mathematics MDPI

Article

Soliton Solutions to Sasa–Satsuma-Type Modified Korteweg–De Vries Equations by Binary Darboux Transformations

Wen-Xiu Ma [1,2,3,4,5]

[1] School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China; mawx@cas.usf.edu
[2] Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[3] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
[4] School of Mathematics, South China University of Technology, Guangzhou 510640, China
[5] Material Science Innovation and Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa

Abstract: Sasa–Satsuma (SS)-type integrable matrix modified Korteweg–de Vries (mKdV) equations are derived from two group constraints, involving the replacement of the spectral matrix in the Ablowitz–Kaup–Newell–Segur matrix eigenproblems with its matrix transpose and its Hermitian transpose. Using the Lax pairs and dual Lax pairs of matrix eigenproblems as a foundation, binary Darboux transformations are constructed. These transformations, initiated with a zero seed solution, facilitate the generation of soliton solutions for the SS-type integrable matrix mKdV equations presented.

Keywords: soliton hierarchy; lax pair; group constraints; darboux transformations;soliton solutions

MSC: 35Q55; 37K15; 37K40

check for updates

Citation: Ma, W.-X. Soliton Solutions to Sasa–Satsuma-Type Modified Korteweg–De Vries Equations by Binary Darboux Transformations. *Mathematics* **2024**, *12*, 3643. https://doi.org/10.3390/math12233643

Academic Editor: Victor Orlov

Received: 19 October 2024
Revised: 8 November 2024
Accepted: 20 November 2024
Published: 21 November 2024

1. Introduction

Soliton theory represents a dynamic area within mathematical physics, offering diverse methods for solving nonlinear integrable equations. Key approaches include the Hirota direct approach, the inverse scattering technique, Darboux transformations (DT), the Riemann–Hilbert approach, Lie symmetry analysis, and Painlevé singularity confinement analysis [1–3]. Integral to these methods is the concept of an integrable equation, which is linked to two matrix eigenproblems known as a Lax pair. A binary Darboux transformation (DT) is derived from both a Lax pair and its adjoint, termed the dual Lax pair, which are equivalent representations of the same integrable equation [4,5]. Matrix eigenproblems play a crucial role not only in constructing DTs and binary DTs, but also in designing inverse scattering transforms and Riemann–Hilbert problems [1–3].

In the framework of (1+1)-dimensional integrable equations, where t and x are independent variables and $p = p(t, x)$ represents a column vector of unknown variables, spatial and temporal matrix eigenproblems are defined as follows:

$$-i\phi_x = \mathcal{E}\phi = \mathcal{E}(p,z)\phi, \tag{1}$$

and

$$-i\phi_t = \mathcal{F}\phi = \mathcal{F}(p,z)\phi, \tag{2}$$

where i denotes the imaginary unit, ϕ is a column eigenfunction, and $\mathcal{E}$ and $\mathcal{F}$ are square spectral matrices depending on p and the spectral parameter z. The consistency condition of these eigenproblems yields an integrable evolution equation

$$p_t = X(p), \tag{3}$$

through the flatness condition

$$\mathcal{E}_t - \mathcal{F}_x + i[\mathcal{E}, \mathcal{F}] = 0, \tag{4}$$

where $[\cdot, \cdot]$ denotes the matrix bracket. Flatness equations are known for their elegant algebraic structures, ensuring the existence of a limitless array of symmetries for the considered nonlinear integrable model [6]. Obviously, the dual Lax pair of the matrix eigenproblems, defined as

$$i\tilde{\phi}_x = \tilde{\phi}\mathcal{E} = \tilde{\phi}\mathcal{E}(p, z), \tag{5}$$

and

$$i\tilde{\phi}_t = \tilde{\phi}\mathcal{F} = \tilde{\phi}\mathcal{F}(p, z), \tag{6}$$

where $\tilde{\phi}$ is a row eigenfunction, results in an identical flatness condition (4) without generating any extra conditions. All of the Lax pair of eigenproblems and the dual Lax pair eigenproblems are utilized in our discussion to formulate binary DTs.

