Physica D 430 (2022) 133078

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Riemann–Hilbert problems and inverse scattering of nonlocal real reverse-spacetime matrix AKNS hierarchies

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ARTICLE INFO

Article history:
Received 16 March 2021
Received in revised form 6 September 2021
Accepted 27 October 2021
Available online 8 November 2021
Communicated by T. Grava

Keywords:
Integrable hierarchy
Matrix spectral problem
Real reverse-spacetime reduction
Riemann–Hilbert problem
Inverse scattering transform
Soliton solution

ABSTRACT

We would like to propose a kind of nonlocal real reverse-spacetime integrable hierarchies of PT-symmetric matrix AKNS equations through nonlocal symmetry reductions on the potential matrix, and formulate their associated Riemann–Hilbert problems to determine generalized Jost solutions of arbitrary-order matrix spectral problems. The Sokhotski–Plemelj formula is applied in transforming the associated Riemann–Hilbert problems into Gelfand–Levitan–Marchenko type integral equations. The Riemann–Hilbert problems corresponding to the reflectionless case are solved explicitly, where eigenvalues could equal adjoint eigenvalues, and thus, soliton solutions are presented for the resulting nonlocal real reverse-spacetime integrable PT-symmetric matrix AKNS equations.

1. Introduction

It is known that non-Hermitian Hamiltonians endowed with an unbroken PT symmetry can possess a real spectrum, and such Hamiltonians form a new kind of quantum mechanics [1,2]. PT symmetries also play an important role in mathematical physics, and can keep integrability of equations of mathematical physics. An equation in (1+1)-dimensions is called to be PT-symmetric, if it is invariant under a parity-time transformation (e.g., $x \to -x$, $t \to -t$, $i \to -i$). Recently, in soliton theory, PT symmetries are extensively applied to matrix spectral problems, and PT-symmetric zero curvature equations present nonlocal integrable equations, which have already become one of active research areas. A PT-symmetric reduction is called to be real (or complex) if it does not involve (or involves) the complex conjugate of the potential. Based on zero curvature equations, PT-symmetric reductions can be classified into four types of real reverse-time, real reverse-spacetime, complex reverse-space and complex reverse-spacetime.

Notably, five examples of scalar nonlocal integrable nonlinear Schrödinger (NLS) equations and modified Korteweg–de Vries (mKdV) equations have been identified as significant models for understanding nonlocal nonlinear physical phenomena in nonlinear PT-symmetric media. Those are the nonlocal real reverse-time NLS equation, the nonlocal real reverse-spacetime NLS equation, the nonlocal complex reverse-space NLS equation, the nonlocal real reverse-spacetime mKdV equation and the nonlocal complex reverse-spacetime mKdV equation [3,4]. The inverse scattering technique has been successfully used to solve nonlocal nonlinear integrable equations, under either zero or nonzero boundary conditions [5–8]. Moreover, the Hirota bilinear method [9,10] and Darboux transformations [11,12], are shown to be effective in constructing N-soliton solutions of nonlocal models. Some multicomponent [8,13] and higher dimensional [14] generalizations of nonlocal integrable equations and hierarchies of nonlocal integrable equations [7] have also been proposed and studied.

Riemann–Hilbert problems are another powerful approach in soliton theory and are used to generate soliton solutions to integrable equations, which are closely connected with Darboux transformations and the inverse scattering transforms of associated integrable equations. Indeed, many integrable equations, both local and nonlocal, have been studied through the corresponding Riemann–Hilbert problems systematically. Interesting examples include the multiple wave interaction equations [15], the general coupled nonlinear Schrödinger equations [16], the Harry Dym equation [17], the generalized Sasa–Satsuma equation [18], and multicomponent mKdV equations [19] in the local case;

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https://doi.org/10.1016/j.physd.2021.133078
0167-2789/

and the nonlocal scalar NLS equations [20], nonlocal multicomponent NLS equations [8,21] and hierarchies [7], and nonlocal multicomponent mKdV equations [22] in the nonlocal case.

In this paper, we would like to propose and analyze a class of nonlocal real reverse-spacetime integrable hierarchies of PT-symmetric matrix AKNS equations by nonlocal symmetry reductions for the AKNS matrix spectral problems with matrix potentials, and formulate their associated Riemann–Hilbert problems, which generate their inverse scattering transforms and soliton solutions. Two of the nonlocal integrable matrix AKNS equations that we will analyze are

$$ip_t(x,t) = p_{xx}(x,t) + 2\gamma p(x,t)p^T(-x,-t)p(x,t), \tag{1.1}$$

and

$$\begin{aligned} p_t(x,t) &= p_{xxx}(x,t) + 3\gamma p(x,t)p^T(-x,-t)p_x(x,t) \\ &\quad + 3\gamma p_x(x,t)p^T(-x,-t)p(x,t), \end{aligned} \tag{1.2}$$

in which $p = (p_{jk})_{m\times n}$ is a matrix potential and $\gamma \neq 0$ is an arbitrary constant. Here m and n are two arbitrary natural numbers.

