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N-soliton solutions and the Hirota conditions in (2+1)-dimensions

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Abstract
We compute *N*-soliton solutions and analyze the Hirota *N*-soliton conditions, in (2+1)-dimensions, based on the Hirota bilinear formulation. An algorithm to check the Hirota conditions is proposed by comparing degrees of the polynomials generated from the Hirota function in *N* wave vectors. A weight number is introduced while transforming the Hirota function to achieve homogeneity of the resulting polynomial. Applications to three integrable equations: the (2+1)-dimensional KdV equation, the Kadomtsev–Petviashvili equation, the (2+1)-dimensional Hirota–Satsuma–Ito equation, are made, thereby providing proofs of the existence of *N*-soliton solutions in the three model equations.

Keywords *N*-Soliton solution · Hirota *N*-soliton condition · (2+1)-Dimensional integrable equations

Mathematics Subject Classification 35Q51 · 35Q53 · Secondary · 37K40

1 Introduction

It is known that *N*-soliton solutions are a kind of exact solutions to weakly nonlinear dispersive wave equations (Ablowitz and Segur 1981; Novikov et al. 1984). Particularly, solitons superimposed in fibers can be applied to optical communications (Hasegawa 1989 and 1990). Many interesting solutions in mathematical physics, including breather, peakon, complexiton, lump and rogue wave solutions, are special reductions of *N*-soliton solutions in different situations. The Hirota bilinear method is a standard and powerful technique to present *N*-soliton solutions (Hirota 2004). The innovative concept of bilinear derivatives is a generalized idea to deal with nonlinear differential equations, and through bilinear forms, it becomes much more natural to generate *N*-soliton solutions.

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Hirota bilinear derivatives are defined by Hirota (2004):

$$D_x^m f \cdot g = \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} (\partial_x^i f)(\partial_x^{m-i} g), \quad m \geq 1, \tag{1.1}$$

and more generally, bilinear partial derivatives with multiple variables are similarly defined:

$$(D_x^m D_t^n f \cdot g)(x,t) = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x',t')|_{x'=x,t'=t}, \quad m,n \geq 1. \tag{1.2}$$

When $f = g$, we obtain Hirota bilinear expressions:

$$D_x^{2m-1} f \cdot f = 0,\ D_x^{2m} f \cdot f = \sum_{i=1}^{2m} (-1)^{2m-i} \binom{2m}{i} (\partial_x^i f)(\partial_x^{2m-i} f), \quad m \geq 1, \tag{1.3}$$

and similarly, bilinear partial derivative expressions:

$$D_x^m D_t^n f \cdot f = \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} (\partial_x^i \partial_t^j f)(\partial_x^{m-i} \partial_t^{n-j} f), \quad m,n \geq 1. \tag{1.4}$$

By means of Hirota bilinear expressions, we can define Hirota bilinear equations. Take an even polynomial $P(x_1, x_2, \cdots, x_M)$ in M variables, and assume that no constant term exists, i.e., $P(\mathbf{0}) = P(0,0,\ldots,0) = 0$. The associated Hirota bilinear equation is defined by

$$P(D_{x_1}, D_{x_2}, \ldots, D_{x_M}) f \cdot f = 0, \tag{1.5}$$

all terms of which are Hirota bilinear expressions.

For example, the bilinear Kadomtsev–Petviashvili equation is

$$B(f) := (D_x^4 + D_x D_t + D_y^2) f \cdot f = 2(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2 + f_{xt} f - f_x f_t + f_{yy} f - f_y^2) = 0,$$

which is transformed into the standard Kadomtsev–Petviashvili equation

$$N(u) := (u_t + 6 u u_x + u_{xxx})_x + u_{yy} = 0,$$

under the logarithmic derivative transformation $u = 2(\ln f)_{xx}$. The link is $N(u) = (B(f)/f^2)_{xx}$.

