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INVERSE SCATTERING AND SOLITON SOLUTIONS OF NONLOCAL REVERSE-SPACETIME NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. The paper presents nonlocal reverse-spacetime PT-symmetric multicomponent nonlinear Schrödinger (NLS) equations under a specific nonlocal group reduction, and generates their inverse scattering transforms and soliton solutions by the Riemann-Hilbert technique. The Sokhotski-Plemelj formula is used to determine solutions to a class of associated Riemann-Hilbert problems and transform the systems that generalized Jost solutions need to satisfy. A formulation of solutions is developed for the Riemann-Hilbert problems associated with the reflectionless transforms, and the corresponding soliton solutions are constructed for the presented nonlocal reverse-spacetime PT-symmetric NLS equations.

1. INTRODUCTION

Nonlocal integrable equations are one of new hot topics in soliton theory. Particularly, a couple of scalar nonlocal nonlinear Schrödinger (NLS) equations and modified Korteweg-de Vries (mKdV) equations have been primary models which are identified as significant to understanding nonlocal nonlinear phenomena and providing a foundation for fostering more innovative research that advances the theory [1, 2]. The inverse scattering technique has been shown to be powerful in solving those nonlocal nonlinear equations, under either zero or nonzero boundary conditions [3–5]. Moreover, Darboux transformations [6–8] and the Hirota bilinear method [9] are used to construct their N-soliton solutions. A few higher-dimensional [10] and multicomponent [5, 11] generalizations of nonlocal integrable equations have also been proposed and studied. Such nonlinear integrable equations share the PT symmetry, i.e., they are invariant under the parity-time transformation $(x \to -x,\ t \to -t,\ i \to -i)$.

Riemann-Hilbert problems have been successfully used to formulate the inverse scattering transforms and generate soliton solutions to both local and nonlocal

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integrable equations. Various integrable equations have been studied by analyzing the associated Riemann-Hilbert problems systematically. Illustrative examples include the multiple wave interaction equations [12], the general coupled nonlinear Schrödinger equations [13], the Harry Dym equation [14], the generalized Sasa-Satsuma equation [15], multicomponent mKdV equations [16], the nonlocal reverse-space scalar NLS equation [17], and nonlocal reverse-time multicomponent NLS equations [5]. In this paper, we would like to present a class of nonlocal reverse-spacetime PT-symmetric multicomponent integrable NLS equations under a specific nonlocal group symmetry reduction, and analyze their inverse scattering transforms and soliton solutions within the formulation of Riemann-Hilbert problems.

The rest of the paper is organized as follows. In Section 2, we make a nonlocal group reduction to generate nonlocal reverse-spacetime PT-symmetric NLS equations from the multicomponent AKNS matrix spectral problems. In Section 3, we analyze their inverse scattering transforms by establishing a class of associated Riemann-Hilbert problems. In Section 4, we develop a formulation of solutions to the Riemann-Hilbert problems with the identity jump matrix and construct N-soliton solutions from the reflectionless inverse scattering transforms. The final section presents a conclusion and a few concluding remarks.

2. Nonlocal reverse-spacetime NLS equations

Let n be an arbitrary natural number. Assume that λ stands for a spectral parameter, and u is a $2n$-dimensional potential

$$u = u(x,t) = (p, q^T)^T,\ p = (p_1, p_2, \dots, p_n),\ q = (q_1, q_2, \dots, q_n)^T. \tag{2.1}$$

It is known that the multicomponent local NLS equations are associated with the multicomponent AKNS matrix spectral problems (see, e.g., [18]):

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ -i\phi_t = V\phi = V(u,\lambda)\phi, \tag{2.2}$$

with the Lax pair

$$U = \lambda\Lambda + P,\ V = \lambda^2\Omega + Q. \tag{2.3}$$

The involved four matrices are defined by $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2 I_n)$, $\Omega = \mathrm{diag}(\beta_1, \beta_2 I_n)$, and

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix},\ Q = \frac{\beta}{\alpha}\lambda \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2} \begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix}, \tag{2.4}$$

where I_n is the identity matrix of size n, $\alpha_1, \alpha_2, \beta_1$, and β_2 are arbitrary constants, and $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$. It is obvious that if $p_j = q_j = 0,\ 2 \le j \le n$, the matrix spectral problems in (2.2) reduce to the original AKNS ones [19]. The compatibility condition of (2.2), i.e., the zero curvature equation

$$U_t - V_x + i[U,V] = 0, \tag{2.5}$$

presents the following multicomponent local NLS equations:

