Nonlinear Analysis 71 (2009) e1716–e1726

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Variational identities and applications to Hamiltonian structures of soliton equations

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ARTICLE INFO

MSC:
37K05
35Q51
37K10
37K30
35Q58

Keywords:
Variational identities
Hamiltonian structures
Zero curvature equations

ABSTRACT

This is an introductory report concerning our recent research on Hamiltonian structures. We will discuss variational identities associated with continuous and discrete spectral problems, and their applications to Hamiltonian structures of soliton equations. Our illustrative examples are the AKNS hierarchy and the Volterra lattice hierarchy associated with semisimple Lie algebras, and two hierarchies of their integrable couplings associated with non-semisimple Lie algebras. The resulting Hamiltonian structures generate infinitely many commuting symmetries and conservation laws for the four soliton hierarchies. The presented variational identities can be applied to Hamiltonian structures of other soliton hierarchies.

1. Introduction

Matrix spectral problems and zero curvature equations play an important role in exploring the mathematical properties of associated soliton equations [1–3]. If Lax pairs are taken from non-semisimple Lie algebras, soliton equations come in a triangular form [4,5], due to the fact that general Lie algebras can be decomposed into semi-direct sums of semisimple Lie algebras and solvable Lie algebras [6]. Such soliton equations are called integrable couplings [7,8]. There are plenty of examples of both continuous and discrete integrable couplings [4,5,7–13], and the existing results exhibit diverse mathematical structures that soliton equations possess. The semi-direct sum decomposition of Lie algebras provides a practical way to analyze soliton equations, particularly, multi-component soliton equations [11,12,14], allowing for more classifications of integrable equations supplementing existing theories [15,16], for example, classifications within the areas of symmetry reductions [17,18] and Lax pairs [19]. In this introductory report, we would like to discuss Hamiltonian structures of soliton equations associated with general Lie algebras.

Let g be a matrix loop algebra. We assume that a pair of matrix continuous spectral problems

$$\begin{cases} \phi_x = U\phi = U(u,\lambda)\phi, \\ \phi_t = V\phi = V\left(u, u_x, \dots, \dfrac{\partial^{m_0} u}{\partial x^{m_0}}; \lambda\right)\phi, \end{cases} \tag{1.1}$$

where ϕ_x and ϕ_t denote the partial derivatives with respect to x and t, $U, V \in g$ are called a Lax pair, λ is a spectral parameter and m_0 is a natural number indicating the differential order, determines (see, say, [20,21]) a continuous soliton equation

$$u_t = K(u) = K(x, t, u, u_x, \dots), \tag{1.2}$$

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doi:10.1016/j.na.2009.02.045

through their isospectral (i.e., $\lambda_t = 0$) compatibility condition (i.e., continuous zero curvature equation):

$$U_t - V_x + [U, V] = 0. \tag{1.3}$$

This means that a triple (U, V, K) satisfies

$$U'[K] - V_x + [U, V] = 0, \quad \text{where } U'[K] = U'(u)[K] = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} U(u + \varepsilon K). \tag{1.4}$$

There exist rich algebraic structures for such triples in both the isospectral case [22,23] and the non-isospectral case [24–26].

Similarly, it is assumed that a pair of matrix discrete spectral problems

$$\begin{cases} E\phi = U\phi = U(u, \lambda)\phi, \\ \phi_t = V\phi = V(u, Eu, E^{-1}u, \dots, u^{(m)}, \dots; \lambda)\phi, \end{cases} \tag{1.5}$$

where $u = u(n, t)$ is the potential, E is the shift operator $(Ef)(n) = f(n + 1)$, $u^{(m)}(n, t) = (E^m u)(n, t) = u(n + m, t)$, and $U, V \in g$ are a Lax pair, determines (see, say, [27,28]) a discrete soliton equation

$$u_t = K = K(n, t, u, Eu, E^{-1}u, \dots), \tag{1.6}$$

through their isospectral (i.e., $\lambda_t = 0$) compatibility condition (i.e., discrete zero curvature equation):

$$U_t = (EV)U - VU. \tag{1.7}$$

This means that a triple (U, V, K) satisfies

$$U'[K] = (EV)U - VU, \tag{1.8}$$

where $U'[K]$ denotes the Gateaux derivative as before. Algebraic structures for such triples were systematically discussed [28] and applied to the construction of isospectral flows [28] and non-isospectral flows [29].

A continuous (or discrete) Hamiltonian equation [30] is as follows:

$$u_t = K(u, u_x, \dots) \quad [\text{or } K(u, Eu, E^{-1}u, \dots)] = J\frac{\delta \mathcal{H}}{\delta u}, \tag{1.9}$$

where J is a Hamiltonian operator (see, say, [20,28,31] for details) and $\mathcal{H}$ is a Hamiltonian functional $\mathcal{H} = \int H[u]\,\mathrm{d}x$(or $\mathcal{H} = \sum_{n\in\mathbb{Z}} H[u]$). A Hamiltonian equation links its conserved functionals with its symmetries [32]:

$$\text{Conserved functional } \mathcal{I} \rightarrow \text{ adjoint symmetry } \frac{\delta \mathcal{I}}{\delta u} \rightarrow \text{ symmetry } J\frac{\delta \mathcal{I}}{\delta u}.$$

Moreover, there is a Lie homomorphism between the Lie algebra of functionals and the Lie algebra of vector fields [33,34]:

$$J\frac{\delta}{\delta u}\{\mathcal{I}_1, \mathcal{I}_2\} = \left[J\frac{\delta \mathcal{I}_1}{\delta u}, J\frac{\delta \mathcal{I}_2}{\delta u}\right]. \tag{1.10}$$

If the Lie algebra g is semisimple, then Hamiltonian structures of the soliton equations (1.2) and (1.6) can be generated by the so-called trace identities [20,27]. However, if we start from non-semisimple Lie algebras, the Killing forms involved in the trace identities are degenerate. Therefore, the trace identities, unfortunately, do not work all the time. To solve this problem, we get rid of some conditions required in the trace identities and present variational identities associated with general Lie algebras.