We would like to discuss N-soliton solutions and derive the corresponding Hirota conditions in (2+1)-dimensions. An algorithm will be proposed for verifying the Hirota N-soliton conditions by comparing degrees of the polynomials generated from the Hirota function in N wave vectors. Applications will be made for the three integrable equations in (2+1)-dimensions: the (2+1)-dimensional KdV equation, the Kadomtsev–Petviashvili equation and the (2+1)-dimensional Hirota–Satsuma–Ito equation, thereby providing proofs of the existence of N-soliton solitons to those three (2+1)-dimensional equations.

2 Bilinear formulation of *N*-soliton solutions

2.1 N-soliton solutions

Let us state N wave vectors as follows:

$$\mathbf{k}_i = (k_{1,i}, k_{2,i}, \dots, k_{M,i}), \quad 1 \le i \le N, \tag{2.1}$$

where $k_{1,i}, k_{2,i}, \dots, k_{M,i}$, $1 \le i \le N$, are constants. An N-soliton solution to a Hirota bilinear equation (1.5) is given by Hirota (1971):

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} a_{ij} \mu_i \mu_j \right), \tag{2.2}$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_N)$, $\mu = 0, 1$ means that each μ_i takes 0 or 1, and

$$\eta_i = k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{M,i}x_M + \eta_{i,0}, \quad 1 \le i \le N, \tag{2.3}$$

$$e^{a_{ij}} = A_{ij} := -\frac{P(\mathbf{k}_i - \mathbf{k}_j)}{P(\mathbf{k}_i + \mathbf{k}_j)}, \quad 1 \le i < j \le N, \tag{2.4}$$

$\eta_{i,0}$'s being arbitrary phase shifts.

2.2 Hirota N-soliton condition

Let us define

$$H(\mathbf{k}_{i_1}, \dots, \mathbf{k}_{i_n}) = \sum_{\sigma=\pm 1} P\left(\sum_{r=1}^{n} \sigma_r \mathbf{k}_{i_r} \right) \prod_{1 \le r < s \le n} P(\sigma_r \mathbf{k}_{i_r} - \sigma_s \mathbf{k}_{i_s}) \sigma_r \sigma_s, \quad 1 \le n \le N, \tag{2.5}$$

where $1 \le i_1 < \cdots < i_n \le N$, $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, and $\sigma = \pm 1$ means that each σ_r takes 1 or -1. We call these functions the Hirota functions.

Using the basic properties

$$P(D_{x_1}, \cdots, D_{x_M}) e^{\eta_i} \cdot e^{\eta_j} = P(\mathbf{k}_i - \mathbf{k}_j) e^{\eta_i + \eta_j}, \tag{2.6}$$

and

$$P(D_{x_1}, \cdots, D_{x_M}) e^{\eta_n} f \cdot e^{\eta_n} g = e^{2\eta_n} P(D_{x_1}, \cdots, D_{x_M}) f \cdot g, \tag{2.7}$$

where η_i, η_j and η_n are arbitrary linear functions, we can work out the following expression.

Theorem 2.1 *Let f be defined by* (2.2), *and* $\hat{\xi}$ *mean that no* ξ *is involved. Then we have*