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} + 2pqp),\ q_t = \frac{\beta}{\alpha^2} i(q_{xx} + 2qpq). \tag{2.6}$$

To introduce nonlocal counterparts, we make a specific nonlocal group reduction for the spectral matrix

$$U^T(-x,-t,\lambda) = CU(x,t,\lambda)C^{-1},\ C = \begin{bmatrix} 1 & 0 \\ 0 & \Sigma \end{bmatrix},\ \Sigma^T = \Sigma, \tag{2.7}$$

where T stands for the matrix transpose and Σ is an arbitrary constant invertible symmetric matrix. This group reduction means that

$$P^T(-x,-t) = CP(x,t)C^{-1}, \tag{2.8}$$

which equivalently leads to

$$q(x,t) = \Sigma^{-1} p^T(-x,-t). \tag{2.9}$$

Under this potential reduction, we have

$$V^T(-x,-t,\lambda) = CV(x,t,\lambda)C^{-1},\ Q^T(-x,-t,\lambda) = CQ(x,t,\lambda)C^{-1}. \tag{2.10}$$

All those certainly imply that the group reduction in (2.7) agrees with the zero curvature equation (2.5). Therefore, from the multicomponent local NLS equations (2.6), we obtain the following multicomponent nonlocal reverse-spacetime integrable NLS equations:

$$ip_t(x,t) = \frac{\beta}{\alpha^2}[p_{xx}(x,t) + 2p(x,t)\Sigma^{-1}p^T(-x,-t)p(x,t)], \tag{2.11}$$

where Σ is an arbitrary invertible symmetric matrix. When $n=1$, we can obtain two scalar examples (see, e.g., [2]):

$$ip_t(x,t) = -p_{xx}(x,t) + 2\sigma p^2(x,t)p(-x,-t),\ \sigma = \pm 1. \tag{2.12}$$

Those equations possess the PT symmetry: if $p(x,t)$ is a solution, so is $p^*(-x,-t)$, where $*$ denotes the complex conjugate.

3. Inverse scattering transforms

In this section, we discuss the scattering and inverse scattering for the nonlocal reverse-spacetime NLS equations (2.11) by developing Riemann-Hilbert problems associated with the spectral problems in (2.2) [12] (see also [20, 21]). The results will provide the basis for generating soliton solutions in the next section.

3.1. **Property of eigenfunctions.** Assume that all the potentials satisfy

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|^{m_1}|t|^{m_2}\sum_{j=1}^{n}|p_j|\,dxdt < \infty,\ m_1, m_2 = 0, 1. \tag{3.1}$$

Upon setting $\check{P} = iP$ and $\check{Q} = iQ$, an equivalent pair of matrix spectral problems to (2.2) is given by

$$\psi_x = i\lambda[\Lambda,\psi] + \check{P}\psi, \tag{3.2}$$

$$\psi_t = i\lambda^2[\Omega,\psi] + \check{Q}\psi, \tag{3.3}$$

which are connected with (2.2) via $\psi = \phi \mathrm{e}^{-i\lambda\Lambda x}$. Applying a generalized Liouville's formula [22], we can have $(\det\psi)_x = 0$, since $\mathrm{tr}(\check{P}) = 0$. To develop associated Riemann-Hilbert problems, we adopt the following adjoint equation of the x-part of (2.2) and the adjoint equation of (3.2):

$$i\tilde{\phi}_x = \tilde{\phi}U \tag{3.4}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi}P, \tag{3.5}$$

between which there is a link: $\tilde{\psi} = e^{i\lambda\Lambda x}\tilde{\phi}$.

Let $\psi(\lambda)$ be a matrix eigenfunction of the spatial spectral problem (3.2) associated with an eigenvalue λ. On one hand, it is clear that $C\psi^{-1}(x,t,\lambda)$ is a matrix adjoint eigenfunction associated with the same eigenvalue λ. On the other hand, under the nonlocal potential reduction (2.8), we can have

$$\begin{aligned}
&i[\psi^T(-x,-t,\lambda)C]_x = i[-(\psi_x)^T(-x,-t,\lambda)C] \\
&= i\{-i\lambda[\Lambda,\psi(-x,-t,\lambda)] - \check{P}(-x,-t)\psi(-x,-t,\lambda)\}^T C \\
&= i\{-i\lambda[\psi^T(-x,-t,\lambda),\Lambda] - \psi^T(-x,-t,\lambda)\check{P}^T(-x,-t)\}C \\
&= \lambda[\psi^T(-x,-t,\lambda)C,\Lambda] + \psi^T(-x,-t,\lambda)C[C^{-1}P^T(-x,-t)C] \\
&= \lambda[\psi^T(-x,-t,\lambda)C,\Lambda] + \psi^T(-x,-t,\lambda)CP(x,t),
\end{aligned}$$

and thus, the matrix function

$$\tilde{\psi}(x,t,\lambda) := \psi^T(-x,-t,\lambda)C \tag{3.6}$$

presents another matrix adjoint eigenfunction associated with the same original eigenvalue λ, i.e., $\psi^T(-x,-t,\lambda)C$ solves the adjoint spectral problem (3.5).

