \ i\tilde{\phi}_t = \tilde{\phi}V = \tilde{\phi}V(u,\lambda). \tag{1.4}$$

The corresponding compatibility condition yields the same zero curvature equation as (1.3), and so, it does not bring any additional equations. Both the Lax pair and the adjoint Lax pair form the basis for the subsequent analyses of Riemann–Hilbert problems.

A standard procedure for formulating Riemann–Hilbert problems can be described as follows. It starts from a pair of matrix spectral problems in (1.1) with

$$U = A(\lambda) + P(u,\lambda), \ \ V = B(\lambda) + Q(u,\lambda), \tag{1.5}$$

where A, B are constant commuting square matrices, and P, Q are trace-less square matrices satisfying $\deg_\lambda(P) < \deg_\lambda(A)$ and

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$\deg_\lambda(Q) < \deg_\lambda(B)$. To establish a Riemann–Hilbert problem for the integrable equation (1.2), we adopt the following equivalent Lax pair of matrix spectral problems:

$$\begin{aligned} &\psi_x = i[A(\lambda), \psi] + \check{P}(u, \lambda)\psi, \\ &\psi_t = i[B(\lambda), \psi] + \check{Q}(u, \lambda)\psi, \check{P} = iP, \ \check{Q} = iQ, \end{aligned} \tag{1.6}$$

where ψ is also a square matrix eigenfunction. The equivalence between the matrix spectral problems in (1.1) with (1.5) and the matrix spectral problems in (1.6) follows from the commutativity of A and B. The properties $(\det\psi)_x = (\det\psi)_t = 0$ are two consequences of $\operatorname{tr} P = \operatorname{tr} Q = 0$. There exists a direct connection between the matrix spectral problems in (1.1) with (1.5) and the matrix spectral problems in (1.6):

$$\phi = \psi E_g, \ E_g = e^{iA(\lambda)x + iB(\lambda)t}. \tag{1.7}$$

It is important to note that for the pair of matrix spectral problems in (1.6), we can impose the asymptotic conditions:

$$\psi^{\pm} \to I, \text{ when } x \text{ or } t \to \pm\infty, \tag{1.8}$$

where I denotes the identity matrix. From those two matrix eigenfunctions $\psi^{\pm}$, we can pick the entries to build two generalized matrix Jost solutions $T^{\pm}(x, t, \lambda)$, which are analytic in the upper and lower half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous in the closed upper and lower half-planes $\bar{\mathbb{C}}^+$ and $\bar{\mathbb{C}}^-$, respectively, and establish a Riemann–Hilbert problem on the real line:

$$G^+(x, t, \lambda) = G^-(x, t, \lambda)G_0(x, t, \lambda), \ \lambda \in \mathbb{R}, \tag{1.9}$$

where two unimodular generalized matrix Jost solutions G^+ and G^- and the jump matrix G_0 are all generated from T^+ and T^-, and G^+ and G^- have the same analyticity properties as T^+ and T^-, respectively. The jump matrix G_0 carries all basic scattering data from the scattering matrix $S_g(\lambda)$ of the associated matrix spectral problems, defined through

$$\psi^- E_g = \psi^+ E_g S_g(\lambda). \tag{1.10}$$

Solutions to the resulting Riemann–Hilbert problem (1.9) generate the required generalized matrix Jost solutions to recover the potential of the matrix spectral problems, and thus, solutions to the corresponding integrable equation. Such solutions, G^+ and G^-, can be computed by applying the Sokhotski–Plemelj formula to the difference of G^+ and G^-. Upon observing asymptotic behaviors of the generalized matrix Jost solutions $G^{\pm}$ at infinity of λ, a recovery of the potential is obtained. This also produces the corresponding inverse scattering transforms. Soliton solutions can be presented by solving the reflectionless Riemann–Hilbert problems, i.e., the ones with the identity jump matrix G_0, or computing the corresponding reflectionless inverse scattering transforms.

It is also known that integrable equations can be reduced under group reductions of matrix spectral problems (see, e.g., [8]). The traditional class of such reductions takes the form

$$U^{\dagger}(x, t, \lambda^*) = (U(x, t, \lambda^*))^{\dagger} = CU(x, t, \lambda)C^{-1}, \tag{1.11}$$

where $\dagger$ stands for the Hermitian transpose, C is a constant invertible Hermitian matrix and λ^* is the complex conjugate of λ. In this condition, the crucial point is to replace the spectral parameter λ with its complex conjugate, λ^*, and such reductions work for both the NLS equations and the mKdV equations. There is the unusual class of integrable reductions, which works for the mKdV equations. This class of reductions replaces the spectral parameter λ with its negative, $-\lambda$, in matrix spectral problems, and takes the form

$$U^T(x, t, -\lambda) = (U(x, t, -\lambda))^T = -CU(x, t, \lambda)C^{-1}, \tag{1.12}$$

where T stands for the transpose of a matrix and C is a constant invertible symmetric matrix. Such reductions do not work for the NLS equations. Additionally, it is recognized that the other two replacements $\lambda \to -\lambda^*$ and $\lambda \to \lambda$ only generate nonlocal integrable reductions, together with the reflection transformations of x and t: $(x, t) \to (-x, t)$, $(x, t) \to (x, -t)$ and $(x, t) \to (-x, -t)$ (see, e.g., [9]).

In this paper, we would like to consider two classes of integrable reductions (1.11) and (1.12) simultaneously for the matrix AKNS spectral problems, to generate Sasa–Satsuma type matrix integrable hierarchies, and to formulate Riemann–Hilbert problems and inverse scattering transforms for the resulting reduced matrix integrable equations. We begin with arbitrary-order matrix AKNS spectral problems. The obtained reflectionless Riemann–Hilbert problems are applied to construction of soliton solutions to the corresponding Sasa–Satsuma type matrix integrable hierarchies. The conclusion is given in the last section, together with a few concluding remarks.

2. Sasa–Satsuma type matrix integrable hierarchies

2.1. The matrix AKNS integrable hierarchies revisited

Let us recall the construction of the integrable hierarchies of matrix AKNS equations (see, e.g., [10]).

Assume that $m, n \geq 1$ are two given integers, p, q are two matrix potentials:

$$p = p(x, t) = (p_{jk})_{m\times n}, \ q = q(x, t) = (q_{kj})_{n\times m}, \tag{2.1}$$

I_s denotes the identity matrix of size s, $s \geq 0$, λ is a spectral parameter, and α_1, α_2 and β_1, β_2 are two arbitrary pairs of distinct real constants. Each of the local matrix AKNS integrable hierarchies is generated from the matrix AKNS spectral problems with matrix potentials:

$$-i\phi_x = U\phi = U(u, \lambda)\phi, \ -i\phi_t = V^{[r]}\phi = V^{[r]}(u, \lambda)\phi, \ r \geq 0, \tag{2.2}$$

where the Lax pair of spectral matrices read

$$U = \lambda\Lambda + P, \ V^{[r]} = \lambda^r\Omega + Q^{[r]}, \tag{2.3}$$

in which Λ and Ω are given by

$$\Lambda = \operatorname{diag}(\alpha_1 I_m, \alpha_2 I_n), \ \Omega = \operatorname{diag}(\beta_1 I_m, \beta_2 I_n), \tag{2.4}$$

and the other two involved square matrices of size $m + n$ are defined by

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.5}$$

which is called the potential matrix, and

$$Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \tag{2.6}$$

where $a^{[s]}, b^{[s]}, c^{[s]}$ and $d^{[s]}$ will be defined recursively later.

It is clear that when $m = 1$, the matrix spectral problems in (2.2) reduce to the multicomponent case, and if there are just a pair of nonzero potentials, p_{jk} and q_{kj}, the matrix spectral problems in (2.2) become the standard AKNS case [11].