The rest of the paper is organized as follows. In Section 2, we generate nonlocal real reverse-spacetime integrable hierarchies of PT-symmetric matrix AKNS equations from the classical integrable hierarchies of matrix AKNS equations by making nonlocal symmetry reductions on the potential matrix. In Section 3, we explore a property of eigenfunctions under the nonlocal symmetry reductions, establish associated Riemann–Hilbert problems to determine generalized Jost solutions, and transform the resulting Riemann–Hilbert problems into systems of Gelfand–Levitan–Marchenko type integral equations through the Sokhotski–Plemelj formula. In Section 4, we present a solution formulation, where eigenvalues could equal adjoint eigenvalues, for the Riemann–Hilbert problems with the identity jump matrix, corresponding to the reflectionless inverse scattering transforms, and consequently generate N-soliton solutions to the resulting nonlocal integrable matrix AKNS equations. In the last section, we give a few concluding remarks.

2. Nonlocal integrable matrix AKNS hierarchies

Let $m, n \geq 1$ be two arbitrary integers as in the introduction. Assume that λ is a spectral parameter, and p, q, two matrix potentials:

$$p = p(x,t) = (p_{jk})_{m\times n},\ \ q = q(x,t) = (q_{kj})_{n\times m}. \tag{2.1}$$

The local integrable hierarchies of matrix AKNS equations are generated from the AKNS matrix spectral problems with matrix potentials

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ \ -i\phi_t = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi,\ \ r \geq 0, \tag{2.2}$$

where the Lax pair reads

$$\begin{aligned} &U = \lambda\Lambda + P,\ \ V^{[r]} = \lambda^r\Omega + Q^{[r]},\ \ \Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n), \\ &\Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \end{aligned} \tag{2.3}$$

where I_s is the identity matrix of size s, and $\alpha_1, \alpha_2, \beta_1$ and β_2 are arbitrary constants. The involved other two square matrices of size $m+n$ are given by

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.4}$$

which is called the potential matrix, and

$$Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}, \tag{2.5}$$

where $a^{[s]}, b^{[s]}, c^{[s]}, d^{[s]}$ are recursively defined by

$$b^{[0]} = 0,\ \ c^{[0]} = 0,\ \ a^{[0]} = \beta_1 I_m,\ \ d^{[0]} = \beta_2 I_n, \tag{2.6a}$$

$$b^{[s+1]} = \frac{1}{\alpha}(-ib_x^{[s]} - pd^{[s]} + a^{[s]}p),\ \ s \geq 0, \tag{2.6b}$$

$$c^{[s+1]} = \frac{1}{\alpha}(ic_x^{[s]} + qa^{[s]} - d^{[s]}q),\ \ s \geq 0, \tag{2.6c}$$

$$a_x^{[s]} = i(pc^{[s]} - b^{[s]}q),\ \ d_x^{[s]} = i(qb^{[s]} - c^{[s]}p),\ \ s \geq 1, \tag{2.6d}$$

in which $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$, and we take zero constants of integration to determine the differential polynomials uniquely. In particular, we can work out

$$\begin{aligned} &Q^{[1]} = \frac{\beta}{\alpha}\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \\ &Q^{[2]} = \frac{\beta}{\alpha}\lambda\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2}\begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} Q^{[3]} &= \frac{\beta}{\alpha}\lambda^2\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2}\lambda\begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix} \\ &\quad - \frac{\beta}{\alpha^3}\begin{bmatrix} i(pq_x - p_xq) & p_{xx} + 2pqp \\ q_{xx} + 2qpq & i(qp_x - q_xp) \end{bmatrix}. \end{aligned} \tag{2.8}$$

Obviously, when $m = 1$, the matrix spectral problems in (2.2) reduce to the multicomponent case [23], and when there are just a pair of nonzero potentials, p_{jk} and q_{kj}, the matrix spectral problems in (2.2) become the standard AKNS case [24].

It is direct to check that for each pair of fixed integers, m and n, the compatibility conditions of the matrix spectral problems in (2.2), i.e., the zero curvature equations

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0,\ \ r \geq 0, \tag{2.9}$$

lead to an integrable hierarchy of local matrix AKNS equations

$$p_t = i\alpha b^{[r+1]},\ \ q_t = -i\alpha c^{[r+1]},\ \ r \geq 0. \tag{2.10}$$

By a Lax operator algebra theory [25] and the trace identity [26], we can prove that (2.10) defines a hierarchy of commuting flows, which possesses infinitely many conservation laws. The first two nonlinear systems of integrable matrix AKNS equations are the matrix NLS equations ($r = 2$):

$$\begin{cases} p_t = -\dfrac{\beta}{\alpha^2} i(p_{xx} + 2pqp), \\ q_t = \dfrac{\beta}{\alpha^2} i(q_{xx} + 2qpq), \end{cases} \tag{2.11}$$

and the matrix mKdV equations ($r = 3$):

$$\begin{cases} p_t = -\dfrac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp), \\ q_t = -\dfrac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x). \end{cases} \tag{2.12}$$

To derive nonlocal integrable counterparts, let us introduce a group of specific nonlocal symmetry reductions for the spectral matrix U:

$$U^T(-x,-t,\lambda) = CU(x,t,\lambda)C^{-1}, \tag{2.13}$$

where T stands for the matrix transpose and the constant matrix C is a block matrix

$$C = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \tag{2.14}$$

in which $\Sigma_{1,2}$ are two arbitrary invertible and symmetric matrices of sizes m and n, respectively. By (2.3), each nonlocal symmetry reduction means the following reduction on the potential matrix:

$$P^T(-x,-t) = CP(x,t)C^{-1}, \tag{2.15}$$

which leads equivalently to the reverse-spacetime reduction on the two matrix potentials:

$$q(x,t)=\Sigma_2^{-1}p^T(-x,-t)\Sigma_1. \tag{2.16}$$

It follows directly from this potential reduction that

$$\begin{cases} (V^{[r]})^T(-x,-t,\lambda)=CV^{[r]}(x,t,\lambda)C^{-1}, \\ (Q^{[r]})^T(-x,-t,\lambda)=CQ^{[r]}(x,t,\lambda)C^{-1}. \end{cases} \tag{2.17}$$

where $r\geq 0$.