$$
\begin{aligned}
&P(D_{x_1},\cdots,D_{x_M})f\cdot f\\
&=(-1)^{\frac{1}{2}N(N-1)}\frac{H(\mathbf{k}_1,\mathbf{k}_2,\cdots,\mathbf{k}_N)}{\prod_{1\le i<j\le N}P(\mathbf{k}_i+\mathbf{k}_j)}\mathrm{e}^{\eta_1+\eta_2+\cdots+\eta_N}\\
&\quad+\sum_{n=1}^{N-1}(-1)^{\frac{1}{2}(N-n)(N-n-1)}\sum_{1\le i_1<\cdots<i_n\le N}\frac{H(\mathbf{k}_1,\ldots,\hat{\mathbf{k}}_{i_1},\ldots,\hat{\mathbf{k}}_{i_n},\ldots,\mathbf{k}_N)}{\prod_{\substack{1\le i<j\le N\\ i,j\notin\{i_1,\ldots,i_n\}}}P(\mathbf{k}_i+\mathbf{k}_j)}\mathrm{e}^{\eta_1+\cdots+\hat{\eta}_{i_1}+\cdots+\hat{\eta}_{i_n}+\cdots+\eta_N}\\
&\quad+\sum_{n=1}^{N-1}\sum_{1\le i_1<\cdots<i_n\le N}\mathrm{e}^{2(\eta_{i_1}+\cdots+\eta_{i_n}+\sum_{1\le r<s\le n}a_{i_ri_s})}P(D_{x_1},\cdots,D_{x_M})\tilde{f}_{i_1\cdots i_n}\cdot\tilde{f}_{i_1\cdots i_n}
\end{aligned}
\tag{2.8}
$$

with

$$
\tilde{f}_{i_1\cdots i_n}=\sum_{\tilde{\mu}_{i_1\cdots i_n}=0,1}\exp\left(\sum_{\substack{1\le i\le N\\ i\notin\{i_1,\ldots,i_n\}}}\mu_i\tilde{\eta}_i+\sum_{\substack{1\le i<j\le N\\ i,j\notin\{i_1,\ldots,i_n\}}}a_{ij}\mu_i\mu_j\right),\quad \tilde{\eta}_i=\eta_i+\sum_{r=1}^{n}a_{ii_r},
$$

where $\tilde{\mu}_{i_1\cdots i_n}=(\mu_1,\ldots,\hat{\mu}_{i_1}\cdots,\hat{\mu}_{i_n},\ldots,\mu_N)$ *and* $\tilde{\mu}_{i_1\cdots i_n}=0,1$ *means that each* μ_i *in* $\tilde{\mu}_{i_1\cdots i_n}$ *takes* 0 *or* 1.

Based on this theorem, we see that a Hirota bilinear equation (1.5) has an N-soliton solution (2.2) if and only if the condition

$$
H(\mathbf{k}_{i_1},\ldots,\mathbf{k}_{i_n})=0,\quad 1\le i_1<\cdots<i_n\le N,\quad 1\le n\le N,
\tag{2.9}
$$

is satisfied. This is called the Hirota condition for an N-soliton solution, or simply, the N-soliton condition (see Hirota 1980, p. 165). The case of (2.9) with $n=1$ gives the dispersion relations

$$
P(\mathbf{k}_i)=P(k_{1,i},k_{2,i},\ldots,k_{M,i})=0,\quad 1\le i\le N,
\tag{2.10}
$$

due to the even property of P. There are very few studies on this Hirota N-soliton condition (see, e.g., Sawada and Kotera 1974; Newell and Zeng 1986), due to its complexity (Hirota 1980).

Examples The one-soliton condition is just the dispersion relation: $P(\mathbf{k}_1)=0$, which means that $f=1+\mathrm{e}^{\eta_1}$ is a solution. Besides the dispersion relations, the two-soliton condition requires

$$
2(P(\mathbf{k}_1+\mathbf{k}_2)P(\mathbf{k}_1-\mathbf{k}_2)-P(\mathbf{k}_1-\mathbf{k}_2)P(\mathbf{k}_1+\mathbf{k}_2))=0,
\tag{2.11}
$$

which is an identity. Therefore, there always exists a two-soliton solution:

$$
f=1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+A_{12}\mathrm{e}^{\eta_1+\eta_2},
\tag{2.12}
$$

to a Hirota bilinear equation. Taking $N=3$, we see that the three-soliton condition (Hietarinta 1987, 1997) requires

$$\sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} P(\sigma_1\mathbf{k}_1+\sigma_2\mathbf{k}_2+\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_2\mathbf{k}_2) \times P(\sigma_2\mathbf{k}_2-\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_3\mathbf{k}_3)=0,$$