As normal, to compute an associated matrix AKNS integrable hierarchy, we first solve the stationary zero curvature equation

$$W_x = i[U, W], \tag{2.7}$$

for a given spectral matrix U defined as in (2.3). We look for a solution W of the form

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.8}$$

where a, b, c, d are $m \times m$, $m \times n$, $n \times m$, and $n \times n$ matrices, respectively. The stationary zero curvature equation (2.7) precisely presents

$$\begin{cases} a_x = i(pc - bq), \\ b_x = i(\alpha\lambda b + pd - ap), \\ c_x = i(-\alpha\lambda c + qa - dq), \\ d_x = i(qb - cp), \end{cases} \tag{2.9}$$

where $\alpha = \alpha_1 - \alpha_2$. We take W as a formal Laurent series:

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{s=0}^{\infty} W_s \lambda^{-s}, \quad W_s = W_s(p,q) = \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}, \ s \geq 0, \tag{2.10}$$

and then, the system (2.9) leads equivalently to the following recursion relations:

$$b^{[0]} = 0, \ c^{[0]} = 0, \ a_x^{[0]} = 0, \ d_x^{[0]} = 0, \tag{2.11a}$$

$$b^{[s+1]} = \frac{1}{\alpha}(-ib_x^{[s]} - pd^{[s]} + a^{[s]}p), \ s \geq 0, \tag{2.11b}$$

$$c^{[s+1]} = \frac{1}{\alpha}(ic_x^{[s]} + qa^{[s]} - d^{[s]}q), \ s \geq 0, \tag{2.11c}$$

$$a_x^{[s]} = i(pc^{[s]} - b^{[s]}q), \ d_x^{[s]} = i(qb^{[s]} - c^{[s]}p), \ s \geq 1. \tag{2.11d}$$

Let us now take the initial values:

$$a^{[0]} = \beta_1 I_m, \ d^{[0]} = \beta_2 I_n, \tag{2.12}$$

which implies that

$$V^{[r]} = (\lambda W)_+ := \sum_{s=0}^{r} \lambda^{r-s} W_s, \ r \geq 0; \tag{2.13}$$

and zero constants of integration in (2.11d), which means that we require

$$W_s|_{p,q=0} = 0, \ s \geq 1. \tag{2.14}$$

In this way, with $a^{[0]}$ and $d^{[0]}$ given by (2.12), one can uniquely determine all matrices W_s, $s \geq 1$, defined recursively. For example, we can work out that

$$b^{[1]} = \frac{\beta}{\alpha}p, \ c^{[1]} = \frac{\beta}{\alpha}q, \ a^{[1]} = 0, \ d^{[1]} = 0; \tag{2.15a}$$

$$b^{[2]} = -\frac{\beta}{\alpha^2}ip_x, \ c^{[2]} = \frac{\beta}{\alpha^2}iq_x, \ a^{[2]} = -\frac{\beta}{\alpha^2}pq, \ d^{[2]} = \frac{\beta}{\alpha^2}qp; \tag{2.15b}$$

$$\begin{cases} b^{[3]} = -\dfrac{\beta}{\alpha^3}(p_{xx} + 2pqp), \ c^{[3]} = -\dfrac{\beta}{\alpha^3}(q_{xx} + 2qpq), \\ a^{[3]} = -\dfrac{\beta}{\alpha^3}i(pq_x - p_xq), \ d^{[3]} = -\dfrac{\beta}{\alpha^3}i(qp_x - q_xp); \end{cases} \tag{2.15c}$$

$$\begin{cases} b^{[4]} = \dfrac{\beta}{\alpha^4}i(p_{xxx} + 3pqp_x + 3p_xqp), \\ c^{[4]} = -\dfrac{\beta}{\alpha^4}i(q_{xxx} + 3q_xpq + 3qpq_x), \\ a^{[4]} = \dfrac{\beta}{\alpha^4}[3(pq)^2 + pq_{xx} - p_xq_x + p_{xx}q], \\ d^{[4]} = -\dfrac{\beta}{\alpha^4}[3(qp)^2 + qp_{xx} - q_xp_x + q_{xx}p]; \end{cases} \tag{2.15d}$$

where $\beta = \beta_1 - \beta_2$. Particularly, we can have

$$Q^{[1]} = \frac{\beta}{\alpha}\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} = \frac{\beta}{\alpha}P, \tag{2.16}$$

$$Q^{[2]} = \frac{\beta}{\alpha}\lambda\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2}\begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix} = \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2}I_{m,n}(P^2 + iP_x), \tag{2.17}$$

and

$$\begin{aligned} Q^{[3]} &= \frac{\beta}{\alpha}\lambda^2\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2}\lambda\begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix} \\ &\quad - \frac{\beta}{\alpha^3}\begin{bmatrix} i(pq_x - p_xq) & p_{xx} + 2pqp \\ q_{xx} + 2qpq & i(qp_x - q_xp) \end{bmatrix} \\ &= \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) \\ &\quad - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3), \ . \end{aligned} \tag{2.18}$$

in which $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. Using (2.11d), we can derive, from (2.11b) and (2.11c), a recursion relation for determining $b^{[s]}$ and $c^{[s]}$:

$$\begin{bmatrix} c^{[s+1]} \\ b^{[s+1]} \end{bmatrix} = \Psi \begin{bmatrix} c^{[s]} \\ b^{[s]} \end{bmatrix}, \ s \geq 1, \tag{2.19}$$

where Ψ is a matrix operator

$$\Psi = \frac{i}{\alpha}\begin{bmatrix} \partial_x + q\partial_x^{-1}(p\,\cdot) + [\partial_x^{-1}(\cdot\,p)]q & -q\partial_x^{-1}(\cdot\,q) - [\partial_x^{-1}(q\,\cdot)]q \\ p\partial_x^{-1}(\cdot\,p) + [\partial_x^{-1}(p\,\cdot)]p & -\partial_x - p\partial_x^{-1}(q\,\cdot) - [\partial_x^{-1}(\cdot\,q)]p \end{bmatrix}. \tag{2.20}$$

The compatibility conditions of the two matrix spectral problems in (2.2), i.e., the zero curvature equations

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \ r \geq 0, \tag{2.21}$$

yield one so-called matrix AKNS integrable hierarchy:

$$p_t = i\alpha b^{[r+1]}, \ q_t = -i\alpha c^{[r+1]}, \ r \geq 0. \tag{2.22}$$

The first two nonlinear integrable equations in the hierarchy give us the AKNS matrix NLS equations:

$$p_t = -\frac{\beta}{\alpha^2}i(p_{xx} + 2pqp), \ q_t = \frac{\beta}{\alpha^2}i(q_{xx} + 2qpq), \tag{2.23}$$

and the AKNS matrix mKdV equations:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp), \ q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \tag{2.24}$$

where the two matrix potentials, p and q, are defined by (2.1).

When $m = 1$ and $n = 2$, the matrix NLS equations (2.23) can be reduced to the Manakov system [12], under a group reduction of type (1.11).

By a Lax operator algebra theory and the trace identity [13], we can directly show that (2.22) defines a hierarchy of commuting flows, each of which possesses a bi-Hamiltonian structure and thus infinitely many commuting conservation laws.

2.2. Sasa–Satsuma type matrix integrable equations

Let us now construct a kind of Sasa–Satsuma type integrable reductions of the general integrable matrix AKNS equations in (2.22).

We take a pair of constant invertible Hermitian matrices Σ_1, Σ_2 and another pair of constant invertible symmetric matrices Δ_1, Δ_2, and introduce two particular reductions for the spectral matrix U defined as in (2.3):

$$U^\dagger(x,t,\lambda^*) = (U(x,t,\lambda^*))^\dagger = \Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{2.25}$$

and

$$U^T(x,t,-\lambda) = (U(x,t,-\lambda))^T = -\Delta U(x,t,\lambda)\Delta^{-1}, \tag{2.26}$$

where Σ, Δ are two constant invertible matrices formed as follows:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \ \Sigma_j^\dagger = \Sigma_j, \ j = 1, 2, \tag{2.27}$$

and

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}, \ \Delta_j^T = \Delta_j, \ j = 1, 2. \tag{2.28}$$

These two group reductions exactly demand

$$P^\dagger(x,t) = \Sigma P(x,t)\Sigma^{-1}, \tag{2.29}$$

and

$$P^T(x,t) = -\Delta P(x,t)\Delta^{-1}, \tag{2.30}$$

which enable us to make the reductions for the matrix potentials:

$$q(x,t) = \Sigma_2^{-1}p^\dagger(x,t)\Sigma_1, \tag{2.31}$$

and

$$q(x,t) = -\Delta_2^{-1}p^T(x,t)\Delta_1, \tag{2.32}$$

respectively. It then follows that to satisfy both group reductions for the spectral matrix U, we need an additional constraint for the matrix potential p:

$$\Sigma_2^{-1}p^\dagger(x,t)\Sigma_1 = -\Delta_2^{-1}p^T(x,t)\Delta_1. \tag{2.33}$$

Moreover, we notice that the reductions in (2.25) and (2.26) guarantee that

$$\begin{cases} W^\dagger(x,t,\lambda^*) = (W(x,t,\lambda^*))^\dagger = \Sigma W(x,t,\lambda)\Sigma^{-1}, \\ W^T(x,t,-\lambda) = (W(x,t,-\lambda))^T = \Delta W(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{2.34}$$

which implies that

$$\begin{cases} V^{[2s+1]\dagger}(x,t,\lambda^*) = (V^{[2s+1]}(x,t,\lambda^*))^\dagger = \Sigma V^{[2s+1]}(x,t,\lambda)\Sigma^{-1}, \\ V^{[2s+1]T}(x,t,-\lambda) = (V^{[2s+1]}(x,t,-\lambda))^T = -\Delta V^{[2s+1]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{2.35}$$

and

$$\begin{cases} Q^{[2s+1]\dagger}(x,t,\lambda^*) = (Q^{[2s+1]}(x,t,\lambda^*))^\dagger = \Sigma Q^{[2s+1]}(x,t,\lambda)\Sigma^{-1}, \\ Q^{[2s+1]T}(x,t,-\lambda) = (Q^{[2s+1]}(x,t,-\lambda))^T = -\Delta Q^{[2s+1]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{2.36}$$

where $s \geq 0$, $V^{[2s+1]}$ is defined as in (2.3) and $Q^{[2s+1]}$ is defined by (2.6).

