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Matrix mKdV Integrable Hierarchies via Two Identical Group Reductions

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Abstract: This paper applies a pair of identical group reductions or similarity transformations to formulate integrable models. An application to the Ablowitz–Kaup–Newell–Segur (AKNS) matrix spectral problems leads to reduced matrix modified Korteweg–de Vries (mKdV) integrable hierarchies. In particular, several illustrative examples of reduced matrix mKdV integrable models are derived from the reduced AKNS matrix spectral problems.

Keywords: integrable hierarchy; lax pair; AKNS matrix spectral problem; zero-curvature equation; group reduction

MSC: 37K10; 35Q51; 37K40

1. Introduction

Integrable models are derived from the Lax pairs of matrix spectral problems [1], with the key step being the selection of an appropriate matrix spatial spectral problem [2]. The inverse scattering transform provides a powerful method for solving initial value problems of integrable models [3,4].

Typical examples, such as the nonlinear Schrödinger (NLS) equation and the modified Korteweg–de Vries (mKdV) equation, can be obtained from the Ablowitz–Kaup–Newell–Segur (AKNS) matrix spectral problems associated with sl(m+n) via a single group reduction or similarity transformation (see, e.g., [5–7]). Furthermore, applying a pair of group reductions or similarity transformations can produce a variety of integrable models [8]. The main challenge lies in balancing the reductions applied to the potentials generated by the two transformations, as this process requires careful attention to maintain the invariance of the associated zero-curvature equations [9].

Group reductions and similarity transformations have been extensively utilized in the construction of nonlocal integrable models that involve reflection points [10]. A comprehensive classification of lower-order integrable models associated with AKNS matrix spectral problems has led to the identification of three types of nonlocal NLS equations and two types of nonlocal mKdV equations [11]. Additionally, various efficient methods have been developed to analyze and solve reduced integrable models, particularly for deriving soliton solutions.

The inverse scattering transform has proven to be a powerful tool for solving initial value problems in nonlocal integrable models [12,13]. Other effective techniques include the Hirota bilinear method, Darboux transformation, Bäcklund transforms, and the Riemann–

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Hilbert method. Furthermore, a variety of intriguing mathematical frameworks have been proposed to address nonlocal reduced integrable models (see, e.g., [11,14–19]).

In this paper, we investigate integrable reductions through a pair of identical group reductions or similarity transformations and explore their applications to AKNS matrix spectral problems, along with the corresponding reduced integrable models. The key contribution of this work is the identification of two similarity transformations that involve diagonal block matrices. In Section 2, we revisit the matrix AKNS spectral problems and their associated integrable mKdV models to lay the foundation for the subsequent analysis. We then introduce the pair of identical group reductions or similarity transformations, which lead to the derivation of reduced matrix mKdV integrable hierarchies. Section 3 illustrates the theory through five specific examples, each selecting distinct sets of block matrices to construct the pair of group reductions. These examples highlight the diversity of reduced AKNS matrix integrable models. Section 4 provides a summary of our findings and concluding remarks.

2. Matrix Integrable mKdV Hierarchies via Group Reductions

2.1. The AKNS Integrable Hierarchies Revisited

Let m, n be two arbitrary natural numbers. We define two matrix potentials, p and q, as follows:

$$p = p(x,t) = (p_{jk})_{m\times n},\ q = q(x,t) = (q_{kj})_{n\times m}, \tag{1}$$

and use $u = u(p,q)$ to denote the dependent variable, which is a vector-valued function of p and q. Then, for all $r \geq 0$, the standard matrix AKNS spectral problems are expressed as follows:

$$-i\phi_x = U\phi,\ -i\phi_t = V^{[r]}\phi, \tag{2}$$

where the Lax pairs are determined by

$$U = U(u,\lambda) = \lambda\Lambda + P,\ \Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix},\ P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{3}$$

and

$$V^{[r]} = V^{[r]}(u,\lambda) = \lambda^r\Omega + Q^{[r]},\ \Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix},\ Q^{[r]} = \sum_{s=0}^{r-1}\lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}. \tag{4}$$

In the above Lax pairs, λ denotes the spectral parameter, I_k is the identity matrix of size k, α_1, α_2 and β_1, β_2 are two pairs of arbitrarily given distinct constants, $Q^{[0]}$ is the $(m+n)$-th-order zero matrix, and the Laurent series

$$W = \sum_{s\geq 0}\lambda^{-s}W^{[s]} = \sum_{s\geq 0}\lambda^{-s}\begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{5}$$

solves the stationary zero-curvature equation

$$W_x = i[U,W], \tag{6}$$

with the initial data $W^{[0]} = \Omega$. This series solution is crucial for generating integrable hierarchies (see, e.g., [20,21]).