in addition to the dispersion relations. This is equivalent to

$$\sum_{(\sigma_1,\sigma_2,\sigma_3)\in S} P(\sigma_1\mathbf{k}_1+\sigma_2\mathbf{k}_2+\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_2\mathbf{k}_2) \times P(\sigma_2\mathbf{k}_2-\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_3\mathbf{k}_3)=0, \tag{2.13}$$

where $S=\{(1,1,1),(1,1,-1),(1,-1,1),(-1,1,1)\}$. The three-soliton solution is given by

$$f=1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+\mathrm{e}^{\eta_3}+A_{12}\mathrm{e}^{\eta_1+\eta_2}+A_{13}\mathrm{e}^{\eta_1+\eta_3} +A_{23}\mathrm{e}^{\eta_2+\eta_3}+A_{123}\mathrm{e}^{\eta_1+\eta_2+\eta_3},\ A_{123}=A_{12}A_{13}A_{23}. \tag{2.14}$$

It is widely believed that the three-soliton condition implies the N-soliton condition, without proof of its accuracy.

2.3 Resonant solitons

If we require a sufficient Hirota N-soliton condition (see Ma and Fan 2011, p 951):

$$P(\mathbf{k}_i-\mathbf{k}_j)=0,\ 1\le i<j\le N, \tag{2.15}$$

we obtain a resonant N-soliton solution:

$$f=1+c_1\mathrm{e}^{\eta_1}+c_2\mathrm{e}^{\eta_2}+\cdots+c_N\mathrm{e}^{\eta_N}, \tag{2.16}$$

where c_i's are arbitrary constants (see also Inc et al. 2019, pS2028 and Hosseini et al. 2020, p3 for examples). All wave vectors $\mathbf{k}_i$'s associated with resonant solutions form a vector space in $\mathbb{R}^M$ (Ma et al. 2012, p7178).

2.4 Properties of the Hirota functions

In order to factor out as more common factors from the Hirota function $H(\mathbf{k}_1,\cdots,\mathbf{k}_N)$ as possible, we will use the following result, which is an automatic consequence of the definition of the Hirota functions.

Theorem 2.2 *The Hirota functions defined by* (2.5) *are symmetric and even functions in the involved wave vectors.*

Taking $\mathbf{k}_2=\pm\mathbf{k}_1$, we have

$$P(\sigma_i\mathbf{k}_i-\mathbf{k}_2)P(\sigma_i\mathbf{k}_i\pm\mathbf{k}_1)=P(\mathbf{k}_i-\mathbf{k}_1)P(\mathbf{k}_i+\mathbf{k}_1) \tag{2.17}$$

in any case of $\sigma_i=\pm 1$, due to the even property of the polynomial P. Using this property, we can derive the following consequence.

Theorem 2.3 *If* $\mathbf{k}_2 = \pm\mathbf{k}_1$, *then we have*

$$H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 2H(\mathbf{k}_3, \cdots, \mathbf{k}_N)P(2\mathbf{k}_1)\prod_{i=3}^{N} P(\mathbf{k}_i - \mathbf{k}_1)P(\mathbf{k}_i + \mathbf{k}_1). \tag{2.18}$$

This theorem will be used to factor out common factors from the Hirota function $H(\mathbf{k}_1, \ldots, \mathbf{k}_N)$, while proving the Hirota N-soliton condition.

3 Applications in (2+1)-dimensions

3.1 A general algorithm

In the (2+1)-dimensional case, we can state the N wave vectors as

$$\mathbf{k}_i = (k_i, l_i, -\omega_i),\ 1 \le i \le N,$$

and we assume that the dispersion relations (2.10) determine all frequencies $\omega_i = \omega(k_i, l_i),\ 1 \le i \le N$. Therefore $P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j)$ are functions of k_i, l_i and k_j, l_j only.