Now, we can see that all those imply that the nonlocal symmetry reductions in (2.13) do not cause any conflict in the zero curvature Eq. (2.9). Therefore, the local matrix AKNS Eqs. (2.10) reduces to the following hierarchy of nonlocal real reverse-spacetime matrix AKNS equations:

$$p_t=i\alpha b^{[r+1]}|_{q=\Sigma_2^{-1}p^T(-x,-t)\Sigma_1},\ r\geq 0, \tag{2.18}$$

which defines a hierarchy of commuting flows possessing infinitely many conservation laws as well. Two examples are the nonlocal real reverse-spacetime matrix NLS equations

$$ip_t(x,t)=\frac{\beta}{\alpha^2}[p_{xx}(x,t)+2p(x,t)\Sigma_2^{-1}p^T(-x,-t)\Sigma_1 p(x,t)], \tag{2.19}$$

and the nonlocal real reverse-spacetime matrix mKdV equations

$$p_t(x,t)=-\frac{\beta}{\alpha^3}[p_{xxx}(x,t)+3p(x,t)\Sigma_2^{-1}p^T(-x,-t)\Sigma_1 p_x(x,t) + 3p_x(x,t)\Sigma_2^{-1}p^T(-x,-t)\Sigma_1 p(x,t)], \tag{2.20}$$

where $\Sigma_{1,2}$ are two arbitrary invertible and symmetric matrices, and p is the matrix potential defined as in (2.1). Up taking special values for α,β and $\Sigma_{1,2}$, those two equations are reduced to the equations in (1.1) and (1.2). When $m=1$ and $\Sigma_1=1$, the two Eqs. (2.18) and (2.20) become the nonlocal real reverse-spacetime multicomponent NLS equations [22] and the nonlocal real reverse-spacetime multicomponent mKdV equations [8], respectively. Two difficulties in solving the nonlinear systems in (2.18) are the reverse-spacetime nonlocality and the higher-dimension of the systems, and will overcome those difficulties through applying the Riemann–Hilbert technique.

3. Riemann-Hilbert problems and inverse scattering

In what follows, we propose a kind of Riemann–Hilbert problems associated with the matrix spectral problems in (2.2) for the integrable hierarchies of nonlocal real reverse-spacetime matrix AKNS Eqs. (2.18) (see, e.g., [15,27,28] for local equations). Solutions to the resulting Riemann–Hilbert problems with the identity jump matrix, corresponding to the reflectionless inverse scattering transforms, generate soliton solutions for the nonlocal matrix AKNS equations in the following section.

3.1. Property of eigenfunctions under the nonlocal reductions

Suppose that all the potentials rapidly vanish when x or $t\to\infty$ or $-\infty$ so that there is no convergence problem involved. Let us consider an equivalent pair of matrix spectral problems to (2.2):

$$\psi_x=i\lambda[\Lambda,\psi]+\check{P}\psi,\ \check{P}=iP, \tag{3.1}$$

$$\psi_t=i\lambda^r[\Omega,\psi]+\check{Q}^{[r]}\psi,\ \check{Q}^{[r]}=iQ^{[r]}, \tag{3.2}$$

which $\psi=\phi\,\mathrm{e}^{-i\lambda\Lambda x-i\lambda^r\Omega t}$ satisfies if ϕ solves (2.2). Applying a generalized Liouville's formula [29], we can have $(\det\psi)_x=0$, because of $\mathrm{tr}(\check{P})=\mathrm{tr}(\check{Q}^{[r]})=0$. To establish associated Riemann–Hilbert problems whose jump matrices can be easily worked out, we adopt the following adjoint spectral problems, besides the spectral problems. The adjoint counterparts of the x-part problem of (2.2) and the spectral problem (3.1) are defined as

$$i\tilde{\phi}_x=\tilde{\phi}U, \tag{3.3}$$

and

$$i\tilde{\psi}_x=\lambda[\tilde{\psi},\Lambda]+\tilde{\psi}P, \tag{3.4}$$

which $\tilde{\phi}=\phi^{-1}$ and $\tilde{\psi}=\psi^{-1}$ satisfy, respectively.

Let $\psi(\lambda)$ be a matrix eigenfunction of the spatial spectral problem (3.1) associated with an eigenvalue λ. First, it is easy to see that $C\psi^{-1}(x,t,\lambda)$ is a matrix adjoint eigenfunction associated with the same eigenvalue λ. Second, under the nonlocal symmetry reductions in (2.13), we can compute that

$$\begin{aligned} & i[\psi^T(-x,-t,\lambda)C]_x=i[-(\psi_x)^T(-x,-t,\lambda)C] \\ & =i\{-i\lambda[\Lambda,\psi(-x,-t,\lambda)]-\check{P}(-x,-t)\psi(-x,-t,\lambda)\}^TC \\ & =i\{-i\lambda[\psi^T(-x,-t,\lambda),\Lambda]-\psi^T(-x,-t,\lambda)\check{P}^T(-x,-t)\}C \\ & =\lambda[\psi^T(-x,-t,\lambda)C,\Lambda]+\psi^T(-x,-t,\lambda)C[C^{-1}P^T(-x,-t)C] \\ & =\lambda[\psi^T(-x,-t,\lambda)C,\Lambda]+\psi^T(-x,-t,\lambda)CP(x,t). \end{aligned}$$

This implies that the matrix function

$$\tilde{\psi}(x,t,\lambda):=\psi^T(-x,-t,\lambda)C \tag{3.5}$$

presents another matrix adjoint eigenfunction associated with the same original eigenvalue λ, namely, $\psi^T(-x,-t,\lambda)C$ solves the adjoint spectral problem (3.4).