Therefore, under the reductions (2.31) and (2.32), the integrable matrix AKNS equations in (2.22) with $r = 2s+1, \ s \geq 0$, reduce to a hierarchy of Sasa–Satsuma type integrable matrix AKNS equations of odd order:

$$p_t = i\alpha b^{[2s+2]}|_{q=\Sigma_2^{-1}p^\dagger\Sigma_1 = -\Delta_2^{-1}p^T\Delta_1}, \ s \geq 0, \tag{2.37}$$

where $p = (p_{jl})_{m\times n}$ satisfies (2.33), Σ_1, Σ_2 are two arbitrary invertible Hermitian matrices of sizes m and n, respectively, and Δ_1, Δ_2 are two arbitrary invertible symmetric matrices of sizes m and n, respectively. Each equation in the hierarchy (2.37) possesses a Lax pair of the reduced spatial and temporal matrix spectral problems in (2.2) with $r = 2s+1, \ s \geq 0$, and infinitely many symmetries and conservation laws reduced from those for the integrable matrix AKNS equations in (2.22) with $r = 2s+1, \ s \geq 0$.

Let us fix $s = 1$, i.e., $r = 3$. Then the reduced matrix integrable equation in (2.37) gives the Sasa–Satsuma type integrable matrix mKdV equation:

$$\begin{aligned} p_t &= -\frac{\beta}{\alpha^3}(p_{xxx} + 3p\Sigma_2^{-1}p^\dagger\Sigma_1 p_x + 3p_x\Sigma_2^{-1}p^\dagger\Sigma_1 p) \\ &= -\frac{\beta}{\alpha^3}(p_{xxx} - 3p\Delta_2^{-1}p^T\Delta_1 p_x - 3p_x\Delta_2^{-1}p^T\Delta_1 p), \end{aligned} \tag{2.38}$$

where p is an $m \times n$ matrix potential satisfying (2.33).

In what follows, let us present a few examples of these Sasa–Satsuma type integrable matrix mKdV equations, by taking different values for m, n and different choices for Σ, Δ. If we consider $m = 1$ and $n = 2$, and take

$$\Sigma_1 = 1, \ \Sigma_2^{-1} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \ \Delta_1 = 1, \ \Delta_2^{-1} = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}, \tag{2.39}$$

where σ and δ are real constants and satisfy $\sigma^2 = \delta^2 = 1$. Then the potential constraint (2.33) tells

$$p_2 = -\sigma\delta p_1^*, \tag{2.40}$$

and so the corresponding potential matrix P reads

$$P = \begin{bmatrix} 0 & p_1 & -\sigma\delta p_1^* \\ \sigma p_1^* & 0 & 0 \\ -\delta p_1 & 0 & 0 \end{bmatrix}. \tag{2.41}$$

Then, the corresponding Sasa–Satsuma type integrable mKdV equations become

$$p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6\sigma|p_1|^2 p_{1,x} + 3\sigma p_1(|p_1|^2)_x], \tag{2.42}$$

where $\sigma = \pm 1$ and $|z|$ denotes the absolute value of z. The case of $\sigma = 1$ exactly gives rise to the Sasa–Satsuma mKdV equation studied in [14], whose dark solitons were presented in [15]. A similar deduction with

$$\Sigma_1 = 1, \ \Sigma_2^{-1} = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \ \Delta_1 = 1, \ \Delta_2^{-1} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}, \tag{2.43}$$

where σ and δ are real constants and satisfy $\sigma^2 = \delta^2 = 1$, yields another pair of novel scalar integrable mKdV equations:

$$p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\delta p_1^2 p_{1,x} - 3\delta p_1^*(|p_1|^2)_x], \tag{2.44}$$

where $\delta = \pm 1$.

If we consider $m = 1$ and $n = 4$, and take

$$\begin{aligned} &\Sigma_1 = 1, \ \Sigma_2^{-1} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \end{bmatrix}, \\ &\Delta_1 = 1, \ \Delta_2^{-1} = \begin{bmatrix} 0 & \delta_1 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 \\ 0 & 0 & \delta_2 & 0 \end{bmatrix}, \end{aligned} \tag{2.45}$$

where σ_j and δ_j are real constants and satisfy $\sigma_j^2 = \delta_j^2 = 1, \ j = 1, 2$. Then the potential constraint (2.33) generates

$$p_2 = -\sigma_1\delta_1 p_1^*, \ p_4 = -\sigma_2\delta_2 p_3^*, \tag{2.46}$$

and so the corresponding potential matrix P reads

$$P = \begin{bmatrix} 0 & p_1 & -\sigma_1\delta_1 p_1^* & p_3 & -\sigma_2\delta_2 p_3^* \\ \sigma_1 p_1^* & 0 & 0 & 0 & 0 \\ -\delta_1 p_1 & 0 & 0 & 0 & 0 \\ \sigma_2 p_3^* & 0 & 0 & 0 & 0 \\ -\delta_2 p_3 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.47}$$

This enables us to obtain a class of two-component Sasa–Satsuma type integrable mKdV equations:

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}[p_{1,xxx} + 6(\sigma_1|p_1|^2 + \sigma_2|p_3|^2)p_{1,x} \\ +3(\sigma_1|p_1|^2 + \sigma_2|p_3|^2)_x p_1], \\ p_{3,t} = -\dfrac{\beta}{\alpha^3}[p_{3,xxx} + 6(\sigma_1|p_1|^2 + \sigma_2|p_3|^2)p_{3,x} \\ +3(\sigma_1|p_1|^2 + \sigma_2|p_3|^2)_x p_3], \end{cases} \tag{2.48}$$

where σ_j are real constants and satisfy $\sigma_j^2 = 1,\ j = 1, 2$. The long-time asymptotics of the equation with $\sigma_j = 1,\ j = 1, 2$, has been investigated by the nonlinear steepest descent method in [16,17].

In a similar manner, we can obtain a class of N-component Sasa–Satsuma type integrable mKdV equations. The three-component ones read

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}[p_{1,xxx} + 6(\sigma_1|p_1|^2 + \sigma_2|p_3|^2 + \sigma_3|p_5|^2)p_{1,x} \\ +3(\sigma_1|p_1|^2 + \sigma_2|p_3|^2 + \sigma_3|p_5|^2)_x p_1], \\ p_{3,t} = -\dfrac{\beta}{\alpha^3}[p_{3,xxx} + 6(\sigma_1|p_1|^2 + \sigma_2|p_3|^2 + \sigma_3|p_5|^2)p_{3,x} \\ +3(\sigma_1|p_1|^2 + \sigma_2|p_3|^2 + \sigma_3|p_5|^2)_x p_3], \\ p_{5,t} = -\dfrac{\beta}{\alpha^3}[p_{5,xxx} + 6(\sigma_1|p_1|^2 + \sigma_2|p_3|^2 + \sigma_3|p_5|^2)p_{5,x} \\ +3(\sigma_1|p_1|^2 + \sigma_2|p_3|^2 + \sigma_3|p_5|^2)_x p_5], \end{cases} \tag{2.49}$$

where σ_j are real constants and satisfy $\sigma_j^2 = 1,\ j = 1, 2, 3$. This equation with $\sigma_j = 1,\ j = 1, 2, 3$, has been studied by the Riemann–Hilbert method in [18].

3. Riemann–Hilbert problems

3.1. Properties of eigenvalues and eigenfunctions

Note that the reduction in (2.25) (or (2.26)) guarantees that λ is an eigenvalue of the matrix spectral problems in (2.2) if and only if $\hat{\lambda} = \lambda^*$ (or $\hat{\lambda} = -\lambda$) is an adjoint eigenvalue, i.e., it satisfies the adjoint matrix spectral problems:

$$i\tilde{\phi}_x = \tilde{\phi}U = \tilde{\phi}U(u, \hat{\lambda}),\ \ i\tilde{\phi}_t = \tilde{\phi}V^{[r]} = \tilde{\phi}V^{[r]}(u, \hat{\lambda}), \tag{3.1}$$

where $r = 2s + 1,\ s \geq 0$. Accordingly, we can assume to have eigenvalues $\lambda:\ \mu,\ -\mu^*,\ i\nu$, and adjoint eigenvalues $\lambda^*:\ \mu^*,\ -\mu,\ -i\nu$ (or $-\lambda:\ -\mu,\ \mu^*,\ -i\nu$), where $\mu \notin i\mathbb{R}$ and $\nu \in \mathbb{R}$.