On one hand, we assume under the substitution

$$l_i = l_i k_i^w,\ 1 \le i \le N, \tag{3.1}$$

for some integer weight w that $P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j)$ and $P(\sigma_1\mathbf{k}_1 + \cdots + \sigma_N\mathbf{k}_N)$ can be simplified into rational functions as follows:

$$P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j) = \frac{\sigma_i\sigma_j k_i k_j Q_1(k_i, l_i, k_j, l_j, \sigma_i, \sigma_j)}{Q_2(k_i, l_i, k_j, l_j)}, \tag{3.2}$$

where Q_1 and Q_2 are two polynomial functions, and

$$P(\sigma_1\mathbf{k}_1 + \cdots + \sigma_N\mathbf{k}_N) = \frac{Q_3(k_1, l_1, \ldots, k_N, l_N, \sigma_1, \cdots, \sigma_N)}{Q_4(k_1, l_1, \cdots, k_N, l_N)}, \tag{3.3}$$

where Q_3 and Q_4 are also two polynomial functions.

On the other hand, Theorem 2.3 tells that under the induction assumption, the Hirota function $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ will be zero, if two of the wave vectors $\mathbf{k}_i = \mathbf{k}_j,\ 1 \le i < j \le N$, are equal. Based on the symmetric property in Theorem 2.2, we know that under the substitution (3.1), $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ is still even with respect to $k_i, l_i\ 1 \le i \le N$, when w is even, but $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ is even only with respect to $k_i,\ 1 \le i \le N$, when w is odd. However, in both cases, we can have the simplified form

$$H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = (k_i^2 - k_j^2)^2 g_{ij} + (l_i - l_j)^2 h_{ij}, \quad \text{for } 1 \le i < j \le N, \tag{3.4}$$

where g_{ij} and h_{ij} are rational functions of $k_n, l_n,\ 1 \le n \le N$.

It then follows that the Hirota function $H(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ can be written as

$$H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = \frac{\prod_{1\le i<j\le N} k_i^2 k_j^2 [\prod_{1\le i<j\le N}(k_i^2 - k_j^2)^2 g + \prod_{1\le i<j\le N}(l_i - l_j)^2 h]}{Q_4(k_1, l_1, \ldots, k_N, l_N)\prod_{1\le i<j\le N} Q_2(k_i, l_i, k_j, l_j)} \tag{3.5}$$

under the substitution (3.1), where g and h are homogeneous polynomials of k_n, l_n, $1 \le n \le N$, and g can be nonzero and so we take a nonzero polynomial g below when $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) \neq 0$.

Now, if $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) \neq 0$, we see that the degree of the polynomial

$$\prod_{1\le i<j\le N} k_i^2 k_j^2 [\prod_{1\le i<j\le N} (k_i^2 - k_j^2)^2 g + \prod_{1\le i<j\le N} (l_i - l_j)^2 h]$$

is at least $2N(N-1) + 2N(N-1) = 4N(N-1)$, and so from (3.5), the degree of the polynomial

$$\begin{aligned}\tilde{H}(\mathbf{k}_1, \cdots, \mathbf{k}_N) &:= H(\mathbf{k}_1, \cdots, \mathbf{k}_N) Q_4(k_1, l_1, \ldots, k_N, l_N) \prod_{1\le i<j\le N} Q_2(k_i, l_i, k_j, l_j) \\ &= \sum_{\sigma=\pm 1} Q_3(k_1, l_1, \ldots, k_N, l_N, \sigma_1, \cdots, \sigma_N) \prod_{1\le i<j\le N} \sigma_i \sigma_j k_i k_j Q_1(k_i, l_i, k_j, l_j, \sigma_i, \sigma_j)\end{aligned}$$

should not be less than $4N(N-1)$. Otherwise, this implies that $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 0$, which is what we need to prove in the following three cases: $w = 0, 1, 2$. Therefore, the final task is to compute Q_1 and Q_3, and to check if the degree of the above polynomial $\tilde{H}(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ is less than $4N(N-1)$.