Now, upon imposing the asymptotic conditions that $\psi\to I_{n+1}$, as x or $t\to\infty$ or $-\infty$, the uniqueness of solutions tells that

$$\psi^T(-x,-t,\lambda)=C\psi^{-1}(x,t,\lambda)C^{-1} \tag{3.6}$$

holds for a matrix eigenfunction ψ that solves the spectral problem (3.1). This enables us to conclude that the spectral problem (3.1) has the property (3.6) for its eigenfunctions with the asymptotic condition imposed above, under the nonlocal symmetry reductions in (2.13).

3.2. Associated Riemann-Hilbert problems

Let us now formulate a class of associated Riemann–Hilbert problems with the space variable x. The whole procedure is really the same as the one for the local case [19], since the nonlocal symmetry reductions in (2.13) do not present any problem in reducing the associated Riemann–Hilbert problems in the local unreduced case. However, we present it below for the sake of clarity.

In order to facilitate the expression in our analysis, we assume that

$$\alpha=\alpha_1-\alpha_2<0,\ \beta=\beta_1-\beta_2<0, \tag{3.7}$$

as usual. To establish the scattering problem, let us take the two matrix eigenfunctions $\psi^{\pm}(x,\lambda)$ of (3.1) with the asymptotic conditions

$$\psi^{\pm}\to I_{n+1},\ \text{when } x\to\pm\infty, \tag{3.8}$$

respectively. From $(\det\psi)_x=0$, we obtain that $\det\psi^{\pm}=1$ for all $x\in\mathbb{R}$. Observing that

$$\phi^{\pm}=\psi^{\pm}E,\ E=\mathrm{e}^{i\lambda\Lambda x}, \tag{3.9}$$

are both matrix eigenfunctions of the spectral problems (2.2), they must be linearly dependent, and consequently, we can have

$$\psi^- E = \psi^+ E S(\lambda),\ \ \lambda \in \mathbb{R}, \tag{3.10}$$

for some matrix $S(\lambda)$, which is commonly called the scattering matrix. Further, it follows from $\det \psi^\pm = 1$ that we obtain $\det S(\lambda) = 1$.

Based on the matrix spectral problems in (2.2), we know that the matrix eigenfunctions $\psi^\pm$ satisfy the following Volterra integral equations:

$$\psi^-(\lambda, x) = I_{m+n} + \int_{-\infty}^{x} e^{i\lambda \Lambda (x-y)} \check{P}(y) \psi^-(\lambda, y) e^{i\lambda \Lambda (y-x)}\, dy, \tag{3.11}$$

$$\psi^+(\lambda, x) = I_{m+n} - \int_{x}^{\infty} e^{i\lambda \Lambda (x-y)} \check{P}(y) \psi^+(\lambda, y) e^{i\lambda \Lambda (y-x)}\, dy, \tag{3.12}$$

where the asymptotic conditions (3.8) have been imposed. It therefore follows from the theory of Volterra integral equations that the eigenfunctions $\psi^\pm$ can exist and allow analytical continuations off the real line $\lambda \in \mathbb{R}$ as long as the integrals on the right hand sides converge.

Clearly from the diagonal form of Λ, we can observe that the integral equation for the last n columns of ψ^+ contains only the exponential factor $e^{i\alpha\lambda(x-y)}$, which decays exponentially because of $y > x$ in the integral, when λ takes values in the upper half-plane $\mathbb{C}^+$, and the integral equation for the first m columns of ψ^- contains only the exponential factor $e^{-i\alpha\lambda(x-y)}$, which decays exponentially because of $y < x$ in the integral, when λ takes values in the upper half-plane $\mathbb{C}^+$. In this way, those $m+n$ columns are analytical in the upper half-plane $\mathbb{C}^+$ and continuous in the closed upper half-plane $\bar{\mathbb{C}}^+$. By a similar argument, we can show that the last n columns of ψ^- and the first m columns of ψ^+ are analytical in the lower half-plane $\mathbb{C}^-$ and continuous in the closed lower half-plane $\bar{\mathbb{C}}^-$.

In order to generate two generalized matrix Jost solutions, denoted by T^+ and T^-, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous in $\bar{\mathbb{C}}^+$ and $\bar{\mathbb{C}}^-$, respectively, let us write

$$\psi^\pm = (\psi_1^\pm, \psi_2^\pm, \ldots, \psi_{m+n}^\pm), \tag{3.13}$$

where $\psi_j^\pm$ denote the jth columns of $\phi^\pm$ $(1 \le j \le m+n)$, and

$$\tilde{\psi}^\pm = \begin{bmatrix} \tilde{\psi}^{\pm,1} \\ \tilde{\psi}^{\pm,2} \\ \vdots \\ \tilde{\psi}^{\pm,m+n} \end{bmatrix}, \tag{3.14}$$

where $\tilde{\psi}^{\pm,j}$ denote the jth rows of $\tilde{\psi}^\pm$ $(1 \le j \le m+n)$. We also denote

$$H_1 = \operatorname{diag}(I_m, \underbrace{0, \ldots, 0}_{n}),\ \ H_2 = \operatorname{diag}(\underbrace{0, \ldots, 0}_{m}, I_n). \tag{3.15}$$

Then, we can take the generalized matrix Jost solution T^+ as

$$T^+ = T^+(x, \lambda) = (\psi_1^-, \ldots, \psi_m^-, \psi_{m+1}^+, \ldots, \psi_{m+n}^+) = \psi^- H_1 + \psi^+ H_2, \tag{3.16}$$

which is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \bar{\mathbb{C}}^+$.