Let us now analyze the triangular form of soliton equations associated with general Lie algebras. An arbitrary Lie algebra $\bar{g}$ takes a semi-direct sum of a semisimple Lie algebra g and a solvable Lie algebra g_c:

$$\bar{g} = g \Subset g_c, \tag{1.11}$$

and we begin with such a Lie algebra $\bar{g}$ of square matrices. The notion of semi-direct sums means that g and g_c satisfy

$$[g, g_c] \subseteq g_c,$$

where $[g, g_c] = \{[A, B] \mid A \in g,\ B \in g_c\}$. It is clear that g_c is an ideal Lie sub-algebra of $\bar{g}$. The subscript c indicates a contribution to the construction of integrable couplings. Then, choose a pair of enlarged continuous matrix spectral problems

$$\begin{cases} \bar{\phi}_x = \bar{U}\bar{\phi} = \bar{U}(\bar{u}, \lambda)\bar{\phi}, \\ \bar{\phi}_t = \bar{V}\bar{\phi} = \bar{V}\left(\bar{u}, \bar{u}_x, \dots, \dfrac{\partial^{m_0}\bar{u}}{\partial x^{m_0}}; \lambda\right)\bar{\phi}, \end{cases} \tag{1.12}$$

where the enlarged Lax pair is given as follows:

$$\bar{U} = U + U_c, \qquad \bar{V} = V + V_c, \quad U, V \in g, \quad U_c, V_c \in g_c. \tag{1.13}$$

Obviously, under the soliton equation (1.2), the corresponding enlarged continuous zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0$$

equivalently gives rise to

$$\begin{cases} U_t - V_x + [U, V] = 0, \\ U_{c,t} - V_{c,x} + [U, V_c] + [U_c, V] + [U_c, V_c] = 0. \end{cases} \tag{1.14}$$

Similarly, we can have a pair of enlarged discrete matrix spectral problems

$$\begin{cases} E\bar{\phi} = \bar{U}\bar{\phi} = \bar{U}(\bar{u}, \lambda)\bar{\phi}, \\ \bar{\phi}_t = \bar{V}\bar{\phi} = \bar{V}(\bar{u}, E\bar{u}, E^{-1}\bar{u}, \ldots; \lambda)\bar{\phi}, \end{cases} \tag{1.15}$$

where the enlarged Lax pair is given as in (1.13). We also require that the closure property between g and g_c under the matrix multiplication

$$gg_c, g_cg \subseteq g_c, \tag{1.16}$$

where $g_1g_2 = \{AB | A \in g_1,\ B \in g_2\}$, to guarantee that the discrete zero curvature equation over semi-direct sums of Lie algebras can generate coupling systems. Now, it is easy to see that under the soliton equation (1.6), the corresponding enlarged discrete zero curvature equation

$$\bar{U}_t = (E\bar{V})\bar{U} - \bar{V}\bar{U} \tag{1.17}$$

equivalently gives rise to

$$\begin{cases} U_t = (EV)U - VU, \\ U_{c,t} = [(EV)U_c - U_cV] + [(EV_c)U - UV_c] + [(EV_c)U_c - U_cV_c]. \end{cases} \tag{1.18}$$

In the systems (1.14) and (1.18), the first equations exactly present the soliton (1.2) and (1.6), and thus, the systems (1.14) and (1.18) provide the coupling systems for the (1.2) and (1.6), respectively. The detailed algebraic structures on both continuous and discrete integrable couplings can be found in [35,36]. If the solvable Lie algebra g_c is zero, i.e., $g_c = \{0\}$, then integrable couplings reduce to soliton equations associated with semisimple Lie algebras. More generally, semi-direct sums of block matrix Lie algebras yield soliton equations in block form. The analysis given here shows the basic idea of generating integrable couplings by using semi-direct sums of Lie algebras, proposed in [4,5].

Now, the basic question for us is how to construct Hamiltonian structures for integrable couplings, namely soliton equations associated with semi-direct sums of Lie algebras. A bilinear form $\langle \cdot, \cdot \rangle$ on a vector space is said to be non-degenerate when if $\langle A, B \rangle = 0$ for all vectors A, then $B = 0$, and if $\langle A, B \rangle = 0$ for all vectors B, then $A = 0$. The Killing form on a Lie algebra g' is non-degenerate iff g' is semisimple, and the Killing form satisfies $\mathrm{tr}(\mathrm{ad}_A \mathrm{ad}_A) = 0$ for all $A \in \mathcal{D}g' = [g', g']$ iff g' is solvable, where $[\cdot, \cdot]$ denotes the Lie bracket of g' and $\mathrm{ad}_A B = [A, B]$. Semi-direct sums of Lie algebras with non-zero solvable Lie algebras $g_c \neq \{0\}$ are non-semisimple, and thus, the Killing forms are always degenerate on semi-direct sums of Lie algebras with $g_c \neq \{0\}$. This is why the trace identities (see [37,38,27,21]) can not be used to establish Hamiltonian structures for integrable couplings associated with semi-direct sums of Lie algebras with $g_c \neq \{0\}$.