Suppose that all the potentials sufficiently rapidly vanish when $x \to \pm\infty$ or $t \to \pm\infty$. For the matrix spectral problems in (2.2) with $r = 2s + 1,\ s \geq 0$, we can impose the asymptotic behavior: $\phi \sim \mathrm{e}^{i\lambda\Lambda x + i\lambda^{2s+1}\Omega t}$, when $x, t \to \pm\infty$. Therefore, if we take the transformation

$$\phi = \psi E_g,\ \ E_g = \mathrm{e}^{i\lambda\Lambda x + i\lambda^{2s+1}\Omega t}, \tag{3.2}$$

then we can achieve the canonical asymptotic conditions: $\psi \to I_{m+n}$, when $x, t \to \infty$ or $-\infty$. The equivalent pair of matrix spectral problems to (2.2) with $r = 2s + 1,\ s \geq 0$, reads

$$\psi_x = i\lambda[\Lambda, \psi] + \check{P}\psi,\ \ \check{P} = iP, \tag{3.3}$$

$$\psi_t = i\lambda^{2s+1}[\Omega, \psi] + \check{Q}^{[2s+1]}\psi,\ \ \check{Q}^{[2s+1]} = iQ^{[2s+1]}. \tag{3.4}$$

Applying a generalized Liouville's formula [19], we can get

$$\det\psi = 1, \tag{3.5}$$

since $(\det\psi)_x = 0$ due to $\operatorname{tr}\check{P} = \operatorname{tr}\check{Q}^{[2s+1]} = 0$.

Recall that the adjoint equation of the x-part of (2.2) and the adjoint equation of (3.3) read

$$i\tilde{\phi}_x = \tilde{\phi}U, \tag{3.6}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi}P, \tag{3.7}$$

respectively. Obviously, there exist the links: $\tilde{\phi} = \phi^{-1}$ and $\tilde{\psi} = \psi^{-1}$, and each pair of adjoint matrix spectral problems and equivalent adjoint matrix spectral problems do not create any additional condition.

Let $\psi(\lambda)$ be a matrix eigenfunction of the spatial spectral problem (3.3) associated with an eigenvalue λ. Then $\Sigma\psi^{-1}(\lambda)$ and $\Delta\psi^{-1}(\lambda)$ are two matrix adjoint eigenfunctions associated with the same eigenvalue λ. Under the group reduction in (2.25), we have

$$\begin{aligned} &i[\psi^\dagger(\lambda^*)\Sigma]_x = i[(\psi_x)^\dagger(\lambda^*)\Sigma] \\ &= i\{(-i)\lambda[\psi^\dagger(\lambda^*), \Lambda] + (-i)\psi^\dagger(\lambda^*)P^\dagger(-x, t)\}\Sigma \\ &= \lambda[\psi^\dagger(\lambda^*), \Lambda]\Sigma + \psi^\dagger(\lambda^*)\Sigma[\Sigma^{-1}P^\dagger\Sigma] \\ &= \lambda[\psi^\dagger(\lambda^*)\Sigma, \Lambda] + \psi^\dagger(\lambda^*)\Sigma P. \end{aligned}$$

This implies that the matrix

$$\tilde{\psi}(\lambda) := \psi^\dagger(\lambda^*)\Sigma, \tag{3.8}$$

presents another matrix adjoint eigenfunction associated with the same original eigenvalue λ. That is to say that $\psi^\dagger(\lambda^*)\Sigma$ solves the adjoint spectral problem (3.7). Thus, upon observing the asymptotic conditions for the matrix eigenfunction ψ, we see that by the uniqueness of solutions, $\psi(\lambda)$ satisfies

$$\psi^\dagger(\lambda^*) = \Sigma\psi^{-1}(\lambda)\Sigma^{-1}, \tag{3.9}$$

when $\psi \to I_{m+n},\ x$ or $t \to \infty$ or $-\infty$.

Similarly, based on the group reduction in (2.26), we can find that

$$\tilde{\psi}(\lambda) = \psi^T(-\lambda)\Delta \tag{3.10}$$

gives a new matrix adjoint eigenfunction associated with λ and satisfies

$$\psi^T(-\lambda) = \Delta\psi^{-1}(\lambda)\Delta^{-1}. \tag{3.11}$$

3.2. Riemann–Hilbert problems

Let us now start to formulate a class of associated Riemann–Hilbert problems with the variable x. In order to describe the problems explicitly, let us make the assumptions:

$$\alpha = \alpha_1 - \alpha_2 < 0,\ \ \beta = \beta_1 - \beta_2 < 0. \tag{3.12}$$

In the scattering problem, we first take the two matrix eigenfunctions $\psi^\pm(x, \lambda)$ of (3.3) with the asymptotic conditions:

$$\psi^\pm \to I_{m+n},\ \text{ when } x \to \pm\infty, \tag{3.13}$$

respectively. Then from (3.5), it follows that $\det\psi^\pm = 1$ for all $x \in \mathbb{R}$. Since

$$\phi^\pm = \psi^\pm E,\ \ E = \mathrm{e}^{i\lambda\Lambda x}, \tag{3.14}$$

are both matrix eigenfunctions of the x-part of (2.2), they must be linearly dependent, and consequently, we have

$$\psi^- E = \psi^+ E S(\lambda),\ \ \lambda \in \mathbb{R}, \tag{3.15}$$

where $S(\lambda)$ is the so-called scattering matrix. Note that $\det S(\lambda) = 1$, because of $\det\psi^\pm = 1$.

As normal, by the method of variation in parameters, we can transform the x-part of (2.2) into the following Volterra integral equations for $\psi^{\pm}$ [7]:

$$\psi^{\pm}(\lambda,x)=I_{m+n}-\int_{x}^{\pm\infty}\mathrm{e}^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^{\pm}(\lambda,y)\mathrm{e}^{i\lambda\Lambda(y-x)}\,dy, \quad (3.16)$$

where the asymptotic conditions (3.13) have been applied. Now, by the Neumann series [20] in the theory of Volterra integral equations, we can show that the eigenfunctions $\psi^{\pm}$ exist and allow analytic continuations off the real axis $\lambda\in\mathbb{R}$ as long as the integrals on their right hand sides converge (see, e.g., [21]). From the diagonal form of Λ and the first assumption in (3.12), we can prove that the integral equation for the first m columns of ψ^- contains only the exponential factor $\mathrm{e}^{-i\alpha\lambda(x-y)}$, which decays because of $y<x$ in the integral, while λ takes values in the upper half-plane $\mathbb{C}^+$, and the integral equation for the last n columns of ψ^+ contains only the exponential factor $\mathrm{e}^{i\alpha\lambda(x-y)}$, which also decays because of $y>x$ in the integral, while λ takes values in the upper half-plane $\mathbb{C}^+$. As a consequence, we know that these $m+n$ columns are analytic in the upper half-plane $\mathbb{C}^+$ and continuous in the closed upper half-plane $\bar{\mathbb{C}}^+$. In a similar manner, we can see that the last n columns of ψ^- and the first m columns of ψ^+ are analytic in the lower half-plane $\mathbb{C}^-$ and continuous in the closed lower half-plane $\bar{\mathbb{C}}^-$.

In what follows, we show how to prove the above statements. Let us express

$$\psi^{\pm}=(\psi_1^{\pm},\psi_2^{\pm},\ldots,\psi_{m+n}^{\pm}), \quad (3.17)$$

that is, $\psi_j^{\pm}$ denotes the jth column of $\phi^{\pm}$ ($1\le j\le m+n$). We would like to prove that ψ_j^-, $1\le j\le m$, and ψ_j^+, $m+1\le j\le m+n$, are analytic at $\lambda\in\mathbb{C}^+$ and continuous at $\lambda\in\bar{\mathbb{C}}^+$; and ψ_j^+, $1\le j\le m$, and ψ_j^-, $m+1\le j\le m+n$, are analytic at $\lambda\in\mathbb{C}^-$ and continuous at $\lambda\in\bar{\mathbb{C}}^-$. Below, we only to prove the result for ψ_j^+, $1\le j\le m$, and the proofs for the other eigenfunctions follow analogously.