Now, by checking the asymptotic properties of adjoint eigenfunctions, the uniqueness of solutions tells us

$$\psi^T(-x,-t,\lambda) = C\psi^{-1}(x,t,\lambda)C^{-1} \tag{3.7}$$

for a matrix eigenfunction ψ that satisfies $\psi \to I_{n+1}$, when x or $t \to \infty$ or $-\infty$. This implies that the spectral problem (3.2) (or the adjoint spectral problem (3.5)) has the involution property (3.7) for its eigenfunctions, under the group reduction in (2.7).

3.2. **Riemann-Hilbert problems.** Let us now formulate a class of associated Riemann-Hilbert problems with the space variable x. The procedure is actually the same as the one for the local case [16], but we present it below for ease of reference.

In order to facilitate the expression below, we assume that

$$\alpha = \alpha_1 - \alpha_2 < 0,\ \beta = \beta_1 - \beta_2 < 0. \tag{3.8}$$

Otherwise, some changes are needed in determining generalized Jost solutions (a kind of combinations of Jost solutions). To establish the scattering problem, we take the two matrix eigenfunctions $\psi^\pm(x,\lambda)$ of (3.2) with the asymptotic conditions

$$\psi^\pm \to I_{n+1},\ \text{when } x \to \pm\infty, \tag{3.9}$$

respectively. Since $(\det\psi)_x = 0$, we see that $\det\psi^\pm = 1$ for all $x \in \mathbb{R}$. Because

$$\phi^\pm = \psi^\pm E,\ E = e^{i\lambda\Lambda x}, \tag{3.10}$$

are both matrix eigenfunctions of the spectral problems (2.2), they must be linearly dependent, and accordingly, we have

$$\psi^- E = \psi^+ E S(\lambda),\ \lambda \in \mathbb{R}, \tag{3.11}$$

where $S(\lambda) = (s_{jl})_{(n+1)\times(n+1)}$ is traditionally called the scattering matrix. Note that $\det S(\lambda) = 1$ since $\det \psi^{\pm} = 1$; and that by (3.7), we have $(\psi^{\pm})^T(-x,-t,\lambda) = C(\psi^{\pm})^{-1}(x,t,\lambda)C^{-1}$, and further

$$S^T(\lambda) = CS(\lambda)C^{-1}. \tag{3.12}$$

It is known that we can turn the x-part of (2.2) into the following Volterra integral equations for $\psi^{\pm}$ [12]:

$$\psi^-(x,\lambda) = I_{n+1} + \int_{-\infty}^{x} \mathrm{e}^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^-(\lambda,y)\mathrm{e}^{i\lambda\Lambda(y-x)}\,dy, \tag{3.13}$$

$$\psi^+(x,\lambda) = I_{n+1} - \int_{x}^{\infty} \mathrm{e}^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^+(\lambda,y)\mathrm{e}^{i\lambda\Lambda(y-x)}\,dy, \tag{3.14}$$

where the asymptotic conditions (3.9) have been applied. Now, the theory of Volterra integral equations shows that the eigenfunctions $\psi^{\pm}$ could exist and allow analytical continuations off the real line $\lambda \in \mathbb{R}$ as long as the integrals on the right-hand sides converge, which can be observed on the basis of (3.1).

First, in order to determine two generalized Jost solutions, denoted by T^+ and T^-, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ (the upper and lower half-planes) and continuous in $\bar{\mathbb{C}}^+$ and $\bar{\mathbb{C}}^-$ (the closed upper and lower half-planes), respectively, we express

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \dots, \psi_{n+1}^{\pm}), \tag{3.15}$$

where $\psi_j^{\pm}$ denotes the jth column of $\phi^{\pm}$ $(1 \le j \le n+1)$. Then, we take the generalized matrix Jost solution T^+ as

$$T^+ = T^+(x,\lambda) = (\psi_1^-, \psi_2^+, \dots, \psi_{n+1}^+) = \psi^- H_1 + \psi^+ H_2, \tag{3.16}$$

which is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \bar{\mathbb{C}}^+$. Here we denote

$$H_1 = \mathrm{diag}(1, \underbrace{0,\dots,0}_{n}),\ H_2 = \mathrm{diag}(0, \underbrace{1,\dots,1}_{n}). \tag{3.17}$$