In this report, we would like to generalize the trace identities to semi-direct sums of Lie algebras to construct Hamiltonian structures of general soliton equations. The key point is that for a bilinear form $\langle \cdot, \cdot \rangle$ on a given Lie algebra g', we get rid of the invariance property

$$\langle \rho(A), \rho(B) \rangle = \langle A, B \rangle$$

under an isomorphism ρ of the Lie algebra g', but keep the symmetric property

$$\langle A, B \rangle = \langle B, A \rangle$$

and the invariance property under the Lie bracket

$$\langle A, [B, C] \rangle = \langle [A, B], C \rangle,$$

where $[\cdot, \cdot]$ is the Lie bracket of g', or the invariance property under the multiplication

$$\langle A, BC \rangle = \langle AB, C \rangle,$$

where g' is assumed to be an algebra and AB and BC are two products in that algebra.

We can have plenty of non-degenerate bilinear forms satisfying the required properties on semi-direct sums of Lie algebras. In what follows, we would like to show that there exist variational identities under non-degenerate, symmetric and invariant bilinear forms, which allow us to generate Hamiltonian structures of soliton equations associated with semi-direct sums of Lie algebras. Applications to the AKNS case and the Volterra lattice case furnishes Hamiltonian structures of the AKNS hierarchy and the Volterra lattice hierarchy and Hamiltonian structures of two hierarchies of their integrable couplings associated with semi-direct sums of Lie algebras. The results also ensures that the algorithms [4,5] to enlarge integrable equations using semi-direct sums of Lie algebras are efficient in presenting integrable couplings possessing Hamiltonian structures. A few of concluding remarks on coupling integrable couplings and super generalizations are given in the final section.

2. Variational identities

2.1. Variational identities on general Lie algebras

Variational identities:

Let g be a loop algebra, either semisimple or non-semisimple, and $U = U(u, \lambda)$ and $V = V(u, \lambda)$ be taken from g. Then the following continuous (or discrete) variational identity holds:

$$\frac{\delta}{\delta u}\int \left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle \mathrm{d}x \left[\text{or } \frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle\right] = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left\langle V, \frac{\partial U}{\partial u}\right\rangle, \tag{2.1}$$

where γ is a constant, $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and invariant bilinear form on g, and $U, V \in g$ satisfy the stationary zero curvature equation

$$V_x = [U, V] \; [\text{or } (EV)(EU) = UV\,]. \tag{2.2}$$

The detailed proofs of the continuous and discrete variational identities are given in [39,31], respectively. If the loop algebra g is a semisimple matrix Lie algebra, then the Killing form $\langle A, B\rangle = \mathrm{tr}(AB)$ provides the required bilinear form and the variational identities under the Killing form become the so-called trace identities [20,27,37]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(V\frac{\partial U}{\partial \lambda}\right) \mathrm{d}x \left[\text{ or } \frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\mathrm{tr}\left(V\frac{\partial U}{\partial \lambda}\right)\right] = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\left(V\frac{\partial U}{\partial u}\right). \tag{2.3}$$

These trace identities and their variants have been applied to various soliton equations including many physically significant soliton equations such as the KdV equation, the AKNS equations, the Toda lattice equation and the Volterra lattice equation (see, say, [20,21,28,38,40–44]).

Properties of bilinear forms:

- The non-degenerate property means that if $\langle A, B\rangle = 0$ for all $A \in g$ but a fixed $B \in g$ or for all B but a fixed $A \in g$, then $B = 0$ or $A = 0$.
- The symmetric property reads

$$\langle A, B\rangle = \langle B, A\rangle, \quad A, B \in g. \tag{2.4}$$

- The invariance property under the multiplication reads

$$\langle A, BC\rangle = \langle AB, C\rangle, \quad A, B, C \in g, \tag{2.5}$$

  where g is assumed to be an algebra and AB and BC are two products in that algebra g. This invariance condition is required in the discrete variational identity [31].
- The invariance property under the Lie bracket reads

$$\langle A, [B, C]\rangle = \langle [A, B], C\rangle, \quad A, B, C \in g. \tag{2.6}$$

  This invariance condition is required in the continuous variational identity [39]. If the algebra g is associative, then g forms a Lie algebra under the commutator bracket

$$[A, B] = AB - BA. \tag{2.7}$$

  Taking this Lie bracket (2.7) defined via an associate product, the invariance condition (2.6) is weaker than the invariance condition (2.5) clearly. In other words, the property (2.5) implies the property (2.6).
- The invariance property under Lie isomorphisms reads

$$\langle \rho(A), \rho(B)\rangle = \langle A, B\rangle, \quad A, B \in g, \tag{2.8}$$

  where ρ is a Lie isomorphism of g.

Two observations:

- The Killing form: If the Lie algebra g is semisimple, then all bilinear forms on g, which are non-degenerate, symmetric and invariant under the Lie bracket and Lie isomorphisms, are the Killing forms up to a constant multiplier.
- Integrable couplings: An arbitrary Lie algebra $\bar{g}$ takes the following form of semi-direct sums

$$\bar{g} = g \Subset g_c,$$

  where g is a semisimple Lie algebra and g_c is a solvable Lie algebra. When g_c is non-zero, i.e., $g_c \neq \{0\}$, this generates integrable couplings associated with non-semisimple Lie algebras.

These two statements show that the trace identities can not work for non-semisimple Lie algebras, and only the variational identities work for general Lie algebras which yield integrable couplings.