From the Volterra integral equation (3.16), we see that

$$\psi_j^+(\lambda,x)=\mathrm{e}_j-\int_x^{\infty}R_1(\lambda,x,y)\psi_j^+(\lambda,y)\,dy,\ \ 1\le j\le m, \quad (3.18)$$

and

$$\psi_j^+(\lambda,x)=\mathrm{e}_j-\int_x^{\infty}R_2(\lambda,x,y)\psi_j^+(\lambda,y)\,dy,\ \ m+1\le j\le m+n, \quad (3.19)$$

where e_j, $1\le j\le m+n$, are standard basis vectors of $\mathbb{R}^{m+n}$ and the matrices R_1 and R_2 are given by

$$R_1(\lambda,x,y)=i\begin{bmatrix}0 & p(y)\\ \mathrm{e}^{-i\alpha\lambda(x-y)}q(y) & 0\end{bmatrix},$$
$$R_2(\lambda,x,y)=i\begin{bmatrix}0 & \mathrm{e}^{i\alpha\lambda(x-y)}p(y)\\ q(y) & 0\end{bmatrix}.$$

Let us prove that for each $1\le j\le m$, the solution to (3.18) is determined by the Neumann series

$$\sum_{k=0}^{\infty}\phi_{j,k}^+(\lambda,x), \quad (3.20)$$

where

$$\phi_{j,0}^+(\lambda,x)=\mathrm{e}_j,\ \ \phi_{j,k+1}^+(\lambda,x)=-\int_x^{\infty}R_1(\lambda,x,y)\phi_{j,k}^+(\lambda,y)\,dy,\ \ k\ge 1. \quad (3.21)$$

This will be true if we can prove that the Neumann series converges uniformly for $x\in\mathbb{R}$ and $\lambda\in\bar{\mathbb{C}}^-$. Based on (3.21), an application of the mathematical induction yields

$$|\phi_{j,k}^+(\lambda,x)|\le\frac{1}{k!}\Big(\int_x^{\infty}\|P(y)\|dy\Big)^k,\ \ 1\le j\le m,\ \ k\ge 0,$$

for $x\in\mathbb{R}$ and $\lambda\in\bar{\mathbb{C}}^-$, where $|\cdot|$ denotes the Euclidean norm for vectors and $\|\cdot\|$ stands for the Frobenius norm for square matrices. By the Weierstrass M-test, this estimation guarantees that

$$\phi_j^+(\lambda,x)=\sum_{k=0}^{\infty}\phi_{j,k}^+(\lambda,x),\ \ 1\le j\le m, \quad (3.22)$$

uniformly converges for $\lambda\in\bar{\mathbb{C}}^-$ and $x\in\mathbb{R}$, and all $\phi_j^+(\lambda,x)$, $1\le j\le m$, are continuous with respect to λ in $\bar{\mathbb{C}}^-$, since so are all $\phi_{j,k}^+(\lambda,x)$, $1\le j\le m$, $k\ge 0$.

We now consider the differentiability of $\phi_j^+(\lambda,x)$, $1\le j\le m$, with respect to λ in $\mathbb{C}^-$ (similarly, we can prove the differentiability with respect to x in $\mathbb{R}$). Fix an integer $1\le j\le m$ and a complex number μ in $\mathbb{C}^-$. Choose a disk $B_\rho(\mu)=\{\lambda\in\mathbb{C}\mid|\lambda-\mu|\le\rho\}$ with a radius $\rho>0$ such that $B_\rho(\mu)\subseteq\mathbb{C}^-$, and then we can have a constant $C(\rho)>0$ such that $|\alpha x\mathrm{e}^{-i\alpha\lambda x}|\le C(\rho)$ for $\lambda\in B_\rho(\mu)$ and $x\le 0$. We define the following Neumann series

$$\sum_{k=0}^{\infty}\phi_{j,\lambda,k}^+(\lambda,x) \quad (3.23)$$

where $\phi_{j,\lambda,0}^+=0$ and $\phi_{j,\lambda,k}^+=0$, $k\ge 1$, are defined recursively by

$$\phi_{j,\lambda,k+1}^+(\lambda,x)=-\int_x^{\infty}R_{1,\lambda}(\lambda,x,y)\phi_{j,k}^+(\lambda,y)\,dy$$
$$-\int_x^{\infty}R_1(\lambda,x,y)\phi_{j,\lambda,k}^+(\lambda,y)\,dy,\ \ k\ge 0, \quad (3.24)$$

with $\phi_{j,k}^+$, $k\ge 0$, being defined by (3.21) and $R_{1,\lambda}$ being given by

$$R_{1,\lambda}(\lambda,x,y)=\frac{\partial}{\partial\lambda}R_1(\lambda,x,y)=\begin{bmatrix}0 & 0\\ \alpha(x-y)\mathrm{e}^{-i\alpha\lambda(x-y)}q(y) & 0\end{bmatrix}.$$

We can easily verify by the mathematical induction that

$$|\phi_{j,\lambda,k}^+(\lambda,x)|\le\frac{1}{k!}\Big\{\big[C(\rho)+1\big]\int_x^{\infty}\|P(y)\|dy\Big\}^k,\ \ k\ge 0,$$

for $x\in\mathbb{R}$ and $\lambda\in B_\rho(\mu)$. Therefore, by the Weierstrass M-test, the Neumann series defined by (3.23) converges uniformly for $x\in\mathbb{R}$ and $\lambda\in B_\rho(\mu)$, and by the term-by-term differentiability theorem, it converges to the derivative of ϕ_j^+ with respect to λ, since $\psi_{j,\lambda,k}^+=\frac{\partial}{\partial\lambda}\phi_{j,k}^+$, $k\ge 0$. It follows that ϕ_j^+ is analytic at any point $\lambda\in B_\rho(\mu)$, and thus, particularly at the point μ. This tells that all ϕ_j^+, $1\le j\le m$, are analytic with respect to λ in $\mathbb{C}^-$. The required proof is finished.

Now, based on these analyses, we can then define the generalized matrix Jost solution T^+ as follows:

$$T^+=T^+(x,\lambda)=(\psi_1^-,\ldots,\psi_m^-,\psi_{m+1}^+,\ldots,\psi_{m+n}^+)$$
$$=\psi^-H_1+\psi^+H_2, \quad (3.25)$$

where

$$H_1=\mathrm{diag}(I_m,\underbrace{0,\ldots,0}_{n}),\ \ H_2=\mathrm{diag}(\underbrace{0,\ldots,0}_{m},I_n), \quad (3.26)$$

and know that T^+ is analytic with respect to λ in $\mathbb{C}^+$ and continuous with respect to λ in $\bar{\mathbb{C}}^+$. The generalized matrix Jost solution

$$(\psi_1^+,\ldots,\psi_m^+,\psi_{m+1}^-,\ldots,\psi_{m+n}^-)=\psi^+H_1+\psi^-H_2$$

is analytic with respect to λ in $\mathbb{C}^-$ and continuous with respect to λ in $\bar{\mathbb{C}}^-$.

To construct the other generalized matrix Jost solution T^-, we adopt the analytic counterpart of T^+ in the lower half-plane $\mathbb{C}^-$, which can be generated from the adjoint counterparts of the matrix spectral problems. Note that the inverse matrices $\tilde{\phi}^\pm = (\phi^\pm)^{-1}$ and $\tilde{\psi}^\pm = (\psi^\pm)^{-1}$ solve those two adjoint equations, respectively. Thus, upon expressing $\tilde{\psi}^\pm$ as

$$\tilde{\psi}^\pm = \begin{bmatrix} \tilde{\psi}^{\pm,1} \\ \tilde{\psi}^{\pm,2} \\ \vdots \\ \tilde{\psi}^{\pm,m+n} \end{bmatrix}, \tag{3.27}$$

that is, $\tilde{\psi}^{\pm,j}$ denotes the jth row of $\tilde{\psi}^\pm$ ($1 \le j \le m+n$), we can prove by similar arguments that we can define the generalized matrix Jost solution T^- as the adjoint matrix solution of (3.7), i.e.,

$$T^- = \begin{bmatrix} \tilde{\psi}^{-,1} \\ \vdots \\ \tilde{\psi}^{-,m} \\ \tilde{\psi}^{+,m+1} \\ \vdots \\ \tilde{\psi}^{+,m+n} \end{bmatrix} = H_1\tilde{\psi}^- + H_2\tilde{\psi}^+ = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1}, \tag{3.28}$$

which is analytic with respect to λ in $\mathbb{C}^-$ and continuous with respect to λ in $\bar{\mathbb{C}}^-$. The other generalized matrix Jost solution of (3.7),

$$\begin{bmatrix} \tilde{\psi}^{+,1} \\ \vdots \\ \tilde{\psi}^{+,m} \\ \tilde{\psi}^{-,m+1} \\ \vdots \\ \tilde{\psi}^{-,m+n} \end{bmatrix} = H_1\tilde{\psi}^+ + H_2\tilde{\psi}^- = H_1(\psi^+)^{-1} + H_2(\psi^-)^{-1},$$

is analytic with respect to λ in $\mathbb{C}^+$ and continuous with respect to λ in $\bar{\mathbb{C}}^+$.