To define the other generalized matrix Jost solution T^-, i.e., the analytic counterpart of T^+ in the lower half-plane $\mathbb{C}^-$, we use the adjoint matrix spectral problems. We remark that when ϕ and ψ solve the two spectral problems, the inverse matrices $\tilde{\phi} = \phi^{-1}$ and $\tilde{\psi} = \psi^{-1}$ solve the corresponding two adjoint spectral problems, respectively. It is advantageous for an explicit formulation of Riemann–Hilbert problems to take the other generalized matrix Jost solution T^- as the adjoint matrix solution of (3.4), i.e.,

$$T^- = \begin{bmatrix} \tilde{\psi}^{-,1} \\ \vdots \\ \tilde{\psi}^{-,m} \\ \tilde{\psi}^{+,m+1} \\ \vdots \\ \tilde{\psi}^{+,m+n} \end{bmatrix} = H_1 \tilde{\psi}^- + H_2 \tilde{\psi}^+ = H_1 (\psi^-)^{-1} + H_2 (\psi^+)^{-1}, \tag{3.17}$$

which is analytic for $\lambda \in \mathbb{C}^-$ and continuous for $\lambda \in \bar{\mathbb{C}}^-$.

It further follows from $\det \psi^\pm = 1$, the above definitions of $T^\pm$, the asymptotic conditions in (3.8) assumed from the outset, and the scattering relation (3.10) between ψ^+ and ψ^- that

$$\lim_{x\to\infty} T^+(x, \lambda) = \begin{bmatrix} S_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix},\ \ \lambda \in \bar{\mathbb{C}}^+, \qquad \lim_{x\to\infty} T^-(x, \lambda) = \begin{bmatrix} \hat{S}_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix},\ \ \lambda \in \bar{\mathbb{C}}^-, \tag{3.18}$$

and

$$\det T^+(x, \lambda) = S_{11}(\lambda),\ \ \det T^-(x, \lambda) = \hat{S}_{11}(\lambda), \tag{3.19}$$

where we split $S(\lambda)$ and $S^{-1}(\lambda)$ as follows:

$$S(\lambda) = \begin{bmatrix} S_{11}(\lambda) & S_{12}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{bmatrix}, \qquad S^{-1}(\lambda) = (S(\lambda))^{-1} = \begin{bmatrix} \hat{S}_{11}(\lambda) & \hat{S}_{12}(\lambda) \\ \hat{S}_{21}(\lambda) & \hat{S}_{22}(\lambda) \end{bmatrix}, \tag{3.20}$$

$S_{11}, \hat{S}_{11}$ being $m \times m$ matrices, $S_{12}, \hat{S}_{12}$ being $m \times n$ matrices, $S_{21}, \hat{S}_{21}$ being $n \times m$ matrices, and $S_{22}, \hat{S}_{22}$ being $n \times n$ matrices.

Now making use of the particular forms of T^+ and T^-, we can take the following two unimodular generalized matrix Jost solutions:

$$\begin{cases} G^+(x, \lambda) = T^+(x, \lambda) \begin{bmatrix} S_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix},\ \ \lambda \in \bar{\mathbb{C}}^+; \\ (G^-)^{-1}(x, \lambda) = \begin{bmatrix} \hat{S}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} T^-(x, \lambda),\ \ \lambda \in \bar{\mathbb{C}}^-. \end{cases} \tag{3.21}$$

Those two generalized matrix Jost solutions present the required matrix Riemann–Hilbert problems on the real line for the nonlocal real reverse-spacetime matrix AKNS Eqs. (2.18):

$$G^+(x, \lambda) = G^-(x, \lambda) G_0(x, \lambda),\ \ \lambda \in \mathbb{R}, \tag{3.22}$$

where based on (3.10) and (3.21), the jump matrix G_0 reads

$$G_0(x, \lambda) = E \begin{bmatrix} \hat{S}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} \tilde{S}(\lambda) \begin{bmatrix} S_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} E^{-1}, \tag{3.23}$$

in which from the definition of G^+ and G^-, we know that the matrix $\tilde{S}(\lambda)$ has the following factorization:

$$\tilde{S}(\lambda) = (H_1 + H_2 S(\lambda))(H_1 + S^{-1}(\lambda) H_2). \tag{3.24}$$

A direct computation leads to

$$\tilde{S}(\lambda) = \begin{bmatrix} I_m & \hat{S}_{12} \\ S_{21} & I_n \end{bmatrix}. \tag{3.25}$$

Finally, directly from the Volterra integral Eqs. (3.11) and (3.12), we can have the canonical normalization conditions for the associated Riemann–Hilbert problems:

$$G^{\pm}(x,\lambda)\to I_{n+1},\ \text{ when } \lambda\in\bar{\mathbb{C}}^{\pm}\to\infty. \tag{3.26}$$

Moreover, based on the involution property (3.6), we can directly show that

$$(G^{+})^{T}(-x,-t,\lambda)=C(G^{-})^{-1}(x,t,\lambda)C^{-1}, \tag{3.27}$$

and thus, the jump matrix G_0 satisfies the corresponding involution property

$$G_0^{T}(-x,-t,\lambda)=CG_0(x,t,\lambda)C^{-1}. \tag{3.28}$$

It is also worth noting that the jump matrix G_0, defined by (3.23) and (3.25), carries all basic scattering data from the scattering matrix $S(\lambda)$ that we need to construct the inverse scattering transforms.