Formulas for the constant γ:

- The continuous case: Let $V_x = [U, V]$. If $|\langle V, V\rangle| \neq 0$, then the constant γ in the variational identity (2.1) is given by

$$\gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\langle V, V\rangle|. \tag{2.9}$$

- The discrete case: Let $(EV)(EU) = UV$ and $\Gamma = VU$. If $|\langle \Gamma, \Gamma\rangle| \neq 0$, then the constant γ in the variational identity (2.1) is given by

$$\gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\langle \Gamma, \Gamma\rangle|. \tag{2.10}$$

The detailed analysis on the computation of the constant γ can be found in [31,39]. The formulas provide a systematical way to compute the constant γ in the variational identities and give an answer to the open question raised in [38].

2.2. *Constructing symmetric invariant bilinear forms*

Non-semisimple Lie algebras:

As an example, take a semi-direct sum of matrix Lie algebras $\bar{g} = g \Subset g_c$:

$$g = \left\{ \mathrm{diag}(A_0, A_0) \,\middle|\, A_0 = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right\}, \qquad g_c = \left\{ \begin{bmatrix} 0 & A_1 \\ 0 & 0 \end{bmatrix} \,\middle|\, A_1 = \begin{bmatrix} a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \right\}, \tag{2.11}$$

whose Lie bracket is defined by $[A, B] = AB - BA$. Introduce a mapping

$$\delta : \bar{g} \to R^8, \qquad A \mapsto (a_1, \ldots, a_8)^{\mathrm{T}}, \quad A = \begin{bmatrix} A_0 & A_1 \\ 0 & A_0 \end{bmatrix} \in \bar{g}. \tag{2.12}$$

This mapping δ induces a Lie bracket on $\mathbb{R}^8$:

$$[a, b]^{\mathrm{T}} = a^{\mathrm{T}} R(b), \quad a = (a_1, \ldots, a_8)^{\mathrm{T}},\ b = (b_1, \ldots, b_8)^{\mathrm{T}} \in \mathbb{R}^8, \tag{2.13}$$

where $R(b)$ is the square matrix uniquely determined by the Lie bracket of $\bar{g}$:

$$R(b) = \begin{bmatrix} 0 & b_2 & -b_3 & 0 & 0 & b_6 & -b_7 & 0 \\ b_3 & b_4 - b_1 & 0 & -b_3 & b_7 & b_8 - b_5 & 0 & -b_7 \\ -b_2 & 0 & b_1 - b_4 & b_2 & -b_6 & 0 & b_5 - b_8 & b_6 \\ 0 & -b_2 & b_3 & 0 & 0 & -b_6 & b_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2 & -b_3 & 0 \\ 0 & 0 & 0 & 0 & b_3 & b_4 - b_1 & 0 & -b_3 \\ 0 & 0 & 0 & 0 & -b_2 & 0 & b_1 - b_4 & b_2 \\ 0 & 0 & 0 & 0 & 0 & -b_2 & b_3 & 0 \end{bmatrix}. \tag{2.14}$$

Transforming basic properties of bilinear forms:

A general bilinear form on $\mathbb{R}^8$ is given by

$$\langle a, b\rangle = a^{\mathrm{T}} F b, \quad a, b \in \mathbb{R}^8, \tag{2.15}$$

where F is a constant matrix, called the structural matrix of the bilinear form.

The non-degenerate property of this bilinear from iff F is invertible. The symmetric property $\langle a, b\rangle = \langle b, a\rangle$ iff $F^{\mathrm{T}} = F$. The invariance property $\langle a, [b, c]\rangle = \langle [a, b], c\rangle$ iff

$$F(R(b))^{\mathrm{T}} = -R(b)F, \quad \forall b \in \mathbb{R}^8. \tag{2.16}$$

This gives us a system of linear equations on the elements of F. With computer algebra systems such as Maple, Mathematica and Matlab, solving the resulting system leads to the matrix F:

$$F = \begin{bmatrix} \eta_1 & 0 & 0 & \eta_2 & \eta_3 & 0 & 0 & \eta_4 \\ 0 & 0 & \eta_1 - \eta_2 & 0 & 0 & 0 & \eta_3 - \eta_4 & 0 \\ 0 & \eta_1 - \eta_2 & 0 & 0 & 0 & \eta_3 - \eta_4 & 0 & 0 \\ \eta_2 & 0 & 0 & \eta_1 & \eta_4 & 0 & 0 & \eta_3 \\ \eta_3 & 0 & 0 & \eta_4 & \eta_5 & 0 & 0 & \eta_5 \\ 0 & 0 & \eta_3 - \eta_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_3 - \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_4 & 0 & 0 & \eta_3 & \eta_5 & 0 & 0 & \eta_5 \end{bmatrix}, \tag{2.17}$$

where η_i, $1 \le i \le 5$, are arbitrary constants. If we consider the non-semisimple Lie algebra $\bar{g}$ with traceless matrices A_0 and A_1, then the structure matrix of bilinear forms is given by

$$F = \begin{bmatrix} 2\eta_1 & 0 & 0 & 2\eta_2 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_1 & 0 & 0 & \eta_2 & 0 \\ 2\eta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.18}$$

where η_1 and η_2 ate arbitrary constants.