Further, directly from $\det \psi^\pm = 1$ and using the scattering relation (3.15) between ψ^+ and ψ^-, we can have

$$\begin{cases} \lim_{x\to\infty} T^+(x,\lambda) = \begin{bmatrix} S_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}, \ \lambda \in \bar{\mathbb{C}}^+, \\ \lim_{x\to-\infty} T^-(x,\lambda) = \begin{bmatrix} \hat{S}_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}, \ \lambda \in \bar{\mathbb{C}}^-, \end{cases} \tag{3.29}$$

and

$$\det T^+(x,\lambda) = \det S_{11}(\lambda), \ \det T^-(x,\lambda) = \det \hat{S}_{11}(\lambda), \tag{3.30}$$

where we split $S(\lambda)$ and $S^{-1}(\lambda)$ as follows:

$$S(\lambda) = \begin{bmatrix} S_{11}(\lambda) & S_{12}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{bmatrix}, \quad S^{-1}(\lambda) = (S(\lambda))^{-1} = \begin{bmatrix} \hat{S}_{11}(\lambda) & \hat{S}_{12}(\lambda) \\ \hat{S}_{21}(\lambda) & \hat{S}_{22}(\lambda) \end{bmatrix}. \tag{3.31}$$

From (3.29), we know that $S_{11}, \hat{S}_{11}$ are $m\times m$ matrices; and so, $S_{12}, \hat{S}_{12}$ are $m\times n$ matrices, $S_{21}, \hat{S}_{21}$ are $n\times m$ matrices, and $S_{22}, \hat{S}_{22}$ are $n\times n$ matrices, because $S(\lambda)$ is a square matrix of size $m+n$. Based on the uniform convergence of the previous Neumann series, we know that $S_{11}(\lambda)$ and $\hat{S}_{11}(\lambda)$ are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively.

Now, we can define the following two unimodular generalized matrix Jost solutions:

$$\begin{cases} G^+(x,\lambda) = T^+(x,\lambda)\begin{bmatrix} S_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}, \ \lambda \in \bar{\mathbb{C}}^+; \\ (G^-)^{-1}(x,\lambda) = \begin{bmatrix} \hat{S}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} T^-(x,\lambda), \ \lambda \in \bar{\mathbb{C}}^-. \end{cases} \tag{3.32}$$

These two generalized matrix Jost solutions enable us to establish the required matrix Riemann–Hilbert problems on the real line for the Sasa–Satsuma type matrix integrable equations (2.37):

$$G^+(x,\lambda) = G^-(x,\lambda)G_0(x,\lambda), \ \lambda \in \mathbb{R}, \tag{3.33}$$

where, by (3.15), the jump matrix G_0 reads

$$G_0(x,\lambda) = E\begin{bmatrix} \hat{S}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} \tilde{S}(\lambda) \begin{bmatrix} S_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} E^{-1}. \tag{3.34}$$

The matrix $\tilde{S}(\lambda)$ has the following factorization:

$$\tilde{S}(\lambda) = (H_1 + H_2 S(\lambda))(H_1 + S^{-1}(\lambda)H_2), \tag{3.35}$$

which can be shown to be

$$\tilde{S}(\lambda) = \begin{bmatrix} I_m & \hat{S}_{12} \\ S_{21} & I_n \end{bmatrix}. \tag{3.36}$$

Note that for the presented Riemann–Hilbert problems, the canonical normalization conditions:

$$G^\pm(x,\lambda) \to I_{m+n}, \ \text{when } \lambda \in \bar{\mathbb{C}}^\pm \to \infty, \tag{3.37}$$

are consequences of the Volterra integral equations in (3.16). Also, from the properties in (3.9) and (3.11), we can have

$$(G^+)^\dagger(\lambda^*) = \Sigma (G^-)^{-1}(\lambda)\Sigma^{-1}, \tag{3.38}$$

and

$$(G^+)^T(-\lambda) = \Delta (G^-)^{-1}(\lambda)\Delta^{-1}. \tag{3.39}$$

Therefore, the jump matrix G_0 satisfies the following involution properties:

$$G_0^\dagger(\lambda^*) = \Sigma G_0(\lambda)\Sigma^{-1}, \ G_0^T(-\lambda) = \Delta G_0(\lambda)\Delta^{-1}, \ \lambda \in \mathbb{R}. \tag{3.40}$$

3.3. Evolution of the scattering data

In order to complete the direct scattering transforms, we take the derivative of (3.15) with time t and use the temporal matrix spectral problems:

$$\psi_t^\pm = i\lambda^{2s+1}[\Omega, \psi^\pm] + iQ^{[2s+1]}\psi^\pm, \tag{3.41}$$

where $s \ge 0$ is fixed. It then follows that the scattering matrix S satisfies the following evolution law:

$$S_t = i\lambda^{2s+1}[\Omega, S]. \tag{3.42}$$

This yields the time evolution of the time-dependent scattering coefficients:

$$S_{12} = S_{12}(t,\lambda) = S_{12}(0,\lambda)\,e^{i\beta\lambda^{2s+1}t}, \quad S_{21} = S_{21}(t,\lambda) = S_{21}(0,\lambda)\,e^{-i\beta\lambda^{2s+1}t}, \tag{3.43}$$

and all other scattering coefficients are independent of time t.

3.4. Gelfand–Levitan–Marchenko type equations

To obtain Gelfand-Levitan-Marchenko type integral equations to determine the generalized matrix Jost solutions, we transform the associated Riemann–Hilbert problems in (3.33) into the following problems:

$$\begin{cases} G^+ - G^- = G^- v, \ v = G_0 - I_{m+n}, \ \text{on } \mathbb{R}, \\ G^\pm \to I_{m+n} \text{ as } \lambda \in \bar{\mathbb{C}}^\pm \to \infty, \end{cases} \tag{3.44}$$

where each jump matrix G_0 is defined by (3.34) and (3.36).

Let $G(\lambda) = G^\pm(\lambda)$ for $\lambda \in \mathbb{C}^\pm$. Suppose that G has simple poles off $\mathbb{R}$: $\{\xi_j\}_{j=1}^R$, to avoid the spectral singularity, where R is an arbitrary natural number. Introduce

$$\tilde{G}^\pm(\lambda) = G^\pm(\lambda) - \sum_{j=1}^R \frac{G_j}{\lambda - \xi_j}, \ \lambda \in \bar{\mathbb{C}}^\pm; \ \tilde{G}(\lambda) = \tilde{G}^\pm(\lambda), \ \lambda \in \mathbb{C}^\pm, \tag{3.45}$$

where G_j is the residue of G at $\lambda = \xi_j$, i.e.,

$$G_j = \operatorname{res}(G(\lambda), \xi_j) = \lim_{\lambda \to \xi_j} (\lambda - \xi_j) G(\lambda). \tag{3.46}$$

Obviously, we have

$$\begin{cases} \tilde{G}^+ - \tilde{G}^- = G^+ - G^- = G^- v, \ \text{on } \mathbb{R}, \\ \tilde{G}^\pm \to I_{m+n} \text{ as } \lambda \in \bar{\mathbb{C}}^\pm \to \infty. \end{cases} \tag{3.47}$$

Then by applying the Sokhotski–Plemelj formula [22], we obtain the solution of each problem in (3.47):

$$\tilde{G}(\lambda) = I_{m+n} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \lambda} \, d\xi. \tag{3.48}$$

Taking the limit as $\lambda \to \xi_l$ generates

$$\begin{aligned} \text{LHS} &= \lim_{\lambda \to \xi_l} \tilde{G} = F_l - \sum_{j \neq l}^R \frac{G_j}{\xi_l - \xi_j}, \\ \text{RHS} &= I_{m+n} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \xi_l} \, d\xi, \end{aligned} \tag{3.49}$$

where

$$F_l = \lim_{\lambda \to \xi_l} \frac{(\lambda - \xi_l) G(\lambda) - G_l}{\lambda - \xi_l}, \ 1 \le l \le R, \tag{3.50}$$

and accordingly, we obtain

$$I_{m+n} - F_l + \sum_{j \neq l}^R \frac{G_j}{\xi_l - \xi_j} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \xi_l} \, d\xi = 0, \ 1 \le l \le R, \tag{3.51}$$

which define the required Gelfand-Levitan-Marchenko type integral equations.

All these integral equations are used to determine solutions to the associated Riemann–Hilbert problems and thus the generalized matrix Jost solutions. Yet, little is known regarding the existence and uniqueness of solutions. In the reflectionless case, a generalized formulation of solutions, where eigenvalues could equal adjoint eigenvalues, will be presented for the Sasa–Satsuma type matrix integrable equations in the following section.