Obviously, the compatibility conditions of the two matrix spectral problems in (2) are the zero-curvature equations:

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0,\ r \geq 0. \tag{7}$$

Together with (3) and (4), these present a matrix AKNS integrable hierarchy:

$$p_t = i\alpha b^{[r+1]},\ q_t = -i\alpha c^{[r+1]},\ r \geq 0, \tag{8}$$

where $\alpha = \alpha_1 - \alpha_2$. The simplest case with $m = n = 1$ yields the AKNS integrable hierarchy with scalar potentials [22]. Each system within the matrix AKNS integrable hierarchy possesses a bi-Hamiltonian structure, along with infinitely many symmetries and conserved quantities (see, e.g., [23–25]).

When $r = 2s + 1$, $s \geq 1$, the matrix AKNS integrable hierarchy (8) reduces to the matrix mKdV integrable hierarchies. The first (when $s = 1$) integrable model in the resulted matrix mKdV integrable hierarches gives the matrix mKdV equations:

$$\begin{cases} p_t = -\dfrac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp), \\ q_t = -\dfrac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \end{cases} \tag{9}$$

where $\beta = \beta_1 - \beta_2$. The corresponding Lax matrix $V^{[3]}$ is given by

$$V^{[3]} = \lambda^3\Omega + \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3), \tag{10}$$

where $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. Other significant examples of higher-order matrix AKNS integrable models can also be generated (see, e.g., [26]).

2.2. *Reducing the AKNS Spectral Problems*

In order to introduce a pair of group reductions or similarity transformations, we take two constant invertible symmetric square matrices of order m, Σ_1, Δ_1, and two constant invertible symmetric square matrices of order n, Σ_2, Δ_2. Then, we define two invertible constant square matrix of order $m + n$ as follows:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},\ \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{11}$$

It is easy to determine that both Σ and Δ satisfy the important similarity properties

$$\Sigma\Lambda\Sigma^{-1} = \Delta\Lambda\Delta^{-1} = \Lambda,\ \Sigma\Omega\Sigma^{-1} = \Delta\Omega\Delta^{-1} = \Omega. \tag{12}$$

Based on these properties, we can introduce the following pair of group reductions or similarity transformations:

$$\Sigma U(\lambda)\Sigma^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T,\ \Delta U(\lambda)\Delta^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T, \tag{13}$$

where A^T stands for the matrix transpose of a matrix A. We will show that each of the group reductions or similarity transformations preserves the invariance of the original zero-curvature equations of the mKdV equations.

Let us now check these two reductions carefully to determine what conditions we need to impose. Following the definition of the spectral matrix U, we can see that the two group reductions or similarity transformations yield the following relations for the potential matrix P:

$$\Sigma P \Sigma^{-1} = -P^T,\ \Delta P \Delta^{-1} = -P^T, \tag{14}$$

respectively. Clearly, these reductions give rise to the following pairs of constraints for the two matrix potentials p and q:

$$p^T = -\Sigma_2 q \Sigma_1^{-1},\ q^T = -\Sigma_1 p \Sigma_2^{-1}, \tag{15}$$

and

$$p^T = -\Delta_2 q \Delta_1^{-1},\ q^T = -\Delta_1 p \Delta_2^{-1}, \tag{16}$$

respectively.

Obviously, the two constraints in each of the two pairs, (15) and (16), are compatible because both Σ and Δ are symmetric. To guarantee the compatibility of the two sets of constraints, we must impose one of the following conditions:

$$\Sigma_1 p \Sigma_2^{-1} = \Delta_1 p \Delta_2^{-1}, \tag{17}$$

or

$$\Sigma_2 q \Sigma_1^{-1} = \Delta_2 q \Delta_1^{-1}. \tag{18}$$

Both conditions are equivalent.