Symmetry and invariant bilinear forms:

Now, all symmetric and invariant bilinear forms on $\bar{g}$ are given by

$$\begin{aligned} \langle A, B\rangle_{\bar{G}} &= \langle \delta^{-1}(A), \delta^{-1}(B)\rangle_{\mathbb{R}^8} = (a_1, \ldots, a_8)F(b_1, \ldots, b_8)^{\mathrm{T}} \\ &= (\eta_1 a_1 + \eta_2 a_4 + \eta_3 a_5 + \eta_4 a_8)\, b_1 + [(\eta_1 - \eta_2)\, a_3 + (\eta_3 - \eta_4)\, a_7]\, b_2 \\ &\quad + [(\eta_1 - \eta_2)\, a_2 + (\eta_3 - \eta_4)\, a_6]\, b_3 + (\eta_2 a_1 + \eta_1 a_4 + \eta_4 a_5 + \eta_3 a_8)\, b_4 \\ &\quad + (\eta_3 a_1 + \eta_4 a_4 + \eta_5 a_5 + \eta_5 a_8)\, b_5 + (\eta_3 - \eta_4)\, a_3 b_6 \\ &\quad + (\eta_3 - \eta_4)\, a_2 b_7 + (\eta_4 a_1 + \eta_3 a_4 + \eta_5 a_5 + \eta_5 a_8)\, b_8, \end{aligned} \tag{2.19}$$

which reduce to the Killing type forms on g when $\eta_3 = \eta_4 = \eta_5 = 0$.

It is easy to check that these kinds of bilinear forms are invariant under the matrix multiplication, and thus, they can be applied to both the continuous case and the discrete case. The Killing form on g with

$$\eta_1 = 1, \qquad \eta_2 = \eta_3 = \eta_3 = \eta_4 = 0, \tag{2.20}$$

and two particular non-degenerate bilinear forms on $\bar{g}$ with

$$\eta_1 = \eta_3 = 1, \qquad \eta_2 = \eta_4 = \eta_5 = 0 \tag{2.21}$$

and

$$\eta_1 = \eta_2 = \eta_3 = 1, \qquad \eta_4 = \eta_5 = 0 \tag{2.22}$$

will be used to generate Hamiltonian structures of the AKNS hierarchy and the Volterra lattice hierarchy, and Hamiltonian structures of two hierarchies of their integrable couplings, respectively.

3. Continuous Hamiltonian structures

Let us focus on the case of the AKNS hierarchy. We will show how to use the continuous trace and variational identities to construct Hamiltonian structures of the AKNS hierarchy and a hierarchy of its integrable couplings.

3.1. The AKNS hierarchy

The AKNS spectral problem [45] reads

$$\phi_x = U\phi, \quad U = U(u, \lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}. \tag{3.1}$$

The zero curvature equations $U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0$ lead to the AKNS hierarchy:

$$u_{t_m} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_m} = K_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} -2p \\ 2q \end{bmatrix}, \quad m \ge 0, \tag{3.2}$$

where the hereditary recursion operator [46,47] is given by

$$\Phi = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix}. \tag{3.3}$$

The Lax pairs of the AKNS hierarchy are as follows:

$$U = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix}, \qquad V^{[m]} = (\lambda^m V)_+, \quad m \ge 0, \tag{3.4}$$

where $(P)_+$ denotes the polynomial part of P in λ and V is the following formal solution of the continuous stationary zero curvature equation $V_x = [U, V]$:

$$V = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\geq 0} V_i \lambda^{-i} = \sum_{i\geq 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}, \tag{3.5}$$

with the initial data: $a_0 = -1$ and $b_0 = c_0 = 0$.

Noting that

$$\left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = \mathrm{tr}\left(V \frac{\partial U}{\partial \lambda}\right) = -2a, \qquad \left\langle V, \frac{\partial U}{\partial u} \right\rangle = \mathrm{tr}\left(V \frac{\partial U}{\partial u}\right) = (c, b)^{\mathrm{T}}, \tag{3.6}$$

the continuous trace identity with the constants $\eta_i,\ 1 \leq i \leq 5$, defined by (2.20) gives

$$\frac{\delta \mathcal{H}_m}{\delta u} = \begin{bmatrix} c_{m+1} \\ b_{m+1} \end{bmatrix}, \quad \mathcal{H}_m = \int \frac{2a_{m+2}}{m+1}\, \mathrm{d}x,\ m \geq 0, \tag{3.7}$$

and further, the Hamiltonian structures of the AKNS hierarchy:

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix},\ m \geq 0. \tag{3.8}$$

It then follows that any AKNS system in (3.2) possesses a hierarchy of commuting symmetries $\{K_n\}_{n=0}^{\infty}$ and a hierarchy of commuting conserved functionals $\{\mathcal{H}_n\}_{n=0}^{\infty}$.

3.2. Integrable couplings

To construct a coupling hierarchy of the AKNS hierarchy, we take an enlarged spectral problem of (3.1):

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{bmatrix} U & U_a \\ 0 & U \end{bmatrix} \in g \Subset g_c, \quad U_a = \begin{bmatrix} -\alpha & v_2 \\ v_3 & \alpha \end{bmatrix}, \tag{3.9}$$

where α is a constant and $\bar{u} = (p, q, v_2, v_3)^{\mathrm{T}}$. Then the enlarged zero curvature equations $\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0$ lead to a hierarchy of integrable couplings:

$$\bar{u}_{t_m} = \begin{bmatrix} K_m \\ S_m \end{bmatrix} = (-2b_{m+1}, 2c_{m+1}, -2f_{m+1}, 2g_{m+1})^{\mathrm{T}}, \quad m \geq 0. \tag{3.10}$$