3.2 (2+1)-dimensional KdV equation

The (2+1)-dimensional KdV equation is associated with

$$P(x, y, t) = x^3 y + yt. \tag{3.6}$$

The bilinear (2+1)-dimensional KdV equation reads

$$\begin{aligned}B(f) := &D_y(D_t + D_x^3) f \cdot \\ &f = 2(f_{yt} f - f_y f_t + f_{xxxy} f - 3 f_{xxy} f_x + 3 f_{xy} f_{xx} - f_y f_{xxx}) = 0.\end{aligned} \tag{3.7}$$

This is equivalently transformed to the (2+1)-dimensional KdV equation (Boiti et al. 1986):

$$N(u, v) := u_t + 3(uv)_x + u_{xxx} = 0, \; u_x = v_y, \tag{3.8}$$

under the logarithmic derivative transformations $u = 2(\ln f)_{xy}$ and $v = 2(\ln f)_{xx}$. The link is $N(u, v) = (B(f)/f^2)_x$.

It is easy to work out that

$$\omega_i = k_i^3, \; 1 \le i \le N, \tag{3.9}$$

and

$$Q_1 = -3(\sigma_i k_i - \sigma_j k_j)(\sigma_i l_i - \sigma_j l_j), \; \deg Q_3 = 4, \; Q_2 = 1, \; Q_4 = 1, \tag{3.10}$$

under the substitution (3.1) with $w = 0$. Now if $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) \neq 0$, then the degree of the polynomial $\tilde{H}(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ $(= H(\mathbf{k}_1, \cdots, \mathbf{k}_N))$ is $2N(N-1) + 4 = 2N^2 - 2N + 4$, which could not be greater than $4N(N-1)$ when $N \ge 3$. Therefore, $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 0$, $N \ge 1$.

We point out that the above argument holds true for all

$$P = x^3y + yt + ax^2 + bxy + cy^2 \tag{3.11}$$

with arbitrary constants a, b, c. Additionally, the spatial symmetric version of the (2+1)-dimensional KdV equation (3.8):

$$u_t + 3(uv)_x + 3(uw)_y + u_{xxx} + u_{yyy} = 0,\ u_x = v_y,\ u_y = w_x,$$

has been considered (see Nizhnik 1980, p. 707; Veselov and Novikov 1984, p. 589), and its inverse scattering transform and the algebro-geometric solutions were presented in (Nizhnik 1980; Veselov and Novikov 1984), respectively.

3.3 The Kadomtsev–Petviashvili equation

The Kadomtsev–Petviashvili equation is associated with

$$P(x, y, t) = x^4 + xt + y^2. \tag{3.12}$$

The bilinear Kadomtsev–Petviashvili equation reads

$$\begin{aligned} B(f) &:= (D_x^4 + D_xD_t + D_y^2)f \cdot f \\ &= 2(f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 + f_{xt}f - f_xf_t + f_{yy}f - f_y^2) = 0. \end{aligned} \tag{3.13}$$

This is equivalent to the Kadomtsev–Petviashvili equation (Kadomtsev and Petviashvili 1970):

$$N(u) := (u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0, \tag{3.14}$$

under the logarithmic derivative transformation $u = 2(\ln f)_{xx}$. The link is $N(u) = (B(f)/f^2)_{xx}$.

It is easy to evaluate that

$$\omega_i = \frac{k_i^4 + l_i^2}{k_i},\ 1 \le i \le N, \tag{3.15}$$

and

$$Q_1 = -3(\sigma_i k_i - \sigma_j k_j)^2 + (l_i - l_j)^2,\ \deg Q_3 = 4,\ Q_2 = 1,\ Q_4 = 1, \tag{3.16}$$

under the substitution (3.1) with $w = 1$. Now if $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) \neq 0$, then the degree of the polynomial $\tilde{H}(\mathbf{k}_1, \cdots, \mathbf{k}_N)$ $(= H(\mathbf{k}_1, \cdots, \mathbf{k}_N))$ is $2N(N-1) + 4 = 2N^2 - 2N + 4$, which could not be greater than $4N(N-1)$ when $N \ge 3$. Therefore, $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 0,\ N \ge 1$.

