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Research paper

Lump waves in a generalized KP-like equation with spatially balanced dispersions

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ABSTRACT

This study investigates dispersion-induced lump structures in a generalized (2+1)-dimensional KP-like framework with spatially balanced dispersions. Using a generalized bilinear form of the governing equation, we construct positive quadratic wave solutions via symbolic computation, which give rise to lump structures. Our analysis shows that the stationary points of these quadratic waves align along a straight line in the spatial domain and propagate at constant velocities. Along this characteristic line, the lump wave amplitude vanishes. The formation of these lump waves is governed by the combined influence of four distinct dispersion terms in the model.

1. Introduction

Exact and explicit closed-form solutions play a central role in mathematical physics and engineering, as they offer profound insights and provide general frameworks for tackling complex nonlinear problems. However, obtaining such solutions is often a challenging endeavor. Consequently, much research has focused on either deriving closed-form expressions directly or determining the conditions under which they can exist.

In the context of soliton theory and integrable models, wave structures such as solitons, rogue waves, and lump waves are often constructed through symbolic or computational techniques. These dispersive waves emerge from the intricate interplay between nonlinearity and dispersion, making their construction, whether analytical or numerical, a central focus in the study of wave dynamics across diverse physical and engineering systems.

Two cornerstone techniques in soliton theory and the analysis of integrable models are the inverse scattering transform (IST) [1] and the Hirota bilinear method [2]. The IST serves as a nonlinear generalization of the Fourier transform tailored for integrable systems. It is widely employed to solve initial-value problems for nonlinear equations via their associated Lax pairs [3], and to analyze the asymptotic behavior of dispersive waves in the long-time limit, including solutions that do not contain solitons [4]. In contrast, the Hirota method provides an efficient framework for generating exact wave solutions, such as solitons and lump waves, particularly for nonlinear dispersive equations in (2+1)- and (3+1)-dimensional settings [5–9].

Let x and y denote spatial coordinates, and t represent time. Given a polynomial $P(x,y,t)$, one can formulate a Hirota bilinear differential equation in (2+1)-dimensions:

$$P(D_x, D_y, D_t) f \cdot f = 0, \tag{1.1}$$

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where D_x, D_y and D_t are Hirota's bilinear operators [2], defined by

$$D_x^m D_y^n D_t^k f \cdot f = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k f(x,y,t)f(x',y',t')\Big|_{x'=x,y'=y,t'=t},$$

with m,n,k being nonnegative integers. Using Bell polynomial theory, one can often derive nonlinear PDEs for a scalar function u from Hirota bilinear forms via logarithmic derivative transformations (see, e,g., [10]), for example,

$$u = \beta(\ln f)_{xx},\ \beta(\ln f)_{yy},\ \beta(\ln f)_{xy},\ \beta(\ln f)_x,\ \beta(\ln f)_y, \tag{1.2}$$

where β is an appropriate nonzero constant. Hirota's bilinear method allows for the construction of N-soliton solutions in the exponential superposition form (see, e.g., [5,11]):

$$f = \sum_{\lambda=0,1} \exp\left(\sum_{i=1}^{N} \lambda_i \eta_i + \sum_{i<j} \lambda_i \lambda_j c_{ij}\right), \tag{1.3}$$

where the sum $\sum_{\lambda=0,1}$ runs over all combinations $\lambda_1, \lambda_2, \dots, \lambda_N \in \{0,1\}$. The phase shifts c_{ij} and the wave variables η_i are given by

$$\exp(c_{ij}) = -\frac{P(\omega_j - \omega_i, k_i - k_j, l_i - l_j)}{P(\omega_j + \omega_i, k_i + k_j, l_i + l_j)},\ \text{where } 1 \le i < j \le N, \tag{1.4}$$

and

$$\eta_i = k_i x + l_i y - \omega_i t + \eta_{i,0},\ \text{where } 1 \le i \le N. \tag{1.5}$$

The only requirement for the existence of N-soliton solutions is the dispersion relation:

$$P(-\omega_i, k_i, l_i) = 0,\ \text{where } 1 \le i \le N. \tag{1.6}$$

A central problem is to determine whether a function f of the form (1.3) indeed solves the bilinear Eq. (1.1) under the dispersion conditions (1.6). A systematic algorithm for this verification, with examples in both (1+1)- and (2+1)-dimensional cases, is presented in [11,12].