To summarize, with the condition in (17) or (18), the two identical group reductions or similarity transformations in (13) generate a class of reduced AKNS matrix spectral problems:

$$-i\phi_x = U\phi,\ U = \begin{bmatrix} \alpha_1 \lambda I_m & p \\ -\Sigma_2^{-1} p^T \Sigma_1 & \alpha_2 \lambda I_n \end{bmatrix}, \tag{19}$$

where the square matrix potential p must satisfy (17). Alternatively, we can express the class of reduced AKNS matrix spectral problems as

$$-i\phi_x = U\phi,\ U = \begin{bmatrix} \alpha_1 \lambda I_m & -\Sigma_1^{-1} q^T \Sigma_2 \\ q & \alpha_2 \lambda I_n \end{bmatrix}, \tag{20}$$

where the square matrix potential q must satisfy (18).

2.3. Matrix Integrable mKdV Hierarchies

Let us examine the effects of the solution W, determined by (5), with the initial data

$$W^{[0]} = \Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix}, \tag{21}$$

under the group reductions or similarity transformations given in (13). First, we can readily verify that

$$\Sigma W(\lambda) \Sigma^{-1}|_{\lambda=\infty} = \Delta W(\lambda) \Delta^{-1}|_{\lambda=\infty} = W^T(-\lambda)|_{\lambda=\infty} = (W(-\lambda))^T|_{\lambda=\infty}. \tag{22}$$

From the uniqueness of solutions to the stationary zero-curvature equation, it follows that

$$\Sigma W(\lambda) \Sigma^{-1} = W^T(-\lambda) = (W(-\lambda))^T,\ \Delta W(\lambda) \Delta^{-1} = W^T(-\lambda) = (W(-\lambda))^T. \tag{23}$$

Therefore, for all $s \geq 0$, we can demonstrate that

$$\begin{aligned} \Sigma V^{[2s+1]}(\lambda)\Sigma^{-1} &= -V^{[2s+1]T}(-\lambda) = -(V^{[2s+1]}(-\lambda))^T, \\ \Delta V^{[2s+1]}(\lambda)\Delta^{-1} &= -V^{[2s+1]T}(-\lambda) = -(V^{[2s+1]}(-\lambda))^T. \end{aligned} \tag{24}$$

Consequently, under the group reductions or similarity transformations in (13), we can compute that

$$\begin{aligned} &\Sigma(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Sigma^{-1} \\ &= (-U^T(-\lambda))_t - (-V^{[2s+1]T}(-\lambda))_x + i[-U^T(-\lambda), -V^{[2s+1]T}(-\lambda)] \\ &= -(U_t^T - V_x^{[2s+1]T} + i[V^{[2s+1]T}, U^T])(-\lambda) \\ &= -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(-\lambda))^T, \end{aligned}$$

and

$$\begin{aligned} &\Delta(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Delta^{-1} \\ &= (-U^T(-\lambda))_t - (-V^{[2s+1]T}(-\lambda))_x + i[-U^T(-\lambda), -V^{[2s+1]T}(-\lambda)] \\ &= -(U_t^T - V_x^{[2s+1]T} - i[U^T, V^{[2s+1]T}])(-\lambda) \\ &= -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(-\lambda))^T, \end{aligned}$$

respectively. Accordingly, the matrix AKNS integrable models in (8) with $r = 2s+1$ give rise to a reduced hierarchy of integrable mKdV models:

$$p_t = 2ib^{[2s+2]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}, s \geq 0, \tag{25}$$

where p needs to satisfy (17) or

$$q_t = -2ic^{[2s+2]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}, \ s \geq 0, \tag{26}$$

where q needs to satisfy (18).

The matrix spectral problems, given by (19) and

$$-i\phi_t = V^{[2s+1]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi, \ s \geq 0, \tag{27}$$

form a Lax pair for every member in the reduced integrable hierarchy (25). Alternatively, the matrix spectral problems, given by (20) and

$$-i\phi_t = V^{[2s+1]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}\phi, \ s \geq 0, \tag{28}$$

form a Lax pair for every member in the reduced integrable hierarchy (26).