The N-soliton solutions of the Kadomtsev–Petviashvili equation has been also analyzed in Satsuma (1976), and their interaction spider web structure has been explored (Biondini and Kodama 2003). We remark here parenthetically that the above result is true for all

$$P = x^4 + xt + ax^2 + bxy + cy^2 \tag{3.17}$$

with arbitrary constants a, b, c, and the quasiperiodic multiphase solutions of the Kadomtsev–Petviashvili equation with $c = \pm 1$ can be decomposed into finite-dimensional canonical Hamiltonian systems (Deconinck 2000).

3.4 (2+1)-dimensional Hirota–Satsuma–Ito equation

The Hirota–Satsuma–Ito equation (Hietarinta 1997) is associated with

$$P(x, y, t) = x^3 t + yt + x^2. \tag{3.18}$$

The bilinear (2+1)-dimensional Hirota–Satsuma–Ito equation reads

$$\begin{aligned} B(f) &:= (D_x^3 D_t + D_y D_t + D_x^2) f \cdot f \\ &= 2(f_{xxxt} f - 3 f_{xxt} f_x + 3 f_{xt} f_{xx} - f_t f_{xxx} + f_{yt} f - f_y f_t + f_{xx} f - f_x^2) = 0. \end{aligned} \tag{3.19}$$

This leads equivalent to the (2+1)-dimensional Hirota–Satsuma–Ito equation:

$$N(u) := u_{xx} + u_{ty} + 3(u_t u_x)_x + u_{txxx} = 0, \tag{3.20}$$

under the logarithmic derivative transformation $u = 2(\ln f)_x$. The link is $N(u) = (B(f)/f^2)_x$. If the equation does not depend on y, it becomes the Hirota–Satsuma equation (Hirota and Satsuma 1976). It is also known that the equation passes the three-soliton test (Ito 1980; Hietarinta 1997).

It is direct to compute that

$$\omega_i = \frac{k_i^2}{k_i^3 + l_i}, \quad 1 \le i \le N, \tag{3.21}$$

and

$$\begin{cases} Q_1 = (k_i^2 + k_j^2)^2 + (l_i^2 - l_j^2)^2 + 2k_i^2 l_i^2 + k_i^2 l_j^2 + k_j^2 l_i^2 + 2k_j^2 l_j^2 \\ \qquad - 3\sigma_i k_i \sigma_j k_j (k_i^2 + k_j^2 + l_i^2 + l_j^2), \\ \deg Q_3 = 2(N+1), \ Q_2 = (k_i^2 + l_i^2)(k_j^2 + l_j^2), \ Q_4 = \prod_{i=1}^N (k_i^2 + l_i^2), \end{cases} \tag{3.22}$$

under the substitution (3.1) with $w = 2$. Now if $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) \neq 0$, then the degree of the polynomial

$$\begin{aligned} \tilde{H}(\mathbf{k}_1, \cdots, \mathbf{k}_N) = & \sum_{\sigma=\pm 1} Q_3(k_1, l_1, \ldots, k_N, l_N, \sigma_1, \cdots, \sigma_N) \\ & \times \prod_{1 \le i < j \le N} \sigma_i \sigma_j k_i k_j Q_1(k_i, l_i, k_j, l_j, \sigma_i, \sigma_j) \end{aligned}$$

is $2(N+1) + 3N(N-1) = 3N^2 - N + 2$, which could not be greater than $4N(N-1)$ when $N \ge 4$. Therefore, $H(\mathbf{k}_1, \cdots, \mathbf{k}_N) = 0$, $N \ge 1$, of which the case $N = 3$ can be checked directly, particularly by Maple.

Many special cases of the N-soliton solution of the (2+1)-dimensional Hirota–Satsuma–Ito equation could be reduced to breather, lump and Rogue wave solutions (Liu et al. 2019; Yang et al. 2020).