Its Lax pairs are as follows:

$$\bar{U} = \begin{bmatrix} U & U_a \\ 0 & U \end{bmatrix} \in g \Subset g_c, \qquad \bar{V}^{[m]} = \begin{bmatrix} V^{[m]} & V_a^{[m]} \\ 0 & V^{[m]} \end{bmatrix} \in g \Subset g_c, \quad V_a^{[m]} = (\lambda^m V_a)_+,\ m \geq 0, \tag{3.11}$$

where $(P)_+$ similarly denotes the polynomial part of P in λ and $\bar{V}$ is the following formal solution of the enlarged continuous stationary zero curvature equation $\bar{V}_x = [\bar{U}, \bar{V}]$:

$$\bar{V} = \begin{bmatrix} V & V_a \\ 0 & V \end{bmatrix}, \quad V_a = \begin{bmatrix} e & f \\ g & -e \end{bmatrix} = \sum_{i\geq 0} \begin{bmatrix} e_i & f_i \\ g_i & -e_i \end{bmatrix} \lambda^{-i}, \tag{3.12}$$

with the additional initial data: $e_0 = -1$ and $f_0 = g_0 = 0$. The first nonlinear system in the coupling hierarchy (3.10) is

$$\begin{cases} p_{t_2} = -\frac{1}{2} p_{xx} + p^2 q,\ q_{t_2} = \frac{1}{2} q_{xx} - pq^2, \\ v_{2,t_2} = -\frac{1}{2}(p + v_2)_{xx} - 2\alpha p_x + p(pq + v_3 p + v_2 q) + v_2 pq, \\ v_{3,t_2} = \frac{1}{2}(p + v_3)_{xx} - 2\alpha q_x - (pq + v_3 p + v_2 q)q - v_3 pq. \end{cases} \tag{3.13}$$

Now, with the constants $\eta_i,\ 1 \leq i \leq 5$, defined by (2.21), we can have

$$\left\langle \bar{V}, \frac{\partial \bar{U}}{\partial \lambda} \right\rangle = -2a - 2e, \qquad \left\langle \bar{V}, \frac{\partial \bar{U}}{\partial u} \right\rangle = (c + g, b + f, c, b)^{\mathrm{T}}. \tag{3.14}$$

Thus, the continuous variational identity with $\gamma = 0$ presents

$$\frac{\delta}{\delta \bar{u}} \bar{\mathcal{H}}_m = \begin{bmatrix} c_{m+1} + g_{m+1} \\ b_{m+1} + f_{m+1} \\ c_{m+1} \\ b_{m+1} \end{bmatrix}, \quad \bar{\mathcal{H}}_m = \int \frac{2(a_{m+1} + e_{m+1})}{m+1}\, \mathrm{d}x,\ m \geq 0, \tag{3.15}$$

and further, the Hamiltonian structures of the coupling hierarchy:

$$\bar{u}_{t_m} = \bar{K}_m = \bar{J}\frac{\delta}{\delta \bar{u}}\bar{\mathcal{H}}_m, \quad \bar{J} = \begin{bmatrix} 0 & J \\ J & -J \end{bmatrix}, \; m \geq 0, \tag{3.16}$$

where $\bar{J}$ with J defined in (3.8) is Hamiltonian. It then follows that each integrable coupling in (3.10) has a hierarchy of commuting symmetries $\{\bar{K}_n\}_{n=0}^{\infty}$ and a hierarchy of commuting conserved functionals $\{\bar{\mathcal{H}}_n\}_{n=0}^{\infty}$.

We point out that starting from other semi-direct sums of Lie algebras can yield different hierarchies of integrable couplings of the AKNS hierarchy. It is interesting to us if one can get a hierarchy of five-component integrable Hamiltonian couplings for the AKNS hierarchy.

4. Discrete Hamiltonian structures

Let us now consider the case of the Volterra lattice hierarchy. We will show how to use the discrete trace and variational identities to construct Hamiltonian structures of the Volterra lattice hierarchy and a hierarchy of its integrable couplings.

4.1. The Volterra lattice hierarchy

A spectral problem for the Volterra hierarchy reads [28]:

$$E\phi = U\phi, \quad U = U(u,\lambda) = \begin{bmatrix} 1 & u \\ \lambda^{-1} & 0 \end{bmatrix}, \; \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{4.1}$$

The zero curvature equations $U_{t_m} = (EV^{[m]})U - V^{[m]}U$ lead to the Volterra lattice hierarchy:

$$u_{t_m} = K_m = \Phi^m K_0 = u(a_{m+1}^{(1)} - a_{m+1}^{(-1)}), \quad m \geq 0, \tag{4.2}$$

where $f^{(m)}(n) = (E^m f)(n) = f(n+m)$, the initial vector field K_0 defines the Volterra lattice equation:

$$u_{t_0} = K_0 = u(u^{(-1)} - u^{(1)}), \tag{4.3}$$

and the hereditary recursion operator is given by

$$\Phi = u(1+E^{-1})(-u^{(1)}E^2 + u)(E-1)^{-1}u^{-1}. \tag{4.4}$$

Its Lax pairs are as follows:

$$U = \begin{bmatrix} 1 & u \\ \lambda^{-1} & 0 \end{bmatrix}, \qquad V^{[m]} = (\lambda^{m+1}\Gamma)_+ + \Delta_m, \quad \Delta_m = \begin{bmatrix} 0 & -b_{m+1} \\ 0 & a_{m+1} + a_{m+1}^{(-1)} \end{bmatrix}, \; m \geq 0, \tag{4.5}$$

where Γ is the following formal solution of the discrete stationary zero curvature equation $(E\Gamma)U = U\Gamma$:

$$\Gamma = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\geq 0}\Gamma_i\lambda^{-i} = \sum_{i\geq 0}\begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix}\lambda^{-i}, \tag{4.6}$$

with the initial data: $a_0 = \frac{1}{2}$, $b_0 = u$ and $c_0 = 0$.