Based on a Lax pair and its adjoint in eigenproblems, a binary DT is constructed for an integrable equation under examination, expressed as

$$\phi' = D^+\phi = D^+(p, z)\phi, \; \tilde{\phi}' = \tilde{\phi}D^- = \tilde{\phi}D^-(p, z), \tag{7}$$

where $D^- = (D^+)^{-1}$. This transformation ensures that ϕ' and $\tilde{\phi}'$ solve new matrix eigenproblems as follows:

$$-i\phi'_x = \mathcal{E}'\phi', \; -i\phi'_t = \mathcal{F}'\phi', \tag{8}$$

and their adjoint counterparts as follows:

$$i\tilde{\phi}'_x = \tilde{\phi}'\mathcal{E}', \; i\tilde{\phi}'_t = \tilde{\phi}'\mathcal{F}', \tag{9}$$

in which the updated Lax pair matrices read as follows:

$$\mathcal{E}' = \mathcal{E}(p', z), \; \mathcal{F}' = \mathcal{F}(p', z), \tag{10}$$

and

$$p' = S(p), \tag{11}$$

defines a Bäcklund transformation. For the binary DT formulation, the Darboux matrices D^+ and D^- must satisfy

$$-iD^+_x D^- + D^+\mathcal{E}D^- = \mathcal{E}' = \mathcal{E}(S(p), z), \tag{12}$$

and

$$-iD^+_t D^- + D^+\mathcal{F}D^- = \mathcal{F}' = \mathcal{F}(S(p), z). \tag{13}$$

The Equations (8) and (9) ensure that the novel Lax pair, $\mathcal{E}'$ and $\mathcal{F}'$, satisfy the identical flatness condition (4), where p is replaced by p'. Moreover, p' represents a novel solution to the original integrable model if p does. Hence, (11) provides a Bäcklund transformation of the given integrable model. It is well documented that Darboux transformations and binary DTs have been extensively studied for single- and multi-component integrable equations [4,7–14]. However, research in non-commutative cases, including matrix integrable equations, remains relatively sparse despite notable contributions [15–17].

It is widely recognized that integrable reductions can generate reduced integrable equations under certain conditions, particularly when specific group constraints are applied to the spectral matrix $\mathcal{E}$. One common class of such reductions is expressed as follows:

$$(\mathcal{E}(t, x, z^*))^\dagger = \Pi\mathcal{E}(t, x, z)\Pi^{-1}, \tag{14}$$

where $\dagger$ denotes the Hermitian transpose, $*$ represents the complex conjugate, and Π is a constant Hermitian non-singular matrix (see, e.g., [18–20]). This reduction involves replacing the parameter z with z^* and is effective for either nonlinear Schrödinger (NLS) or modified Korteweg–de Vries (mKdV)-type integrable models. A second class of integrable reductions specific to mKdV-type equations involves replacing z with its negative counterpart $-z$ in given matrix eigenproblems as follows:

$$(\mathcal{E}(t,x,-z))^T = -\Pi\mathcal{E}(t,x,z)\Pi^{-1}, \tag{15}$$

where T denotes the transpose of a matrix and Π is a constant symmetric non-singular matrix. This reduction is effective for mKdV-type equations but is not applicable to NLS-type equations [20]. Other substitutions such as $z \to -z^*$ and $z \to z$ lead to nonlocal integrable reductions, often accompanied by reflections of the independent variables $t \to -t$, $x \to -x$, or $(t,x) \to (-t,-x)$. These transformations are notable for their role in generating distinct forms of integrable reductions across different classes of nonlinear equations.

In this paper, we aim to explore two kinds of integrable reductions, (14) and (15), simultaneously applied to construct Sasa–Satsuma (SS)-type integrable matrix mKdV equations. Our approach will utilize matrix Ablowitz–Kaup–Newell–Segur (AKNS) eigenproblems of an arbitrary order as the foundational framework. The primary objective is to derive binary DTs for the resultant integrable matrix mKdV models. These binary DTs will demonstrate an N-fold decomposition property, particularly when the eigenvalues and dual eigenvalues differ from each other in the regular case. Starting from the zero seed solution, the derived binary DTs will be employed to compute soliton solutions specific to the SS-type integrable matrix mKdV models. The paper will conclude with a summary in the Section 5, accompanied by pertinent concluding remarks on the significance and implications of the findings.