Second, to determine the other generalized Jost solution T^-, i.e., the analytic counterpart of T^+ in the lower half-plane $\mathbb{C}^-$, we adopt the adjoint matrix spectral problems. Notice that the inverse matrices $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and $\tilde{\psi}^{\pm} = (\psi^{\pm})^{-1}$ solve those two adjoint equations, respectively. Therefore, upon writing $\tilde{\psi}^{\pm}$ as

$$\tilde{\psi}^{\pm} = \begin{bmatrix} \tilde{\psi}^{\pm,1} \\ \tilde{\psi}^{\pm,2} \\ \vdots \\ \tilde{\psi}^{\pm,n+1} \end{bmatrix}, \tag{3.18}$$

where $\tilde{\psi}^{\pm,j}$ denotes the jth row of $\tilde{\psi}^{\pm}$ $(1 \le j \le n+1)$, we can take the generalized Jost solution T^- as the adjoint matrix solution of (3.5), i.e.,

$$T^- = \begin{bmatrix} \tilde{\psi}^{-,1} \\ \tilde{\psi}^{+,2} \\ \vdots \\ \tilde{\psi}^{+,n+1} \end{bmatrix} = H_1\tilde{\psi}^- + H_2\tilde{\psi}^+ = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1}, \tag{3.19}$$

which is analytic for $\lambda \in \mathbb{C}^-$ and continuous for $\lambda \in \bar{\mathbb{C}}^-$.

From $\det\psi^{\pm}=1$, the definitions of $T^{\pm}$, and the scattering relation between ψ^+ and ψ^-, we see that

$$\det T^+(x,\lambda)=s_{11}(\lambda),\ \det T^-(x,\lambda)=\hat{s}_{11}(\lambda), \tag{3.20}$$

where $S^{-1}(\lambda)=(S(\lambda))^{-1}=(\hat{s}_{jl})_{(n+1)\times(n+1)}$, and thus

$$\begin{aligned}\lim_{x\to\infty}T^+(x,\lambda)&=\begin{bmatrix} s_{11}(\lambda) & 0\\ 0 & I_n\end{bmatrix},\ \lambda\in\bar{\mathbb{C}}^+;\\ \lim_{x\to\infty}T^-(x,\lambda)&=\begin{bmatrix} \hat{s}_{11}(\lambda) & 0\\ 0 & I_n\end{bmatrix},\ \lambda\in\bar{\mathbb{C}}^-.\end{aligned} \tag{3.21}$$

Now we can define the following two unimodular generalized Jost solutions:

$$\begin{aligned}G^+(x,\lambda)&=T^+(x,\lambda)\begin{bmatrix} s_{11}^{-1}(\lambda) & 0\\ 0 & I_n\end{bmatrix},\ \lambda\in\bar{\mathbb{C}}^+;\\ (G^-)^{-1}(x,\lambda)&=\begin{bmatrix} \hat{s}_{11}^{-1}(\lambda) & 0\\ 0 & I_n\end{bmatrix}T^-(x,\lambda),\ \lambda\in\bar{\mathbb{C}}^-.\end{aligned} \tag{3.22}$$

The required class of matrix Riemann-Hilbert problems on the real line for the nonlocal reverse-spacetime NLS equations (2.11) reads

$$G^+(x,\lambda)=G^-(x,\lambda)G_0(x,\lambda),\ \lambda\in\mathbb{R}, \tag{3.23}$$

where by (3.11), the jump matrix G_0 is

$$\begin{aligned}G_0(x,\lambda)=E&\begin{bmatrix} \hat{s}_{11}^{-1}(\lambda) & 0\\ 0 & I_n\end{bmatrix}\tilde{S}(\lambda)\begin{bmatrix} s_{11}^{-1}(\lambda) & 0\\ 0 & I_n\end{bmatrix}E^{-1},\\ \text{with } &\tilde{S}(\lambda)=(H_1+H_2S(\lambda))(H_1+S^{-1}(\lambda)H_2).\end{aligned} \tag{3.24}$$

In this jump matrix G_0, the matrix $\tilde{S}(\lambda)=(\tilde{s}_{jl})_{(n+1)\times(n+1)}$ can be worked out as follows:

$$\begin{aligned}&\tilde{s}_{1,j+1}=\hat{s}_{1,j+1},\ \tilde{s}_{j+1,1}=s_{j+1,1},\ 1\le j\le n;\\ &\tilde{s}_{jj}=1,\ 1\le j\le n+1;\tilde{s}_{jl}=0,\ \text{otherwise.}\end{aligned} \tag{3.25}$$