Obviously, we can have

$$\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = \mathrm{tr}\left(V\frac{\partial U}{\partial \lambda}\right) = \lambda^{-1}a^{(1)}, \qquad \left\langle V, \frac{\partial U}{\partial u}\right\rangle = \mathrm{tr}\left(V\frac{\partial U}{\partial u}\right) = -\frac{a}{u}, \tag{4.7}$$

and thus, the discrete trace identity with $\gamma = 0$ presents

$$\frac{\delta \mathcal{H}_m}{\delta u} = -\frac{a_{m+1}}{u}, \quad \mathcal{H}_m = -\sum_{n\in\mathbb{Z}}\frac{a_{m+1}(n)}{m+1}, \; m \geq 0, \tag{4.8}$$

and further, the Hamiltonian structures of the Volterra lattice hierarchy:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad J = u(E^{-1} - E)u, \; m \geq 0. \tag{4.9}$$

It now follows that each Volterra lattice equation in (4.2) possesses a hierarchy of commuting symmetries $\{K_n\}_{n=0}^{\infty}$ and a hierarchy of commuting conserved functionals $\{\mathcal{H}_n\}_{n=0}^{\infty}$.

4.2. Integrable couplings

To construct a coupling hierarchy of the Volterra lattice hierarchy, we take an enlarged spectral problem of (4.1):

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{bmatrix} U & U_a \\ 0 & U \end{bmatrix} \in g \Subset g_c, \quad U_a = U_a(v) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}, \tag{4.10}$$

where v is a new dependent variable and $\bar{u} = (u, v)^{\mathrm{T}}$. Then the enlarged zero curvature equations $\bar{U}_{t_m} = (E\bar{V}^{[m]})\bar{U} - \bar{V}^{[m]}\bar{U}$ lead to a hierarchy of integrable couplings:

$$\bar{u}_{t_m} = \begin{bmatrix} u(a^{(1)}_{m+1} - a^{(-1)}_{m+1}) \\ u(e^{(1)}_{m+1} - e^{(-1)}_{m+1}) + v(a^{(1)}_{m+1} - a^{(-1)}_{m+1}) \end{bmatrix}, \quad m \geq 0. \tag{4.11}$$

Its Lax pairs are as follows:

$$\bar{U} = \begin{bmatrix} U & U_a \\ 0 & U \end{bmatrix} \in g \Subset g_c, \qquad \bar{V}^{[m]} = \begin{bmatrix} V^{[m]} & V_a^{[m]} \\ 0 & V^{[m]} \end{bmatrix} \in g \Subset g_c, \tag{4.12}$$

with $V_a^{[m]}$ being defined by

$$V_a^{[m]} = (\lambda^{m+1}\Gamma_a)_+ + \Delta_{m,a}, \quad \Delta_{m,a} = \begin{bmatrix} 0 & -f_{m+1} \\ 0 & e_{m+1} + e^{(-1)}_{m+1} \end{bmatrix}, \ m \geq 0, \tag{4.13}$$

where $\bar{\Gamma}$ is the following formal solution of the enlarged discrete stationary zero curvature equation $(E\bar{\Gamma})\bar{U} = \bar{U}\bar{\Gamma}$:

$$\bar{\Gamma} = \begin{bmatrix} \Gamma & \Gamma_a \\ 0 & \Gamma \end{bmatrix}, \quad \Gamma_a = \Gamma_a(\bar{u}, \lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix} = \sum_{i\geq 0} \begin{bmatrix} e_i & f_i \\ g_i & -e_i \end{bmatrix} \lambda^{-i} \tag{4.14}$$

with the additional initial data: $e_0 = 0, f_0 = v$ and $g_0 = 0$. The first nonlinear system in the coupling hierarchy (4.11) reads

$$u_{t_0} = u(u^{(-1)} - u^{(1)}), \qquad v_{t_0} = v(u^{(-1)} - u^{(1)}) + u(v^{(-1)} - v^{(1)}), \tag{4.15}$$

Now, noticing that with the constants $\eta_i,\ 1 \leq i \leq 5$, defined by (2.22), we can have

$$\left\langle \bar{V}, \frac{\partial \bar{U}}{\partial \lambda} \right\rangle = \lambda^{-1} e^{(1)}, \qquad \left\langle \bar{V}, \frac{\partial \bar{U}}{\partial u} \right\rangle = \frac{va}{u^2} - \frac{e}{u}, \qquad \left\langle \bar{V}, \frac{\partial \bar{U}}{\partial v} \right\rangle = -\frac{a}{u}, \tag{4.16}$$

the discrete variational identity with $\gamma = 0$ presents

$$\frac{\delta}{\delta \bar{u}} \bar{\mathcal{H}}_m = \left(\frac{va_{m+1}}{u^2} - \frac{e_{m+1}}{u}, -\frac{a_{m+1}}{u} \right)^{\mathrm{T}}, \quad \bar{\mathcal{H}}_m = -\sum_{n\in\mathbb{Z}} \frac{e_{m+1}(n)}{m+1}, \ m \geq 0. \tag{4.17}$$

Further, an application of the discrete variational identity with $\gamma = 0$ yields the Hamiltonian structures of the coupling hierarchy:

$$\bar{u}_{t_m} = \bar{K}_m = \bar{J} \frac{\delta}{\delta \bar{u}} \bar{\mathcal{H}}_m, \quad \bar{J} = \begin{bmatrix} 0 & u(E^{-1} - E)u \\ u(E^{-1} - E)u & J_{22} \end{bmatrix}, \ m \geq 0, \tag{4.18}$$

where $J_{22} = u(E^{-1} - E)v + v(E^{-1} - E)u$. It finally follows that each integrable coupling in (4.11) has a hierarchy of commuting symmetries $\{\bar{K}_n\}_{n=0}^{\infty}$ and a hierarchy of commuting conserved functionals $\{\bar{\mathcal{H}}_n\}_{n=0}^{\infty}$.