3.3. Evolution law of the scattering data

To present the direct scattering transforms, let us compute the derivative of Eq. (3.10) with the time variable t, and utilize the temporal spectral problem (3.2) for $\psi^{\pm}$:

$$\psi^{\pm}_t=i\lambda^{r}[\Omega,\psi^{\pm}]+\check{Q}^{[r]}\psi^{\pm},\ r\geq 0. \tag{3.29}$$

Then, a direct computation shows that the scattering matrix S has to obey a matrix evolution law:

$$S_t=i\lambda^{r}[\Omega,S], \tag{3.30}$$

where Ω is defined as in (2.3). This precisely yields the time dependence of the scattering coefficients:

$$S_{12}=S_{12}(t,\lambda)=S_{12}(0,\lambda)e^{i\beta\lambda^{r}t},\ S_{21}=S_{21}(t,\lambda)=S_{21}(0,\lambda)e^{-i\beta\lambda^{r}t}, \tag{3.31}$$

and tells that all other scattering coefficients are independent of the time variable t. Further, we can immediately drive the evolution law for the inverse matrix of $S(\lambda)$, $\hat{S}(\lambda)$, and thus, for the jump matrix $G_0(\lambda)$.

3.4. Gelfand–Levitan–Marchenko type integral equations

In order to derive Gelfand–Levitan–Marchenko type integral equations for the generalized matrix Jost solutions, we transform the Riemann–Hilbert problems in (3.22) as follows:

$$\begin{cases} G^{+}-G^{-}=G^{-}v,\ v=G_0-I_{n+1},\ \text{on } \mathbb{R},\\ G^{\pm}\to I_{n+1} \text{ as } \lambda\in\bar{\mathbb{C}}^{\pm}\to\infty,\end{cases} \tag{3.32}$$

where G_0 is determined by (3.23) and (3.25). Define $G(\lambda)=G^{\pm}(\lambda)$ if $\lambda\in\mathbb{C}^{\pm}$. Assume that G has R simple poles: $\{\mu_j\}_{j=1}^{R}$, where R is an arbitrary natural number, and that the poles are off the real line $\mathbb{R}$ to avoid spectral singularities. Further, let us introduce

$$\tilde{G}^{\pm}(\lambda)=G^{\pm}(\lambda)-\sum_{j=1}^{R}\frac{G_j}{\lambda-\mu_j},\ \lambda\in\bar{\mathbb{C}}^{\pm},\ \tilde{G}(\lambda)=\tilde{G}^{\pm}(\lambda),\ \lambda\in\mathbb{C}^{\pm}, \tag{3.33}$$

where G_j is the residue of $G(\lambda)$ at $\lambda=\mu_j$, i.e.,

$$G_j=\operatorname{res}(G(\lambda),\mu_j)=\lim_{\lambda\to\mu_j}(\lambda-\mu_j)G(\lambda). \tag{3.34}$$

In this way, we have

$$\begin{cases} \tilde{G}^{+}-\tilde{G}^{-}=G^{+}-G^{-}=G^{-}v,\ \text{on } \mathbb{R},\\ \tilde{G}^{\pm}\to I_{n+1} \text{ as } \lambda\in\bar{\mathbb{C}}^{\pm}\to\infty.\end{cases} \tag{3.35}$$

Through the Sokhotski–Plemelj formula [30], we obtain the solution to (3.35):

$$\tilde{G}(\lambda)=I_{n+1}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(G^{-}v)(\xi)}{\xi-\lambda}\,d\xi. \tag{3.36}$$

Now taking the limit as $\lambda\to\mu_l$, we obtain

$$\text{lhs}=\lim_{\lambda\to\mu_l}\tilde{G}=F_l-\sum_{j\neq l}^{R}\frac{G_j}{\mu_l-\mu_j},$$

$$\text{rhs}=I_{n+1}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(G^{-}v)(\xi)}{\xi-\mu_l}\,d\xi,$$

where

$$F_l=\lim_{\lambda\to\mu_l}\frac{(\lambda-\mu_l)G(\lambda)-G_l}{\lambda-\mu_l},\ 1\leq l\leq R. \tag{3.37}$$

When all this is carried through, we arrive at the following Gelfand–Levitan–Marchenko type integral equations:

$$I_{n+1}-F_l+\sum_{j\neq l}^{R}\frac{G_j}{\mu_l-\mu_j}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(G^{-}G_0)(\xi)}{\xi-\mu_l}\,d\xi \\ -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{G^{-}(\xi)}{\xi-\mu_l}\,d\xi=0,\ 1\leq l\leq R. \tag{3.38}$$

These integral equations are used to determine solutions to the associated Riemann–Hilbert problems by (3.32), and hence, produce the generalized matrix Jost solutions. However, little was yet known about the general theory of existence and uniqueness of solutions. Only in the case of identity jump matrices, we will be able to present a specific solution formulation explicitly in the next section.