3.5. Recovery of the potential

To recover the potential matrix P from the generalized matrix Jost solutions, we make an asymptotic expansion

$$G^+(x, t, \lambda) = I_{m+n} + \frac{1}{\lambda} G_1^+(x, t) + O(\frac{1}{\lambda^2}), \ \lambda \to \infty. \tag{3.52}$$

Then, inserting the above asymptotic expansion into the matrix spectral problem (3.3) and comparing constant terms yields

$$P = \lim_{\lambda \to \infty} \lambda [G^+(\lambda), \Lambda] = -[\Lambda, G_1^+]. \tag{3.53}$$

Thus, the potential matrix reads

$$P = \begin{bmatrix} 0 & -\alpha G_{1,12}^+ \\ \alpha G_{1,21}^+ & 0 \end{bmatrix}, \tag{3.54}$$

where we have similarly partitioned the matrix G_1^+ into four blocks as follows:

$$G_1^+ = \begin{bmatrix} G_{1,11}^+ & G_{1,12}^+ \\ G_{1,21}^+ & G_{1,22}^+ \end{bmatrix} = \begin{bmatrix} (G_{1,11}^+)_{n\times n} & (G_{1,12}^+)_{n\times m} \\ (G_{1,21}^+)_{m\times n} & (G_{1,22}^+)_{m\times m} \end{bmatrix}. \tag{3.55}$$

Consequently, the solutions to the matrix AKNS equations (2.22) read

$$p = -\alpha G_{1,12}^+, \ q = \alpha G_{1,21}^+. \tag{3.56}$$

When the reduction conditions in (2.29) and (2.30) are satisfied, the reduced matrix potential p solves the Sasa–Satsuma type matrix integrable equations (2.37).

To conclude, this provides an inverse scattering procedure for computing solutions to the Sasa–Satsuma type matrix integrable equations (2.37), from the scattering matrix $S(\lambda)$, through the jump matrix $G_0(\lambda)$ and the solution $\{G^+(\lambda), G^-(\lambda)\}$ of the associated Riemann–Hilbert problems, to the potential matrix P.

4. Soliton solutions

4.1. General formulation

Let $N \ge 1$ be another given integer. Assume that $\det S_{11}(\lambda)$ has N zeros $\{\lambda \in \mathbb{C}, \ 1 \le k \le N\}$ and $\det \hat{S}_{11}(\lambda)$ has N zeros $\{\hat{\lambda} \in \mathbb{C}, \ 1 \le k \le N\}$.

In order to compute soliton solutions explicitly, we additionally assume that all these zeros, λ_k and $\hat{\lambda}_k$, $1 \le k \le N$, are geometrically simple. Thus, we know that each of $\ker T^+(\lambda_k)$, $1 \le k \le N$, contains only a single basis column vector, which we denote by v_k, $1 \le k \le N$; and each of $\ker T^-(\hat{\lambda}_k)$, $1 \le k \le N$, a single basis row vector, which we denote by $\hat{v}_k$, $1 \le k \le N$. Therefore, we have

$$T^+(\lambda_k) v_k = 0, \ \hat{v}_k T^-(\hat{\lambda}_k) = 0, \ 1 \le k \le N. \tag{4.1}$$

Soliton solutions are associated with the situation where $G_0 = I_{m+n}$ is taken in each Riemann–Hilbert problem in (3.33). Such a situation can be met if we take that $S_{21} = \hat{S}_{12} = 0$, which means that all the reflection coefficients are taken as zero in the scattering problem.

This kind of special Riemann–Hilbert problems with the canonical normalization conditions in (3.37) and the zero structures given in (4.1) can be solved explicitly [7,23], in the case of

$$\{\lambda_k | 1 \le k \le N\} \cap \{\hat{\lambda}_k | 1 \le k \le N\} = \emptyset, \tag{4.2}$$

and therefore, we can present the potential matrix P exactly, which generates soliton solutions. Also, without the condition (4.2), the solutions to the special Riemann–Hilbert problem with the identity jump matrix have been given recently as follows (see, e.g., [24]):

$$\begin{aligned} G^+(\lambda) &= I_{m+n} - \sum_{k,l=1}^N \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l}, \\ (G^-)^{-1}(\lambda) &= I_{m+n} + \sum_{k,l=1}^N \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}, \end{aligned} \tag{4.3}$$

where $M = (m_{kl})_{N\times N}$ is a square matrix with its entries determined by

$$m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, & \text{if } \lambda_l \neq \hat{\lambda}_k, \\ 0, & \text{if } \lambda_l = \hat{\lambda}_k, \end{cases} \quad 1 \le k, l \le N, \tag{4.4}$$

and we require an orthogonal condition

$$\hat{v}_k v_l = 0,\ \text{if}\ \lambda_l = \hat{\lambda}_k,\ 1 \le k, l \le N, \tag{4.5}$$

to ensure that $G^+(\lambda)$ and $G^-(\lambda)$ solve the corresponding reflectionless Riemann–Hilbert problem:

$$(G^-)^{-1}(\lambda)G^+(\lambda) = I_{m+n}. \tag{4.6}$$

Note that the zeros λ_k and $\hat{\lambda}_k$ are constants, i.e., space and time independent, and thus, we can readily determine the spatial and temporal evolutions for the vectors, $v_k(x,t)$ and $\hat{v}_k(x,t)$, $1 \le k \le N$, in the kernels. For example, let us compute the x-derivative of both sides of the first set of equations in (4.1). Applying (3.3) first and then again the first set of equations in (4.1), we obtain

$$T^+(x,\lambda_k)\Big(\frac{dv_k}{dx} - i\lambda_k \Lambda v_k\Big) = 0,\ 1 \le k \le N.$$

Consequently, for each $1 \le k \le N$, $\frac{dv_k}{dx} - i\lambda_k\Lambda v_k$ is in the kernel of $T^+(x,\lambda_k)$, and hence, a constant multiple of v_k, because λ_k is geometrically simple. Without loss of generality, we can simply assume

$$\frac{dv_k}{dx} = i\lambda_k \Lambda v_k,\ 1 \le k \le N. \tag{4.7}$$

The time dependence of v_k:

$$\frac{dv_k}{dt} = i\lambda_k^{2s+1}\Omega v_k,\ 1 \le k \le N, \tag{4.8}$$

can be obtained similarly via applying the t-part of the associated matrix spectral problem, i.e., (3.4). As a consequence of these differential equations, we get

$$v_k(x,t) = \mathrm{e}^{i\lambda_k\Lambda x + i\lambda_k^{2s+1}\Omega t} w_k,\ 1 \le k \le N, \tag{4.9}$$

and completely similarly, we can obtain

$$\hat{v}_k(x,t) = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k\Lambda x - i\hat{\lambda}_k^{2s+1}\Omega t},\ 1 \le k \le N, \tag{4.10}$$

where w_k and $\hat{w}_k$, $1 \le k \le N$, are constant column and row vectors, respectively, but need to satisfy an orthogonal condition:

$$\hat{w}_k w_l = 0,\ \text{if}\ \lambda_l = \hat{\lambda}_k,\ 1 \le k, l \le N, \tag{4.11}$$

in consequence of (4.5).

Now, from the solutions in (4.3), we obtain

$$G_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{4.12}$$

and further, the presentations in (3.56) give the following N-soliton solution to the matrix AKNS equations (2.22):

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2,\ q = -\alpha \sum_{k,l=1}^{N} v_k^2 (M^{-1})_{kl} \hat{v}_l^1. \tag{4.13}$$

Here for each $1 \le k \le N$, we split $v_k = ((v_k^1)^T, (v_k^2)^T)^T$ and $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2)$, where v_k^1 and $\hat{v}_k^1$ are m-dimensional column and row vectors, respectively, and v_k^2 and $\hat{v}_k^2$ are n-dimensional column and row vectors, respectively.

To present N-soliton solutions for the Sasa–Satsuma type matrix integrable equations (2.37), we need to check if G_1^+ defined by (4.12) satisfies the involution properties:

$$(G_1^+)^\dagger = \Sigma G_1^+ \Sigma^{-1},\ (G_1^+)^T = \Delta G_1^+ \Delta^{-1}. \tag{4.14}$$

These mean that the potential matrix P determined by (3.54) satisfies the reduction conditions in (2.29) and (2.30). In this way, the N-soliton solution to the matrix AKNS equations (2.22) is reduced to the N-soliton solution:

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2, \tag{4.15}$$

to the Sasa–Satsuma type matrix integrable equations (2.37).

4.2. Realization

Let us now check how to realize the involution properties in (4.14).