The commuting properties of the resulting reduced integrable hierarchies stem from the Lax operator algebras (see, e.g., [27]). It is important to note that Σ_1, Σ_2, Δ_1, and Δ_2 are arbitrary invertible symmetric square matrices. By selecting appropriate values for these matrices, we can derive a variety of integrable mKdV hierarchies corresponding to the reduced matrix AKNS models discussed above. Note that we can also similarly derive examples of integrable mKdV type equations associated with symmetric spaces as special reductions of sl(m+n) (see, e.g., [28]), and other matrix generalizations of the mKdV equation, integrable via the inverse scattering transform (see, e.g., [29]).

3. Applications

In this section, we apply the general framework to five distinct cases, presenting illustrative examples of reduced matrix AKNS spectral problems and integrable mKdV equations, as formulated above.

Example 1. *Let us begin by considering the case where $m = 1$ and $n = 3$. We select the following specific values for the pairs of matrices:*

$$\Sigma_1 = \sigma,\ \Sigma_2 = \begin{bmatrix} 0 & 0 & \rho \\ 0 & \delta & 0 \\ \gamma & 0 & 0 \end{bmatrix};\ \Delta_1 = \sigma,\ \Delta_2 = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \rho \end{bmatrix}; \tag{29}$$

where $\sigma, \rho, \delta,$ and γ are arbitrary non-zero constants. Then, the group reductions or similarity transformations in (13) lead to the following expression for the spectral matrix U:

$$U = U(u, \lambda) = \begin{bmatrix} \alpha_1\lambda & p \\ q & \alpha_2\lambda I_3 \end{bmatrix} \text{ with } p = [p_1, p_2, \frac{\rho}{\gamma}p_1],\ q = [-\frac{\sigma\rho}{\gamma^2}p_1, -\frac{\sigma}{\delta}p_2, -\frac{\sigma}{\rho}p_1]^T, \tag{30}$$

where $u = (p_1, p_2)^T$. Accordingly, the corresponding reduced matrix integrable mKdV equations are expressed by

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}\{p_{1,xxx} - \dfrac{3\sigma}{\delta\gamma^2}[(2\delta\rho p_1^2 + 2\delta\gamma p_1^2 + \gamma^2 p_2^2)p_{1,x} + \gamma^2 p_1 p_2 p_{2,x}]\}, \\ p_{2,t} = -\dfrac{\beta}{\alpha^3}\{p_{2,xxx} - \dfrac{3\sigma}{\delta\gamma^2}[\delta(\rho+\gamma)p_1 p_2 p_{1,x} + (\delta\rho p_1^2 + \delta\gamma p_1^2 + 2\gamma^2 p_2^2)p_{2,x}]\}, \end{cases} \tag{31}$$

where $\sigma, \rho, \delta,$ and γ are arbitrary non-zero constants.

Example 2. *Let us next consider the case where $m = n = 2$. We select the following specific values for the two pairs of matrices:*

$$\Sigma_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ \Sigma_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix};\ \Delta_1 = \begin{bmatrix} 0 & -2 \\ -2 & 1 \end{bmatrix},\ \Delta_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{32}$$

Then, the group reductions or similarity transformations in (13) yield

$$U = U(u, \lambda) = \begin{bmatrix} \alpha_1\lambda I_2 & p \\ q & \alpha_2\lambda I_2 \end{bmatrix} \text{ with } p = \begin{bmatrix} p_2 & p_1 \\ p_2 & p_1 \end{bmatrix},\ q = \begin{bmatrix} 2p_1 & p_1 \\ 2p_2 & p_2 \end{bmatrix}, \tag{33}$$

where $u = (p_1, p_2)^T$. Consequently, the corresponding reduced matrix integrable mKdV equations are given by

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}[p_{1,xxx} + 9p_1(3p_{1,x}p_2 + p_1 p_{2,x})], \\ p_{2,t} = -\dfrac{\beta}{\alpha^3}[p_{2,xxx} + 9p_2(p_{1,x}p_2 + 3p_1 p_{2,x})]. \end{cases} \tag{34}$$

Example 3. *Let us now consider the second case where $m = n = 2$, and take the following specific values for the pairs of matrices:*

$$\Sigma_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ \Sigma_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix};\ \Delta_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ \Delta_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{35}$$