3.5. Recovery of the potential

As the last part of the inverse scattering transforms, we retrieve the potential matrix P from the generalized matrix Jost solutions. Let us begin with an asymptotic expansion for the generalized matrix Jost solution G^{+}:

$$G^{+}(x,t,\lambda)=I_{n+1}+\frac{1}{\lambda}G_1^{+}(x,t)+O(\frac{1}{\lambda^2}),\ \lambda\to\infty, \tag{3.39}$$

where we have used the canonical normalization condition for G^{+}. Then, inserting the asymptotic expansion into the matrix spectral problem (3.1) and comparing the $O(1)$ terms engender

$$P=\lim_{\lambda\to\infty}\lambda[G^{+}(\lambda),\Lambda]=-[\Lambda,G_1^{+}]. \tag{3.40}$$

To get the potential for the nonlocal equations, one needs to check if an involution property (2.15) holds for P or equivalently if the following involution property holds for G_1^{+}:

$$(G_1^{+})^{T}(-x,-t)=-CG_1^{+}(x,t)C^{-1}. \tag{3.41}$$

Then if the answer is yes, we immediately obtain the solutions to the nonlocal real reverse-spacetime integrable matrix AKNS Eqs. (2.18):

$$p=-\alpha G^{+}_{1,12}, \tag{3.42}$$

where we have similarly partitioned the matrix G_1^{+} into four blocks as follows:

$$G_1^{+}=\begin{bmatrix} G^{+}_{1,11} & G^{+}_{1,12}\\ G^{+}_{1,21} & G^{+}_{1,22}\end{bmatrix}=\begin{bmatrix} (G^{+}_{1,11})_{n\times n} & (G^{+}_{1,12})_{n\times m}\\ (G^{+}_{1,21})_{m\times n} & (G^{+}_{1,22})_{m\times m}\end{bmatrix}. \tag{3.43}$$

Once the solutions $\{G^{+}(\lambda),G^{-}(\lambda)\}$ to the associated Riemann–Hilbert problems are determined, the potential matrix P defined by (3.42), provides solutions to the nonlocal real reverse-spacetime integrable PT-symmetric matrix AKNS Eqs. (2.18).

4. Soliton solutions

In this section, we would like to construct soliton solutions to the nonlocal real reverse-spacetime integrable matrix AKNS Eqs. (2.18). As usual, we avoid solving Gelfand–Levitan–Marchenko type integral equations, and instead, we solve the associated linear systems with the Riemann–Hilbert problems directly. In our analysis, eigenvalues could equal adjoint eigenvalues, which never happened before in the local theory.

Let $N \geq 1$ be another arbitrary integer. Assume that $\det S_{11}(\lambda)$ has N geometrically simple zeros $\{\lambda_k \in \mathbb{C},\ 1 \leq k \leq N\}$, and $\det \hat{S}_{11}(\lambda)$ has N geometrically simple zeros $\{\hat{\lambda}_k \in \mathbb{C},\ 1 \leq k \leq N\}$. Then, each of $\ker T^+(\lambda_k)$, $1 \leq k \leq N$, contains only one single basis column vector, denoted by v_k, $1 \leq k \leq N$; and each of $\ker T^-(\hat{\lambda}_k)$, $1 \leq k \leq N$, one single basis row vector, denoted by $\hat{v}_k$, $1 \leq k \leq N$. Beginning with the two equations,

$$T^+(\lambda_k)v_k = 0,\ \ \hat{v}_k T^-(\hat{\lambda}_k) = 0,\ \ 1 \leq k \leq N, \tag{4.1}$$

we can determine the kernel vectors $v_k, \hat{v}_k$, $1 \leq k \leq N$, by making use of the associated spectral problems that T^+ and T^- satisfy (see, e.g., [19] for the local case).

It is known that when taking the identity jump matrix, the Riemann–Hilbert problems, by (3.22) and (3.23), with the canonical normalization conditions in (3.26) and the zero structures given in (4.1), can be solved explicitly [15,31,32], and then, we can determine the potential matrix P that provides solutions to the nonlocal real reverse-spacetime integrable matrix AKNS equations.

4.1. Solutions to special Riemann-Hilbert problems

As usual, soliton solutions are constructed from the Riemann–Hilbert problems in (3.22) with the identity jump matrix $G_0 = I_{m+n}$, which can be achieved if we assume the zero reflection coefficient conditions

$$S_{21} = \hat{S}_{12} = 0, \tag{4.2}$$

in the inverse scattering problem. Note that in the case of nonlocal integrable equations, we often do not have the condition

$$\{\lambda_k | 1 \leq k \leq N\} \cap \{\hat{\lambda}_k | 1 \leq k \leq N\} = \emptyset, \tag{4.3}$$

and thus, we need to establish a new solution formulation, which generalizes the one for the local case in the literature [15,31,32]. A direct computation shows that solutions to a kind of special Riemann–Hilbert problems without the condition (4.3) can be presented as follows.