Also, under (3.1), the Volterra integral equations (3.13) and (3.14) generate the canonical normalization conditions:

$$G^{\pm}(x,\lambda)\to I_{n+1},\ \text{when } \lambda\in\bar{\mathbb{C}}^{\pm}\to\infty, \tag{3.26}$$

for the associated Riemann-Hilbert problems, and the nonlocal reduction (2.7) leads to the involution property:

$$(G^+)^T(-x,-t,\lambda)=CG^-(x,t,\lambda)C^{-1}. \tag{3.27}$$

It is worth noting that the jump matrix G_0 carries all basic scattering data from the scattering matrix $S(\lambda)$. Solutions to the above Riemann-Hilbert problems depend on poles of $s_{11}(\lambda)$ and $\hat{s}_{11}(\lambda)$, and kernels of T^+ and T^- at those zeros [12], and can be computed, particularly in the reflectionless case, i.e., $\tilde{S}=I_{n+1}$, to obtain soliton solutions [23].

3.3. **Evolution of the scattering data.** To complete the direct scattering transforms, we compute the derivative of the equation (3.11) with the time variable t, and use the temporal spectral problem (3.3) that $\psi^{\pm}$ satisfy. It then follows that the scattering matrix S needs to satisfy an evolution matrix equation:

$$S_t = i\lambda^2[\Omega, S]. \tag{3.28}$$

This exactly tells us that the time-dependent scattering coefficients obey the following evolution laws:

(3.29)
$$\begin{cases} s_{12} = s_{12}(0,\lambda)\mathrm{e}^{i\beta\lambda^2 t},\ s_{13} = s_{13}(0,\lambda)\mathrm{e}^{i\beta\lambda^2 t},\ \ldots,\ s_{1,n+1} = s_{1,n+1}(0,\lambda)\mathrm{e}^{i\beta\lambda^2 t}, \\ s_{21} = s_{21}(0,\lambda)\mathrm{e}^{-i\beta\lambda^2 t},\ s_{31} = s_{31}(0,\lambda)\mathrm{e}^{-i\beta\lambda^2 t},\ \ldots,\ s_{n+1,1} = s_{n+1,1}(0,\lambda)\mathrm{e}^{-i\beta\lambda^2 t}, \end{cases}$$

and that all other scattering coefficients are independent of the time variable t.

3.4. **Transforming the Riemann-Hilbert problems.** To determine the generalized matrix Jost solutions, we change the Riemann-Hilbert problems in (3.23) as follows:

$$\begin{cases} G^+ - G^- = G^- v,\ v = G_0 - I_{n+1},\ \text{on } \mathbb{R}, \\ G^{\pm} \to I_{n+1} \text{ as } \lambda \in \bar{\mathbb{C}}^{\pm} \to \infty. \end{cases} \tag{3.30}$$

Let $G(\lambda) = G^{\pm}(\lambda)$ if $\lambda \in \mathbb{C}^{\pm}$, respectively. Assume that G has R simple poles $\{\mu_j\}_{j=1}^{R}$, where R is an arbitrarily given natural number, and those poles are off the real line $\mathbb{R}$ to avoid spectral singularities. Introduce

$$\tilde{G}^{\pm}(\lambda) = G^{\pm}(\lambda) - \sum_{j=1}^{R} \frac{G_j}{\lambda - \mu_j},\ \lambda \in \bar{\mathbb{C}}^{\pm};\ \tilde{G}(\lambda) = \tilde{G}^{\pm}(\lambda),\ \lambda \in \mathbb{C}^{\pm}; \tag{3.31}$$

where G_j is the residue of $G(\lambda)$ at $\lambda = \mu_j$, i.e.,

$$G_j = \mathrm{res}(G(\lambda), \mu_j) = \lim_{\lambda \to \mu_j} (\lambda - \mu_j)G(\lambda).$$

Then, we have

$$\begin{cases} \tilde{G}^+ - \tilde{G}^- = G^+ - G^- = G^- v,\ \text{on } \mathbb{R}, \\ \tilde{G}^{\pm} \to I_{n+1} \text{ as } \lambda \in \bar{\mathbb{C}}^{\pm} \to \infty. \end{cases} \tag{3.32}$$

Further, by the Sokhotski-Plemelj formula [24], we obtain the solutions

$$\tilde{G}(\lambda) = I_{n+1} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{G^- v(\xi)}{\xi - \lambda}\, d\xi,\ \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.33}$$