Then, the group reductions or similarity transformations in (13) engender the concrete expression for U:

$$U = U(u,\lambda) = \begin{bmatrix} \alpha_1 \lambda I_2 & p \\ q & \alpha_2 \lambda I_2 \end{bmatrix} \text{ with } p = \begin{bmatrix} p_1 & p_1 \\ p_2 & p_2 \end{bmatrix}, \; q = \begin{bmatrix} -p_1 - p_2 & p_2 - p_1 \\ 0 & 0 \end{bmatrix}, \tag{36}$$

where $u = (p_1, p_2)^T$*. Furthermore, the corresponding reduced matrix integrable mKdV equations are given by*

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}\{p_{1,xxx} - 3[(2p_1^2 + 3p_1p_2 - p_2^2)p_{1,x} + p_1(p_1 - p_2)p_{2,x}]\}, \\ p_{2,t} = -\dfrac{\beta}{\alpha^3}\{p_{2,xxx} - 3[(p_1 + p_2)p_2p_{1,x} + (p_1^2 + 3p_1p_2 - 2p_2^2)p_{2,x}]\}. \end{cases} \tag{37}$$

Example 4. *For* $m = n = 2$*, we consider a third case and take the following two pairs of matrices:*

$$\Sigma_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \; \Sigma_2 = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}; \; \Delta_1 = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}, \; \Delta_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{38}$$

Now, the group reductions or similarity transformations in (13) generate

$$U = U(u,\lambda) = \begin{bmatrix} \alpha_1 \lambda I_2 & p \\ q & \alpha_2 \lambda I_2 \end{bmatrix} \text{ with } p = \begin{bmatrix} -p_2 & p_1 \\ p_2 - p_1 & p_2 \end{bmatrix}, \; q = \begin{bmatrix} p_2 & p_1 - p_2 \\ -p_1 & -p_2 \end{bmatrix}, \tag{39}$$

where $u = (p_1, p_2)^T$*. The resulting system of integrable models in this case involves complex interactions between the entries of the potential matrix, exhibiting both nonlinear interactions and differential terms. These interactions reflect the structure of the reduced AKNS matrix spectral problems and the corresponding integrable mKdV equations. The corresponding reduced matrix integrable mKdV equations are determined by*

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}\{p_{1,xxx} - 6[(p_1^2 + p_2^2)p_{1,x} + (2p_1 - p_2)p_2p_{2,x}]\}, \\ p_{2,t} = -\dfrac{\beta}{\alpha^3}\{p_{2,xxx} - 6[(2p_1 - p_2)p_2p_{1,x} + (p_1^2 - 2p_1p_2 + 2p_2^2)p_{2,x}]\}. \end{cases} \tag{40}$$

Example 5. *Finally, let us consider the case where* $m = 2$ *and* $n = 3$*. We select the following specific values for the pairs of matrices:*

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \; \Sigma_2 = \begin{bmatrix} 0 & 0 & \delta_1 \\ 0 & \delta_2 & 0 \\ \delta_3 & 0 & 0 \end{bmatrix}; \; \Delta_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_2 & 0 \end{bmatrix}, \; \Delta_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix}; \tag{41}$$

where $\sigma_1, \sigma_2, \delta_1, \delta_2$*, and* δ_3 *are arbitrary non-zero constants. Then, the group reductions or similarity transformations given in (13) determine*

$$U = U(u,\lambda) = \begin{bmatrix} \alpha_1 \lambda I_2 & p \\ q & \alpha_2 \lambda I_3 \end{bmatrix} \text{ with } p = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_3 & p_2 & p_1 \end{bmatrix}, \; q = \begin{bmatrix} -\dfrac{\sigma_1}{\delta_3}p_3 & -\dfrac{\sigma_2}{\delta_3}p_1 \\ -\dfrac{\sigma_1}{\delta_2}p_2 & -\dfrac{\sigma_2}{\delta_2}p_2 \\ -\dfrac{\sigma_1}{\delta_1}p_1 & -\dfrac{\sigma_2}{\delta_1}p_3 \end{bmatrix}, \tag{42}$$

where $u = (p_1, p_2, p_3)^T$*. Consequently, the corresponding reduced matrix integrable mKdV equations are formulated as*