Another important class of explicit wave solutions in nonlinear integrable models comprises lump waves and rogue waves, which are closely related to solitons and capture a wide spectrum of nonlinear phenomena [13]. Lump waves are rationally localized in space, vanishing at infinity in all directions at a fixed time [13,14]. For instance, the KPI equation admits a rich variety of lump solutions [6], some of which arise in the long-wave limit of multi-soliton configurations [15]. These localized structures are not restricted to integrable systems: lump-type waves also appear in nonintegrable (2+1)-dimensional KP, BKP, and KP–Boussinesq-type extensions [16], and even in certain higher-dimensional linear wave settings via linear superposition [17,18].

A widely used approach for constructing lump solutions is the sum-of-squares ansatz, in which a positive quadratic function is substituted into a bilinear equation [6,13]. Applying logarithmic derivative transformations to such quadratic forms then produces lump solutions for various nonlinear model equations. In this work, we employ this method to study a (2+1)-dimensional generalized KP-like (gKP-like) equation incorporating two sets of nonlinear terms and four distinct dispersive terms. These nonlinear and dispersive effects serve as the balancing mechanisms that sustain lump structures. Lump solutions are derived symbolically using computer algebra systems, and the stationary points of the underlying quadratic function are analyzed to provide insight into the wave dynamics. The paper concludes with remarks and prospective directions for related problems.

2. A generalized KP-like model with spatially balanced dispersions

We consider a class of generalized bilinear differential operators introduced in [19]:

$$\begin{aligned} &D_{p,x}^m D_{p,y}^n D_{p,t}^k f \cdot f \\ &= \left(\frac{\partial}{\partial x} + \alpha_p \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} + \alpha_p \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} + \alpha_p \frac{\partial}{\partial t'}\right)^k f(x,y,t)f(x',y',t')\Big|_{x'=x,y'=y,t'=t}, \end{aligned} \tag{2.1}$$

where

$$\alpha_p^k = (-1)^{r(k)} \text{ where } k \equiv r(k) \bmod p,\ 0 \le r(k) < p. \tag{2.2}$$

For example, choosing $p = 3$ yields the pattern

$$\alpha_3 = -1,\ \alpha_3^2 = \alpha_3^3 = 1,\ \alpha_3^4 = -1,\ \alpha_3^5 = \alpha_3^6 = 1,\ \cdots, \tag{2.3}$$

while taking $p = 5$ gives

$$\alpha_5 = -1,\ \alpha_5^2 = 1,\ \alpha_5^3 = -1,\ \alpha_5^4 = \alpha_5^5 = 1,\ \alpha_5^6 = -1,\ \alpha_5^7 = 1,\ \alpha_5^8 = -1,\ \alpha_5^9 = \alpha_5^{10} = 1,\ \cdots. \tag{2.4}$$

It is noted that when p is even, the generalized derivatives reduce to the standard Hirota bilinear derivatives, whereas when p is odd, they yield new forms of bilinear derivatives.

Setting $p = 3$, we consider a gKP-like bilinear equation:

$$F_{\text{gKP-like}}(f)$$

$$
\begin{aligned}
&:= \left(D_{3,x}^4+\gamma_0 D_{3,x}^3+\gamma_1 D_{3,t}D_{3,x}+\gamma_2 D_{3,t}D_{3,y}+\gamma_3 D_{3,x}^2+\gamma_4 D_{3,y}^2\right) f\cdot f\\
&=2\left[\,3f_{xx}^2+\gamma_0 f f_{xxx}+\gamma_1(f_{tx}f-f_tf_x)+\gamma_2(f_{ty}f-f_tf_y)\right.\\
&\quad\left.+\gamma_3(f_{xx}f-f_x^2)+\gamma_4(f_{yy}f-f_y^2)\,\right]=0,
\end{aligned} \tag{2.5}
$$

where $D_{3,x}$, $D_{3,y}$ and $D_{3,t}$ denote the generalized bilinear derivatives, and γ_i for $0\le i\le 4$ are arbitrary constants. This represents a natural generalization of the standard KP equation. By redefining the dependent variable as

$$
u=2(\ln f)_x, \tag{2.6}
$$

we obtain the corresponding gKP-like model equation:

$$
\begin{aligned}
P_{\text{gKP-like}}(u) := {}& 3u_xu_{xx}+3uu_x^2+\frac{3}{2}u^3u_x+\frac{3}{2}u^2u_{xx}\\
&+\gamma_0\Big[\frac{3}{4}u^2u_x+\frac{3}{2}(uu_x)_x+u_{xxx}\Big]\\
&+\gamma_1u_{tx}+\gamma_2u_{ty}+\gamma_3u_{xx}+\gamma_4u_{yy}=0.
\end{aligned} \tag{2.7}
$$

This new model incorporates two sets of nonlinear terms and four dispersion terms.