Let $N_1, N_2 \ge 0$ be a pair of integers such that $N := 2N_1 + N_2 \ge 1$. First, we take N distinct zeros of $\det T^+(\lambda)$ (or eigenvalues of the spectral problems under the zero potential):

$$\{\lambda_k \,|\, 1 \le k \le N\} = \{\mu_k,\ -\mu_k^*,\ 1 \le k \le N_1;\ i\nu_k,\ 1 \le k \le N_2\} \tag{4.16}$$

and N zeros of $\det T^-(\lambda)$ (eigenvalues of the adjoint spectral problems under the zero potential):

$$\{\hat{\lambda}_k \,|\, 1 \le k \le N\} = \{\mu_k^*,\ -\mu_k,\ 1 \le k \le N_1;\ -i\nu_k,\ 1 \le k \le N_2\}, \tag{4.17}$$

where $\mu_k \notin i\mathbb{R}$ and $\nu_k \in \mathbb{R}$, It is easy to see that $\ker T^+(\lambda_k)$, $1 \le k \le N$, are spanned by

$$v_k = v_k(x,t,\lambda_k) = \mathrm{e}^{i\lambda_k\Lambda x + i\lambda_k^{2s+1}\Omega t} w_k,\ 1 \le k \le N, \tag{4.18}$$

respectively, where w_k, $1 \le k \le N$, are constant column vectors. These column vectors in (4.18) are eigenfunctions of the spectral problems under the zero potential associated with λ_k, $1 \le k \le N$. Furthermore, following the previous analysis in Section 3.1, $\ker T^-(\lambda_k)$, $1 \le k \le N$, are spanned by

$$\hat{v}_k = v_k^\dagger \Sigma = v_{N_1+k}^T \Delta,\ \hat{v}_{N_1+k} = v_{N_1+k}^\dagger \Sigma = v_k^T \Delta,\ 1 \le k \le N_1, \tag{4.19}$$

and

$$\hat{v}_k = v_k^\dagger \Sigma = v_k^T \Delta,\ 2N_1 + 1 \le k \le N, \tag{4.20}$$

respectively. These row vectors $\hat{v}_k$, $1 \le k \le N$, are eigenfunctions of the adjoint spectral problems under the zero potential associated with $\hat{\lambda}_k$, $1 \le k \le N$, respectively. It is direct to see that the choices in (4.19) and (4.20) yield the selections on w_k, $1 \le k \le N$:

$$\begin{cases} w_k^T(\Delta\Sigma^{-1} - \Sigma^*\Delta^{*-1}) = 0,\ 1 \le k \le N_1, \\ w_k = \Delta^{-1}\Sigma^* w_{k-N_1}^*,\ N_1 + 1 \le k \le 2N_1, \\ w_k^\dagger \Sigma = w_k^T \Delta,\ 2N_1 + 1 \le k \le N, \end{cases} \tag{4.21}$$

where $*$ denotes the complex conjugate of a matrix. We emphasize that all these selections aim to satisfy the reduction conditions in (2.29) and (2.30).

To satisfy the orthogonal condition (4.11), we can check the following equivalent orthogonal condition

$$w_k^\dagger \Sigma w_l = 0,\ \text{if}\ \lambda_l = \hat{\lambda}_k,\ 1 \le k, l \le N, \tag{4.22}$$

on the constant columns $\{w_k \,|\, 1 \le k \le N\}$. Interestingly, the situation of $\lambda_k = \hat{\lambda}_k$ occurs only when $\lambda_k = 0$, $2N_1 + 1 \le k \le N$. Since $\alpha_1 \ne \alpha_2$ and $\beta_1 \ne \beta_2$, we can easily observe that the conditions in (4.22) are equivalent to

$$(w_k^1)^\dagger \Sigma_1 w_l^1 = 0,\ (w_k^2)^\dagger \Sigma_2 w_l^2 = 0,\ \text{if}\ \lambda_l = \hat{\lambda}_k,$$
$$\text{where}\ 1 \le k, l \le N, \tag{4.23}$$

in which we split $w_k = ((w_k^1)^T, (w_k^2)^T)^T$, $1 \le k \le N$, as we did for v_k before. All these create the conditions for the orthogonality requirements, which can also be expressed by using the symmetric matrix Δ.

Now, note that if the solutions to the specific Riemann–Hilbert problems, determined by (4.3) and (4.4), satisfy the involution properties in (3.38) and (3.39), then the corresponding matrix G_1^+ possesses the involution properties in (4.14), generated from the group reductions in (2.25) and (2.26). Therefore, when the selections in (4.21) are made and the conditions in (4.23) are satisfied, the formula (4.15), together with (4.3), (4.4), (4.18), (4.19) and (4.20), gives rise to N-soliton solutions to the Sasa–Satsuma type matrix integrable equations (2.37).

When $m = N = 1$ and $n = 4$, let us fix $\alpha = \alpha_1 - \alpha_2 = -1$, take $\lambda_1 = i\nu$, $\hat{\lambda}_1 = -i\nu$, $\nu \in \mathbb{R}$, $\nu \neq 0$, and due to the last requirement in (4.21), choose

$$w_1 = (w_{1,1}, w_{1,2}, \sigma_1\delta_1 w_{1,2}^*, w_{1,4}, \sigma_2\delta_2 w_{1,4}^*)^T,$$

where $w_{1,1}$ is real and $\sigma_j^2 = \delta_j^2 = 1$, $j = 1, 2$. Then we can obtain the following one-soliton solution to the two-component Sasa–Satsuma type integrable mKdV equations in (2.48):

$$p_1 = -\frac{i\sigma_1 w_{1,1} w_{1,2}^* \, \mathrm{e}^{\nu x + (\beta_1-\beta_2)\nu^3 t}}{\nu[|w_{1,1}|^2 \, \mathrm{e}^{-2\nu\alpha_1 x + 2\nu^3\beta_1 t} + 2(\sigma_1|w_{1,2}|^2 + \sigma_2|w_{1,4}|^2)\, \mathrm{e}^{-2\nu\alpha_2 x + 2\nu^3\beta_2 t}]}, \tag{4.24}$$

and

$$p_3 = -\frac{i\sigma_2 w_{1,1} w_{1,4}^* \, \mathrm{e}^{\nu x + (\beta_1-\beta_2)\nu^3 t}}{\nu[|w_{1,1}|^2 \, \mathrm{e}^{-2\nu\alpha_1 x + 2\nu^3\beta_1 t} + 2(\sigma_1|w_{1,2}|^2 + \sigma_2|w_{1,4}|^2)\, \mathrm{e}^{-2\nu\alpha_2 x + 2\nu^3\beta_2 t}]}, \tag{4.25}$$

where ν, $w_{1,2}$ and $w_{1,4}$ need to satisfy

$$(1 - 16\nu^4)(\sigma_1|w_{1,2}|^2 + \sigma_2|w_{1,4}|^2) = 0, \tag{4.26}$$

which comes from the involution properties in (4.14). When there is no σ_2 and $w_{1,4}$, one can reduce this solution to get a one-soliton to the Sasa–Satsuma type mKdV equations in (2.42).

5. Concluding remarks

The paper has explored Sasa–Satsuma type matrix integrable equations, by use of two group reductions of the matrix AKNS spectral problem of arbitrary order, and presented Riemann–Hilbert problems for the resulting Sasa–Satsuma type matrix integrable equations, by taking advantage of the Lax pair and the adjoint Lax pair of matrix spectral problems. The obtained Riemann–Hilbert problems have been applied to soliton solutions of the Sasa–Satsuma type matrix integrable equations, which amends the binary Darboux transformation theory for the Sasa–Satsuma type matrix mKdV equations [25,26].

The crucial step in our analysis is to use two local group reductions simultaneously to generate reduced integrable equations, which also forms the basis for studying the Sasa–Satsuma mKdV equation. In our formulation of Riemann–Hilbert problems, we have taken advantage of a generalized M-matrix, where eigenvalues could be equal to adjoint eigenvalues. Such an introduction of generalized M-matrices is motivated by recent studies on Riemann–Hilbert problems of nonlocal integrable equations (see, for example, [24,27]). The associated generalized formulation of Riemann–Hilbert problems can be applied to both local and nonlocal integrable equations (see, for example, [24,27–31] for nonlocal cases). We point out that there are only those two kinds of group reductions for the matrix AKNS spectral problems which produce reduced local integrable equations. It should be interesting to apply the idea of adopting two group reductions to other matrix spectral problems to explore reduced local integrable equations.

The Riemann–Hilbert technique, which is very effective in generating soliton solutions (see also, e.g., [32–34]), has been recently generalized to solve various initial–boundary value problems of continuous integrable equations on the half-line and the finite interval [35,36]. There are many other powerful approaches to soliton solutions, among which are the Hirota direct method [4], the generalized bilinear technique [37], the Wronskian technique [38,39] and the Darboux transformation [3,40]. It would be significantly important to look for connections among different approaches to exhibit dynamical characteristics of soliton solutions. We would also like to emphasize that it would be particularly interesting to compute various kinds of exact solutions other than solitons to integrable equations, for example, positon and complexiton solutions [41,42], lump and rogue wave solutions [43–51], solitonless solutions [52–54] and algebro-geometric solutions [55,56], in a perspective of Riemann–Hilbert problems. It is another interesting topic for future study to link Riemann–Hilbert problems to generalized integrable counterparts, including integrable couplings, super-symmetric integrable equations and fractional spacetime analogous equations.

Declaration of competing interest

The author declares that there is no known competing financial interest or personal relationship that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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