$$\begin{cases} p_{1,t} = -\frac{\beta}{\alpha^3} p_{1,xxx} + \frac{3\beta}{\alpha^2\delta_1\delta_2\delta_3} \big\{ \big[(2\sigma_1\delta_1 + \sigma_2\delta_1 + \sigma_1\delta_3)\delta_2 p_1 p_3 + \sigma_1\delta_1\delta_3 p_2^2 \big] p_{1,x} \\ \quad + \delta_1\delta_3(\sigma_1 p_1 + \sigma_2 p_3) p_2 p_{2,x} + \big[\delta_2(\sigma_1\delta_3 + \sigma_2\delta_1) p_1^2 + \sigma_2\delta_3(\delta_1 p_2^2 + 2\delta_2 p_3^2) \big] p_{3,x} \big\}, \\ p_{2,t} = -\frac{\beta}{\alpha^3} p_{2,xxx} + \frac{3\beta}{\alpha^2\delta_1\delta_2\delta_3} \big\{ \delta_1\delta_2(\sigma_2 p_1 + \sigma_1 p_3) p_2 p_{1,x} + \delta_2\delta_3(\sigma_1 p_1 + \sigma_2 p_3) p_2 p_{3,x} \\ \quad + \big[\sigma_2\delta_1\delta_2 p_1^2 + \sigma_1\delta_2(\delta_1 + \delta_3) p_1 p_3 + 2(\sigma_1 + \sigma_2)\delta_1\delta_3 p_2^2 + \sigma_2\delta_2\delta_3 p_3^2 \big] p_{2,x} \big\}, \\ p_{3,t} = -\frac{\beta}{\alpha^3} p_{3,xxx} + \frac{3\beta}{\alpha^2\delta_1\delta_2\delta_3} \big\{ \big[\sigma_2\delta_1(2\delta_2 p_1^2 + \delta_3 p_2^2) + \delta_2(\sigma_1\delta_1 + \sigma_2\delta_3) p_3^2 \big] p_{1,x} \\ \quad + \delta_1\delta_3(\sigma_2 p_1 + \sigma_1 p_3) p_2 p_{2,x} + \big[\delta_2(\sigma_1\delta_1 + 2\sigma_1\delta_3 + \sigma_2\delta_3) p_1 p_3 + \sigma_1\delta_1\delta_3 p_2^2 \big] p_{3,x} \big\}, \end{cases} \tag{43}$$

where $\sigma_1, \sigma_2, \delta_1, \delta_2$, and δ_3 are arbitrary non-zero constants.

In this example, introducing the spectral matrix into the system of equations reveals complex nonlinear interactions that determine the structure of the integrable mKdV equations. The parameters $\sigma_1, \sigma_2, \delta_1, \delta_2$, and δ_3 are crucial in influencing the system's dynamics and the way the components interact with each other, whereas the parameters α and β can be scaled out without affecting the essential structure.

The examples provided above highlight the flexibility and depth of the Lax pair formulation in the construction of integrable models. By applying different similarity transformations to the zero-curvature equations, we can generate a wide variety of integrable reductions (see, e.g., [30–33]). These transformations facilitate the investigation of various nonlinear wave phenomena, with potential applications spanning numerous fields of study. Furthermore, these examples contribute to the growing body of research on integrable models related to the 4×4 matrix spectral problems, as detailed in [34–38].

4. Concluding Remarks

This paper investigates a pair of identical group reductions or similarity transformations and applies them to the AKNS matrix spectral problems, resulting in reduced matrix mKdV integrable hierarchies. Five specific examples of reduced AKNS matrix spectral problems and their corresponding integrable mKdV models are provided. An important aspect of this work is the identification of the constraints arising from the two group reductions or similarity transformations. These reductions lead to novel mKdV integrable models and expand the framework of group reductions or similarity transformations established in earlier studies [9,39].

Our examples highlight the versatility of the Lax pair formulation in constructing integrable models, demonstrating how various group reductions or similarity transformations can produce a diverse range of integrable mKdV models, each characterized by unique nonlinear interactions. The selection of diagonal block matrices is crucial in shaping the structure of these systems. The flexibility of the Lax pair approach enables the creation of customized models, serving as a valuable tool for both theoretical investigations and practical applications.

This study serves as an introduction to the construction of integrable models within a more sophisticated framework. The next step in this exploration is to investigate intriguing solution phenomena such as rogue waves, lump waves, and soliton waves (see, e.g., [40–46]). The integrable models presented here offer valuable insight into classifying multi-component integrable systems within the Lax pair formulation. It is hoped that these models will find applications in various fields, including nonlinear optics, water waves, fluid dynamics, and plasma physics.

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