Now, taking the limit as $\lambda \to \mu_l$ generates

$$\text{lhs} = \lim_{\lambda \to \mu_l} \tilde{G} = F_l - \sum_{j \neq l}^{R} \frac{G_j}{\mu_l - \mu_j},\ \text{rhs} = I_{n+1} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \mu_l}\, d\xi,$$

where $F_l = \lim_{\lambda \to \mu_l}[(\lambda - \mu_l)G(\lambda) - G_l]/(\lambda - \mu_l)$, and consequently, we obtain the transformed systems:

$$I_{n+1} - F_l + \sum_{j \neq l}^{R} \frac{G_j}{\mu_l - \mu_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \lambda_l}\, d\xi = 0,\ 1 \le l \le R. \tag{3.34}$$

These systems, like the Gelfand-Levitan-Marchenko equation, are used to determine solutions to the associated Riemann-Hilbert problems, and thus, the generalized matrix Jost solutions. The existence and solvability problem of the systems has yet to be investigated. In the case of soliton solutions, we will present an explicit formulation of solutions to the corresponding Riemann-Hilbert problems later.

3.5. **Recovery of the potential.** To recover the potential matrix P from the generalized matrix Jost solutions, we make an asymptotic expansion for the generalized Jost solution G^+:

$$G^+(x,t,\lambda) = I_{n+1} + \frac{1}{\lambda} G_1^+(x,t) + \mathrm{O}(\frac{1}{\lambda^2}),\ \lambda \to \infty. \tag{3.35}$$

Then plugging the asymptotic expansion into the matrix spectral problem (3.2), all $\mathrm{O}(1)$ terms engender

$$P = \lim_{\lambda\to\infty} \lambda [G^+(\lambda), \Lambda] = -[\Lambda, G_1^+]. \tag{3.36}$$

To get the potential for the nonlocal equations, one needs to check an involution property for G_1^+:

$$(G_1^+)^T(-x,-t) = -C G_1^+(x,t) C^{-1}, \tag{3.37}$$

which is a consequence of the involution property for G in (3.27). Then if so, we obtain the solutions to the nonlocal reverse-spacetime NLS equations (2.11):

$$p_j = -\alpha (G_1^+)_{1,j+1},\ 1 \le j \le n, \tag{3.38}$$

where $G_1^+ = ((G_1^+)_{jl})_{(n+1)\times(n+1)}$.

This finishes the inverse scattering procedure from the scattering matrix $S(\lambda)$, through the jump matrix $G_0(\lambda)$ and the solution $\{G^+(\lambda), G^-(\lambda)\}$ to the associated Riemann-Hilbert problems, to the potential matrix P. The resulting potential matrix P defines exact solutions to the nonlocal reverse-spacetime NLS equations (2.11).

4. Soliton solutions

In this section, we construct N-soliton solutions to the nonlocal reverse-spacetime integrable NLS equations (2.11). We are not going to solve the transformed systems, but directly apply the Riemann-Hilbert technique.

Let N be another arbitrary natural number. Assume that s_{11} has N geometrically simple zeros $\{\lambda_k \in \mathbb{C},\ 1 \le k \le N\}$, and $\hat{s}_{11}$ has other N geometrically simple zeros $\{\hat{\lambda}_k \in \mathbb{C},\ 1 \le k \le N\}$. Upon recalling (3.20), we see that each of $\ker T^+(\lambda_k)$, $1 \le k \le N$, contains only a single basis column vector, denoted by v_k, $1 \le k \le N$; and each of $\ker T^-(\hat{\lambda}_k)$, $1 \le k \le N$, contains only a single basis row vector, denoted by $\hat{v}_k$, $1 \le k \le N$. This way, we have

$$T^+(\lambda_k) v_k = 0,\ \hat{v}_k T^-(\hat{\lambda}_k) = 0,\ 1 \le k \le N, \tag{4.1}$$

from which we can determine those kernel vectors by using the associated spectral problems that T^+ and T^- satisfy [16].

Soliton solutions are generated from the Riemann-Hilbert problems in (3.23) with the identity jump matrix $G_0 = I_{n+1}$ (equivalently, $\tilde{S} = I_{n+1}$), which can be achieved under the zero reflection coefficient conditions $s_{i1} = \hat{s}_{1i} = 0,\ 2 \le i \le n+1$, in the scattering problem. Such Riemann-Hilbert problems with the canonical normalization condition and the zero structures given in (4.1) can be solved explicitly (see, e.g., [12, 23, 25]).