4 Concluding remarks

We have analyzed the Hirota N-soliton conditions for bilinear differential equations in (2+1)-dimensions, and shown the existence of N-soliton solutions to three (2+1)-dimensional integrable equations: the (2+1)-dimensional KdV equation, the Kadomtsev–Petviashvili equation and the (2+1)-dimensional Hirota–Satsuma–Ito equation. It would be interesting to search for other bilinear equations in (2+1)-dimensions, which possess N-soliton solutions. Symbolic computations and theoretical proofs could be combined to achieve new results in the (2+1)-dimensional case.

The $\mathbf{D_{p,x}}$-operators The Hirota bilinear derivatives have been generalized to work with bilinear differential equations involving odd-order derivatives. Particularly, we have the $D_{p,x}$-operators (Ma 2011):

$$D_{p,x}^m D_{p,t}^n f \cdot g = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \alpha_p^{i+j} (\partial_x^{m-i} \partial_t^{n-j} f)(\partial_x^i \partial_t^j g), \quad m, n \geq 0,\ m+n \geq 1, \tag{4.1}$$

where the powers of α_p are determined by

$$\alpha_p^i = (-1)^{r(i)},\ i = r(i) \bmod p,\ i \geq 0, \tag{4.2}$$

with $0 \leq r(i) < p$. The patterns of those powers for $i = 1, 2, 3, \ldots$ read

$$\begin{aligned} p = 3 &: \ -,+,+,-,+,+,\ldots; \\ p = 5 &: \ -,+,-,+,+,-,+,-,+,+,\ldots; \\ p = 7 &: \ -,+,-,+,-,+,+,-,+,-,+,-,+,+,\ldots. \end{aligned}$$

For example, we can have $D_{3,x}$ and $D_{5,x}$ associated with the two smallest odd prime numbers: $p = 3, 5$. The cases of $p = 2k$, $k \in \mathbb{N}$, just present the Hirota case. The corresponding generalized bilinear expressions exhibit new characteristics. For instance, we have

$$D_{3,x}^3 f \cdot f = 2 f_{xxx} f,\ D_{3,x}^4 f \cdot f = 6 f_{xx}^2, \tag{4.3}$$

which is completely different from the Hirota case. Naturally, there are other generalized bilinear derivatives such as $D_{6,x}$ and $D_{9,x}$.

4.1 Resonant *N*-solitons in generalized cases

Resonant N-solitons have been analyzed for generalized bilinear equations (Ma 2013a, b) or trilinear equations (Ma 2013c). A generalized bilinear equation

$$P(D_{p,x_1}, \ldots, D_{p,x_M}) f \cdot f = 0 \tag{4.4}$$

possesses a resonant N-soliton solution (Ma 2013a, b):

$$f = 1 + c_1 e^{\eta_1} + c_2 e^{\eta_2} + \cdots + c_N e^{\eta_N} \tag{4.5}$$

where c_i's are arbitrary constants and η_i's are defined as in Sect. 2, iff

$$P(\mathbf{k}_i + \alpha_p \mathbf{k}_j) + P(\mathbf{k}_j + \alpha_p \mathbf{k}_i) = 0,\ 1 \leq i \leq j \leq N, \tag{4.6}$$

where $\mathbf{k}_i$'s are also defined as in Sect. 2.

4.2 Generalized *N*-soliton condition

We are curious about any example of generalized bilinear equations, which has an *N*-soliton solution. There are many interesting questions in the theory. For example, what is a generalized *N*-soliton condition, i.e., an *N*-soliton condition for generalized bilinear equations? How can one identify generalized bilinear equations, for instance,

$$P(D_{3,x}, D_{3,y}, D_{3,t}) = 0,$$

with $p = 3$, which possess *N*-soliton solutions? All this will help us understand linear and nonlinear wave motion, particularly in water waves and nonlinear optics.

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