We point out that starting from other semi-direct sums of Lie algebras can generate different hierarchies of integrable couplings of the Volterra lattice hierarchy. It is interesting to us how one can construct a hierarchy of three-component integrable Hamiltonian couplings for the Volterra hierarchy.

5. Concluding remarks

We have discussed variational identities associated with general matrix spectral problems and applied them to Hamiltonian structures of soliton equations associated with semisimple Lie algebras and integrable couplings associated with semi-direct sums of Lie algebras. The required conditions for the involved bilinear forms are the non-degenerate, symmetric and invariance properties under the Lie bracket or the multiplication. Illustrative examples includes the AKNS hierarchy and the Volterra lattice hierarchy, and two hierarchies of their integrable couplings. The results show that the approaches for enlarging integrable equations through semi-direct sums of Lie algebras [4,5] are powerful in presenting continuous and discrete integrable Hamiltonian couplings.

Bilinearization of Lax pairs of integrable couplings [48] could engender higher-dimensional Liouville integrable Hamiltonian systems on symplectic manifolds, and integrable couplings with self-consistent sources for given soliton equations. There are more integrable couplings if we couple given integrable couplings [49]. One open question is whether there exist Hamiltonian structures of such integrable couplings. More precisely, let us assume that we have two integrable couplings

$$\begin{cases} u_t = K(u), \\ v_t = S_1(u, v), \end{cases} \qquad \begin{cases} u_t = K(u), \\ v_t = S_2(u, v), \end{cases} \tag{5.1}$$

for a given soliton equation $u_t = K(u)$. Is there is any Hamiltonian structure for a coupled system of integrable couplings

$$\begin{cases} u_t = K(u), \\ v_t = S_1(u, v), \\ w_t = S_2(u, w), \end{cases} \tag{5.2}$$

if $u_t = K(u)$ is Hamiltonian? In particular, it remains an open question to us if there is any Hamiltonian structure for the coupled system:

$$u_t = K(u), \qquad v_t = K'(u)[v], \qquad w_t = K'(u)[w], \tag{5.3}$$

where $K'(u)[X]$ is the Gateaux derivative: $K'(u)[X] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} K(u + \varepsilon X)$ [49].

There is a super-trace identity for super zero curvature equations, which allows us to construct Hamiltonian structures of super soliton equations [50], associated with Lie superalgebras [51]. Let g be a Lie superalgebra over a supercommutative ring. Then the super-trace identity on the Lie superalgebra g reads

$$\frac{\delta}{\delta u} \int \mathrm{str}\left(\mathrm{ad}_V \mathrm{ad}_{\frac{\partial U}{\partial \lambda}}\right) \mathrm{d}x \ \left[\text{or } \frac{\delta}{\delta u} \sum_{n \in \mathbb{Z}} \mathrm{str}\left(\mathrm{ad}_V \mathrm{ad}_{\frac{\partial U}{\partial \lambda}}\right) \right] = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \left(\mathrm{ad}_V \mathrm{ad}_{\frac{\partial U}{\partial u}}\right), \tag{5.4}$$

where $U, V \in g$ solve $V_x = [U, V]$ [or $(EV)(EU) = UV$], and ad_a denotes the adjoint action of $a \in g$ on g: $\mathrm{ad}_a b = [a, b]$, and str is the supertrace. Based on the Lie superalgebra $B(0, 1)$, two applications to the super-AKNS soliton hierarchy and the super-Dirac soliton hierarchy are given in [50].

It is interesting to generalize the super-trace identity to super-symmetric soliton equations. In the simplest super-symmetric case where $D = 1$ and $N = 1$, the super-symmetric derivative is given by $D_x = \partial_\theta + \theta \partial_x$, where x is even but θ is odd. To start from zero curvature equations, let us take the Taylor expansion about $\theta = 0$ for a Lax pair:

$$U = U_1 + \theta U_2, \qquad V = V_1 + \theta V_2, \tag{5.5}$$

where U_1, V_1 are bosonic fields and U_2, V_2 are fermionic fields. Then the zero curvature equation $U_t - V_x + [U, V] = 0$ exactly leads to the so-called integrable coupling:

$$\begin{cases} U_{1,t} - V_{1,x} + [U_1, V_1] = 0, & \text{– bosinic field,} \\ U_{2,t} - V_{2,x} + [U_1, V_2] + [U_2, V_1] = 0, & \text{– fermionic field.} \end{cases}$$

To involve in a business of the super-symmetric derivative D_x, one needs to adjust the standard zero curvature equation. It seems that a first choice to work with is a generalized zero curvature equation

$$U_t - D_x V + [U, V] = 0. \tag{5.6}$$

The nonlinearization of Lax pairs in the super case can yield super finite-dimensional integrable Hamiltonian systems when there are only super dependent variables, and super-symmetric finite-dimensional integrable Hamiltonian systems when there are super independent variables [52]. This will further generate super or super-symmetric soliton equations with self-consistent sources.

Acknowledgments

The work was supported in part by Zhejiang Normal University, the Established Researcher Grant of the University of South Florida, the CAS faculty development grant of the University of South Florida, Chunhui Plan of the Ministry of Education of China, and Wang Kuancheng foundation.

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