However, in the case of nonlocal integrable equations, we often do not have the property

$$\{\lambda_k | 1 \le k \le N\} \cap \{\hat{\lambda}_k | 1 \le k \le N\} = \emptyset, \tag{4.2}$$

and so we need to generalize the solution formulation in the literature. The following theorem offers us a way to determine solutions to this kind of special Riemann-Hilbert problems.

Theorem 4.1 (*Formulation of solutions*)**.** *Let λ_k and $\hat{\lambda}_k$, $1 \le k \le N$, be two sets of N complex numbers, and let v_k and $\hat{v}_k$, $1 \le k \le N$, be $(n+1)$-dimensional column and row vectors, respectively. Suppose that G^+ and G^- are defined by*

$$G^+(\lambda) = I_{n+1} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l}, \ (G^-)^{-1}(\lambda) = I_{n+1} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}, \tag{4.3}$$

where $M = (m_{kl})_{N\times N}$ is a square matrix whose entries are determined by

$$m_{kl} = \begin{cases} \frac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, & \text{if } \lambda_l \ne \hat{\lambda}_k, \\ 0, & \text{if } \lambda_l = \hat{\lambda}_k, \end{cases} \quad 1 \le k, l \le N. \tag{4.4}$$

Then, (a) *the following properties hold:*

$$\Big(\prod_{l=1}^{N} (\lambda - \hat{\lambda}_l) G^+\Big)(\lambda_k) v_k = 0, \ \hat{v}_k \Big(\prod_{l=1}^{N} (\lambda - \lambda_l)(G^-)^{-1}\Big)(\hat{\lambda}_k) = 0, \ 1 \le k \le N; \tag{4.5}$$

(b) *G^+ and G^- satisfy*

$$(G^-(\lambda))^{-1} G^+(\lambda) = I_{n+1}, \tag{4.6}$$

if we require an orthogonal condition

$$\hat{v}_k v_l = 0, \ \text{when } \lambda_l = \hat{\lambda}_k, \ 1 \le k, l \le N. \tag{4.7}$$

Proof. The characteristic properties in (4.5) follow directly from the definitions of $G^\pm$ in (4.3) and M in (4.4). To prove (4.6), let us rewrite

$$G^+ = I_{n+1} - v M^{-1} \hat{R} \hat{v}, \ (G^-)^{-1} = I_{n+1} + v R M^{-1} \hat{v},$$

where

$$\begin{cases} v = (v_1, \dots, v_N), \ R = \mathrm{diag}((\lambda - \lambda_1)^{-1}, \dots, (\lambda - \lambda_N)^{-1}), \\ \hat{v} = (\hat{v}^T, \dots, \hat{v}_N^T)^T, \ \hat{R} = \mathrm{diag}((\lambda - \hat{\lambda}_1)^{-1}, \dots, (\lambda - \hat{\lambda}_N)^{-1}). \end{cases}$$

Then, taking (4.7) into consideration, we can check the (k,l)-th entries as follows:

$$(\hat{R}\hat{v}vR)_{kl} = (MR - \hat{R}M)_{kl} = 0, \ \text{when } \lambda_l = \hat{\lambda}_k;$$

$$(\hat{R}\hat{v}vR)_{kl} = (MR - \hat{R}M)_{kl} = \frac{\hat{v}_k v_l}{(\lambda - \hat{\lambda}_k)(\lambda - \hat{\lambda}_l)}, \ \text{otherwise};$$

and thus, we see that $\hat{R}\hat{v}vR = MR - \hat{R}M$. Further, by using this equality, we can easily verify (4.6). The proof is finished. □

If the condition (4.2) holds, then the above result reduces to the one presented in [12,23,25]. If $\lambda_k \in \mathbb{C}^+$ and $\hat{\lambda}_k \in \mathbb{C}^-$, $1 \le k \le N$, then G^+ and G^- are meromorphic in the upper and lower half-planes, respectively, and thus, they provide solutions to the Riemann-Hilbert problems with the identity jump matrix on the real line.