In the special case $\gamma_1=\gamma_4=1$ and all other coefficients set to zero, the equation reduces to the standard KP-like equation:

$$
3u_xu_{xx}+3uu_x^2+\frac{3}{2}u^3u_x+\frac{3}{2}u^2u_{xx}+u_{tx}+u_{yy}=0., \tag{2.8}
$$

whose rational solutions have been studied in [20].

The model Eq. (2.7) is exactly related to the generalized bilinear Eq. (2.5) via

$$
P_{\text{gKP-like}}(u)=\Big[\frac{F_{\text{gKP-like}}(f)}{f^2}\Big]_x. \tag{2.9}
$$

Hence, u, defined by (2.6), satisfies the nonlinear generalized model Eq. (2.7) whenever f solves the bilinear Eq. (2.5).

Several questions remain concerning the integrability of this model, including whether it supports lump solutions, which are characteristic of integrable systems. In the following section, we address this issue, with a particular focus on a class of lump solutions shaped by the dispersion terms.

3. Lump waves governed by dispersion

We proceed to construct lump wave solutions of the gKP-like model Eq. (2.7) by performing symbolic computations on the corresponding generalized bilinear Eq. (2.5). In particular, we show that all four dispersion terms play a crucial role in generating lump waves and analyze the stationary points of the resulting quadratic function.

3.1. Application of the sum-of-squares ansatz

The sum-of-squares ansatz is a well-established method for constructing lump solutions of nonlinear evolution equations in higher dimensions, as illustrated in [6]. The approach begins by expressing the dependent variable as a logarithmic derivative of a positive quadratic function. The quadratic function is generally represented as a sum of squared linear terms augmented by a constant:

$$
f=\xi_1^2+\xi_2^2+a_9,\ \ \xi_1=a_1x+a_2y+a_3t+a_4,\ \ \xi_2=a_5x+a_6y+a_7t+a_8, \tag{3.1}
$$

which guarantees a rationally localized solution in all spatial directions. Substituting this form into the generalized bilinear representation of the target equation reduces the problem to solving an algebraic system for the nine parameters a_i. This framework provides a systematic basis for generating general lump wave structures of lower order in (2+1)-dimensional settings [13], with symbolic computation used to determine the coefficients.

Substituting the function f from (3.1) into the generalized bilinear Eq. (2.5) yields a system of algebraic equations. Solving this system with computer algebra provides explicit expressions for a_3, a_7 and a_9:

$$
\begin{aligned}
a_3=-&\frac{1}{(a_1\gamma_1+a_2\gamma_2)^2+(a_5\gamma_1+a_6\gamma_2)^2}\big[a_1(a_1^2+a_5^2)\gamma_1\gamma_3+a_2(a_2^2+a_6^2)\gamma_2\gamma_4\\
&+(a_1a_2^2-a_1a_6^2+2a_2a_5a_6)\gamma_1\gamma_4+(a_1^2a_2+2a_1a_5a_6-a_2a_5^2)\gamma_2\gamma_3\big],
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
a_7=-&\frac{1}{(a_1\gamma_1+a_2\gamma_2)^2+(a_5\gamma_1+a_6\gamma_2)^2}\big[a_5(a_1^2+a_5^2)\gamma_1\gamma_3+a_6(a_2^2+a_6^2)\gamma_2\gamma_4\\
&+(2a_1a_2a_6-a_2^2a_5+a_5a_6^2)\gamma_1\gamma_4+(2a_1a_2a_5-a_1^2a_6+a_5^2a_6)\gamma_2\gamma_3\big],
\end{aligned} \tag{3.3}
$$

and

$$
a_9=-\frac{3(a_1^2+a_5^2)^2\big[(a_1\gamma_1+a_2\gamma_2)^2+(a_5\gamma_1+a_6\gamma_2)^2\big]}{(a_1a_6-a_2a_5)^2(\gamma_1^2\gamma_4+\gamma_2^2\gamma_3)}, \tag{3.4}
$$

while all other parameters can be chosen arbitrarily. The frequency parameters a_3 and a_7 encode the dispersion relations in (2+1)-dimensional nonlinear dispersive wave systems, whereas the constant a_9 depends intricately on the wave numbers and plays a key role in shaping lump waves. Similar of higher-order dispersion relations have appeared in studies of lump waves associated with the second flow of the integrable KP hierarchy [21], and related dynamical behaviors have been investigated in various generalized KP-type models (see, e.g., [22,23]).