To incorporate the indicated requirements, let us define G^+ and G^- by

$$G^+(\lambda) = I_{m+n} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l},\ \ (G^-)^{-1}(\lambda) = I_{m+n} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}, \tag{4.4}$$

where $M = (m_{kl})_{N\times N}$ is a square matrix with its entries:

$$m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, & \text{when } \lambda_l \neq \hat{\lambda}_k, \\ 0, & \text{when } \lambda_l = \hat{\lambda}_k, \end{cases} \quad 1 \leq k, l \leq N. \tag{4.5}$$

This M-matrix is different from the traditional one in the literature, since we included the case of $\lambda_l = \hat{\lambda}_k$, which often occurs in the case of nonlocal integrable equations. We can now show that

$$\Big(\prod_{l=1}^{N}(\lambda - \hat{\lambda}_l) G^+\Big)(\lambda_k) v_k = 0,\ \ \hat{v}_k \Big(\prod_{l=1}^{N}(\lambda - \lambda_l)(G^-)^{-1}\Big)(\hat{\lambda}_k) = 0,\ \ 1 \leq k \leq N, \tag{4.6}$$

and G^+ and G^- satisfy

$$(G^-)^{-1}(\lambda) G^+(\lambda) = I_{m+n}, \tag{4.7}$$

if we additionally require an orthogonal condition

$$\hat{v}_k v_l = 0,\ \ \text{when } \lambda_l = \hat{\lambda}_k,\ \ 1 \leq k, l \leq N. \tag{4.8}$$

Obviously, if the condition (4.3) holds, then the above result reduces to the one presented in the literature (see, e.g., [15,31,32]). If $\lambda_k \in \mathbb{C}^+$ and $\hat{\lambda}_k \in \mathbb{C}^-$, $1 \leq k \leq N$, then the above two matrices G^+ and G^- are meromorphic in the upper and lower half-planes, respectively, and thus, they present solutions to the Riemann–Hilbert problems with the identity jump matrix on the real line. However, the general case provides more diverse solutions.

4.2. Soliton solutions

To compute soliton solutions to the nonlocal real reverse-spacetime integrable matrix AKNS Eqs. (2.18), we have to check if the involution condition (2.15) or (3.41) is satisfied. To implement this condition, motivated by the previous property of eigenfunctions in Section 3.1, we can assume [19] that $\ker T^+(\lambda_k)$ and $\ker T^-(\lambda_k)$, $1 \leq k \leq N$, are spanned by

$$v_k(x,t) = v_k(x,t,\lambda_k) = \mathrm{e}^{\mathrm{i}\lambda_k \Lambda x + \mathrm{i}\lambda_k^r \Omega t} w_k,\ \ 1 \leq k \leq N, \tag{4.9}$$

and

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\lambda}_k) = \hat{w}_k \mathrm{e}^{-\mathrm{i}\hat{\lambda}_k \Lambda x - \mathrm{i}\hat{\lambda}_k^r \Omega t} C,\ \ 1 \leq k \leq N, \tag{4.10}$$

respectively. Here w_k and $\hat{w}_k$, $1 \leq k \leq N$, are arbitrary column and row vectors, but the orthogonal condition (4.8) requires

$$\hat{w}_k C w_l = 0,\ \ \text{when } \lambda_l = \hat{\lambda}_k,\ \ 1 \leq k, l \leq N. \tag{4.11}$$

Now, based on the above construction, we can directly verify under (4.11) that G_1^+ satisfies the involution property (3.41), i.e., the solution to the special Riemann–Hilbert problem, determined by (4.4) and (4.5), satisfies (3.27), and thus, the matrix potential defined by (3.42) presents soliton solutions to the nonlocal real reverse-spacetime integrable PT-symmetric matrix AKNS Eqs. (2.18):

$$p = \alpha \sum_{k,l=1}^{N} v_{k,1} (M^{-1})_{kl} \hat{v}_{l,2}, \tag{4.12}$$

where $M = (m_{kl})_{N\times N}$ is defined by (4.5), and we split v_k and $\hat{v}_k$, determined by (4.9) and (4.10), as $v_k = ((v_{k,1})^T, (v_{k,2})^T)^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2})$, with $v_{k,1}$ and $\hat{v}_{k,1}$ being m-dimensional column and row vectors, respectively, and $v_{k,2}$ and $\hat{v}_{k,2}$ being n-dimensional column and row vectors, respectively.

5. Concluding remarks

Nonlocal integrable hierarchies of real reverse-spacetime PT-symmetric matrix AKNS equations were presented from a group of nonlocal symmetry reductions, and their associated Riemann–Hilbert problems were formulated via the matrix AKNS spectral problems and adjoint matrix AKNS spectral problems. The Sokhotski–Plemelj formula was used to transform the associated Riemann–Hilbert problems into systems of Gelfand–Levitan–Marchenko type integral equations. Soliton solutions to the nonlocal real reverse-spacetime integrable matrix AKNS equations were generated from the Riemann–Hilbert problems with the identity jump matrix (or equivalently the reflectionless inverse scattering transforms).

It is also worth remarking that it would be particularly interesting to find a certain kind of connections among different solution approaches, including the Hirota direct method [33], the Wronskian technique [34,35] and the Darboux transformation [36]. Moreover, various recent studies have exhibited great richness of other kinds of solutions to nonlinear dispersive wave equations, including lump and rogue wave solutions and their interaction solutions [37–42], algebro–geometric solutions [43,44] and solitonless solutions [45,46]. Absolutely, it would be very intriguing to explore those exact solutions to nonlinear dispersive wave equations, particularly, lump and rogue wave solutions, through the Riemann–Hilbert technique. The kinetic mechanism of soliton propagation and interaction in the nonlocal theory also certainly deserves further investigations.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Wen-Xiu Ma: Conceptualization, Methodology, Writing, Visualization, Investigation, Validation.

Acknowledgments

The work was supported in part by NSFC, China under the grants 11975145, 11972291 and 51771083, and the Natural Science Foundation for Colleges and Universities in Jiangsu Province, China (17 KJB 110020).

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