To present soliton solutions to the nonlocal reverse-spacetime NLS equations (2.11), we need to formulate two generalized Jost solutions. Motivated by the previous property of eigenfunctions, we take N zeros of $s_{11}(=\det T^+(\lambda))$: $\lambda_k \in \mathbb{C},\ 1 \le k \le N$, and N zeros of $\hat{s}_{11}(=\det T^-(\lambda))$: $\hat{\lambda}_k \in \mathbb{C},\ 1 \le k \le N$; and assume [16] that $\ker T^+(\lambda_k)$ and $\ker T^-(\lambda_k)$, $1 \le k \le N$, are spanned by

$$v_k(x,t) = v_k(x,t,\lambda_k) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^2 \Omega t} w_k,\ 1 \le k \le N, \tag{4.8}$$

and

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\lambda}_k) = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^2 \Omega t} C,\ 1 \le k \le N, \tag{4.9}$$

respectively. Here the constant column vectors w_k's and row vectors $\hat{w}_k$'s are arbitrary but need to satisfy

$$\hat{w}_k C w_l = 0,\ \text{when } \lambda_l = \hat{\lambda}_k,\ 1 \le k,l \le N, \tag{4.10}$$

which is a consequence of the orthogonal condition (4.7).

Finally, recalling (3.22), if the solutions to the specific Riemann-Hilbert problems, determined by (4.3) and (4.4), satisfy the involution property (3.27), then all matrices, G_1^+, with many free parameters satisfy the corresponding involution property (3.37). Therefore, by (3.38), we obtain N-soliton solutions to the nonlocal reverse-spacetime NLS equations (2.11), which we summarize as follows.

Theorem 4.2 (*N-soliton solutions*). *Let $\lambda_k,\ 1 \le k \le N$, and $\hat{\lambda}_k,\ 1 \le k \le N$, be two sets of arbitrary N complex numbers, and let $w_k, 1 \le k \le N$, and $\hat{w}_k,\ 1 \le k \le N$, be two sets of arbitrary constant $(n+1)$-dimensional complex column and row vectors, respectively. Assume that the w_k's and $\hat{w}_k$'s satisfy the conditional pairwise orthogonal condition* (4.10)*, and that the solutions to the Riemann-Hilbert problems with the identity jump matrix on the real line, determined by* (4.3) *and* (4.4)*, satisfy the involution property* (3.27)*. Then we have the N-soliton solutions to the nonlocal reverse-spacetime integrable NLS equations* (2.11)*:*

$$p_j = \alpha \sum_{k,l=1}^{N} v_{k,1} (M^{-1})_{kl} \hat{v}_{l,j+1},\ 1 \le j \le n, \tag{4.11}$$

where M is defined by (4.4)*, and $v_k = (v_{k,1}, v_{k,2}, \dots, v_{k,n+1})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \dots, \hat{v}_{k,n+1})$ are determined by* (4.8) *and* (4.9)*, respectively.*

When $n = N = 1$, we denote $w_1 = (w_{1,1}, w_{1,2})^T$ and $\hat{w}_1 = (\hat{w}_{1,1}, \hat{w}_{1,2})$. A direct computation leads to the following one-soliton solution to the nonlocal reverse-spacetime integrable NLS equation (2.12):

$$p = -\frac{(\lambda_1 - \hat{\lambda}_1) w_{1,1} \hat{w}_{1,2}\, \mathrm{e}^{-i(\hat{\lambda}_1 x + \hat{\lambda}_1^2 t)}}{\sigma w_{1,1} \hat{w}_{1,1} + w_{1,2} \hat{w}_{1,2}\, \mathrm{e}^{i(\lambda_1 - \hat{\lambda}_1) x + i(\lambda_1^2 - \hat{\lambda}_1^2) t}}, \tag{4.12}$$

where λ_1 and $\hat{\lambda}_1$ are arbitrary, and $w_{1,1}^2 + \sigma w_{1,2}^2 = \hat{w}_{1,1}^2 + \sigma \hat{w}_{1,2}^2 = 0$, which comes from the involution condition.

5. Concluding remarks

Nonlocal reverse-spacetime nonlinear Schrödinger (NLS) equations were presented and their inverse scattering transforms were analyzed. A class of associated Riemann-Hilbert problems is the basis for our discussion. The Sokhotski-Plemelj formula was used to determine solutions to the associated Riemann-Hilbert problems, and soliton solutions to the nonlocal reverse-spacetime integrable NLS equations were constructed from the Riemann-Hilbert problems with the identity jump matrix or equivalently the reflectionless inverse scattering transforms.

It is worth pointing out that it would be interesting to determine any certain kind of connections among different solution methods, including the Riemann-Hilbert technique [12, 23, 25], the Hirota direct method [26], the Wronskian technique [27, 28], and the Darboux transformation [29, 30]. Moreover, various studies have exhibited great richness of other kinds of solutions in nonlinear dispersive waves recently, including lump solutions [31–33], Rossby wave solutions [34], solitonless solutions [35, 36], and algebro-geometric solutions [37, 38]. It would also be very interesting to understand how to construct those exact solutions through the Riemann-Hilbert perspective or a larger basic perspective.

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