All expressions, (3.2), (3.3) and (3.4), are simplified by means of symbolic computation. To ensure the parameters are well-defined, the dispersion coefficients must satisfy the following condition:

$$\gamma_1^2\gamma_4 + \gamma_2^2\gamma_3 \neq 0, \tag{3.5}$$

which guarantees that

$$\gamma_1^1 + \gamma_2^2 \neq 0. \tag{3.6}$$

Moreover, the wave numbers must satisfy the determinant condition

$$a_1a_6 - a_2a_5 \neq 0, \tag{3.7}$$

which further leads to

$$a_1^2 + a_5^2 \neq 0, \;\; a_2^2 + a_6^2 \neq 0, \tag{3.8}$$

thereby ensuring that the solution u, defined through the logarithmic derivative transformation (2.6), decays to zero as $x^2+y^2 \to \infty$, confirming its spatial localization.

With respect to positivity, a necessary and sufficient condition on the dispersion coefficients for the solution is

$$\gamma_1^2\gamma_4 + \gamma_2^2\gamma_3 < 0, \tag{3.9}$$

which can be verified from the expression (3.4) for a_9.

In summary, the construction of lump wave solutions via the logarithmic derivative transformation requires the two essential conditions (3.7) and (3.9). The former ensures the well-posedness of u in the entire spatial–temporal domain and guarantee its localization, while the later establishes the positivity of the solution. Under these conditions, the resulting u indeed represents a genuine lump wave solution.

3.2. Characteristic path of stationary points

We now compute the stationary points of the quadratic function f defined in (3.1). These points satisfy the system

$$f_x(x(t), y(t), t) = 0, \;\; f_y(x(t), y(t), t) = 0.$$

Since f is quadratic in x and y, this reduces to the linear system:

$$a_1\xi_1 + a_5\xi_2 = 0, \;\; a_2\xi_1 + a_6\xi_2 = 0,$$

where ξ_1 and ξ_2 are defined as in (3.1). Under the non-degeneracy condition (3.7), we obtain

$$\xi_1 = a_1x + a_2y + a_3t + a_4 = 0, \;\; \xi_2 = a_5x + a_6y + a_7t + a_8 = 0. \tag{3.10}$$

Solving (3.10) for x and y as functions of t gives the trajectories of the stationary points:

$$x(t) = \frac{[(a_1^2 + a_5^2)\gamma_3 - (a_2^2 + a_6^2)\gamma_4]\gamma_1 + 2(a_1a_2 + a_5a_6)\gamma_2\gamma_3}{(a_1\gamma_1 + a_2\gamma_2)^2 + (a_5\gamma_1 + a_6\gamma_2)^2}t + \frac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \tag{3.11}$$

$$y(t) = \frac{2(a_1a_2 + a_5a_6)\gamma_1\gamma_4 - [(a_1^2 + a_5^2)\gamma_3 - (a_2^2 + a_6^2)\gamma_4]\gamma_2}{(a_1\gamma_1 + a_2\gamma_2)^2 + (a_5\gamma_1 + a_6\gamma_2)^2}t - \frac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}. \tag{3.12}$$

These formulas describe the time evolution of the stationary points of f in the spatial plane.

These stationary points define a straight-line characteristic trajectory, along which x and y advance uniformly, and the lump wave u vanishes.

4. Concluding remarks

Through symbolic computation using computer algebra systems, lump waves for a (2+1)-dimensional generalized KP-type model were obtained. The stationary points of the quadratic function specify a characteristic curve, along which the wave vanishes.

Lump waves arise in a wide range of physical and mathematical settings, underscoring both their versatility and the challenges involved in modeling nonlinear dispersive phenomena. Previous studies have explored lump solutions in linear wave models [17,18], as well as in nonlinear, nonintegrable systems in (2+1)-dimensions [24–29] and (3+1)-dimensions [22,30]. Their construction often relies on Hirota bilinear forms and related generalizations, which provide effective frameworks for analyzing such localized structures [13].

Lump waves interact in rich ways with other coherent structures in (2+1)-dimensional integrable models, such as homoclinic and heteroclinic waves [31–33], while N-soliton solutions and integrability have been widely studied using Riemann–Hilbert, bi-Hamiltonian methods and neural network-based approaches [34–41]. The properties and dynamics of lump waves in generalized (2+1)-dimensional integrable systems and multi-component integrable systems (see, e.g., [42–46]) remain open and important questions.

In conclusion, investigating lump waves offers deeper insight into nonlinear dispersive dynamics and may guide applications in physical and engineering systems where localized, coherent, and energy-concentrated structures are crucial.

Declaration of competing interest

The author declares that there is no known competing financial interest or personal relationship that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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