2. SS-Type Integrable Matrix mKdV Models

2.1. The Integrable Matrix AKNS Hierarchy Revisited

Let us revisit the procedure for constructing the hierarchy of integrable matrix AKNS equations. Assume $m,n \geq 1$ are two integers, z represents the spectral parameter, I_k denotes the $k \times k$ identity matrix ($k \in \mathbb{N}$), and r,s are two matrix potentials:

$$r = r(t,x) = (r_{jl})_{m\times n},\ s = s(t,x) = (s_{lj})_{n\times m}. \tag{16}$$

The local integrable matrix AKNS hierarchy originates from the matrix AKNS eigenproblems characterized by the matrix potentials

$$-i\phi_x = \mathcal{E}\phi = \mathcal{E}(p,z)\phi,\ -i\phi_t = \mathcal{F}^{[k]}\phi = \mathcal{F}^{[k]}(p,z)\phi,\ k \geq 0, \tag{17}$$

whose Lax pair reads

$$\mathcal{E} = z\Sigma + A(p),\ \mathcal{F}^{[k]} = z^k\Delta + B^{[k]}(p), \tag{18}$$

with Σ, Δ defined by

$$\Sigma = \mathrm{diag}(\gamma_1 I_m, \gamma_2 I_n), \tag{19}$$

$$\Delta = \mathrm{diag}(\delta_1 I_m, \delta_2 I_n). \tag{20}$$

Here, γ_1, γ_2, and δ_1, δ_2 are pairs of distinct arbitrary real numbers. In addition, the other two $m+n$th-order matrices are determined as follows:

$$A(p) = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix}, \tag{21}$$

referred to as the potential matrix, and

$$B^{[k]}(p) = \sum_{l=0}^{k-1} z^l \begin{bmatrix} e^{[s]} & f^{[s]} \\ g^{[s]} & h^{[s]} \end{bmatrix}, \tag{22}$$

where four sequences of differential polynomial matrices, $e^{[s]}, f^{[s]}, g^{[s]}$, and $h^{[s]}$, are determined in a recursive fashion by

$$f^{[0]} = 0,\ g^{[0]} = 0,\ e^{[0]} = \delta_1 I_m,\ h^{[0]} = \delta_2 I_n, \tag{23a}$$

$$f^{[l+1]} = \frac{1}{\gamma}(-ig_x^{[l]} - rh^{[l]} + e^{[l]}r),\ l \geq 0, \tag{23b}$$

$$g^{[l+1]} = \frac{1}{\gamma}(ig_x^{[k]} + se^{[l]} - h^{[l]}s),\ l \geq 0, \tag{23c}$$

$$e_x^{[l]} = i(rg^{[l]} - f^{[l]}s),\ h_x^{[l]} = i(sf^{[l]} - g^{[l]}r),\ l \geq 1, \tag{23d}$$

with $\gamma = \gamma_1 - \gamma_2$, $\delta = \delta_1 - \delta_2$. It is required that the constants of integration vanish, i.e., $e^{[l]}|_{r=0,s=0} = 0$, $h^{[l]}|_{r=0,s=0} = 0$, $l \geq 1$), in order to uniquely determine the differential polynomial matrices $B^{[l]}$, $l \geq 1$. For instance, using this approach, one can present

$$B^{[1]} = \frac{\delta}{\gamma}\begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix},\ B^{[2]} = \frac{\delta}{\gamma}z\begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix} - \frac{\delta}{\gamma^2}\begin{bmatrix} rs & ir_x \\ -is_x & -sr \end{bmatrix}, \tag{24}$$

and

$$B^{[3]} = \frac{\delta}{\gamma}z^2\begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix} - \frac{\delta}{\gamma^2}z\begin{bmatrix} rs & ir_x \\ -is_x & -sr \end{bmatrix} - \frac{\delta}{\gamma^3}\begin{bmatrix} i(rs_x - r_x s) & r_{xx} + 2rsr \\ s_{xx} + 2srs & i(sr_x - s_x r) \end{bmatrix}. \tag{25}$$

It is evident that for $m = 1$, the matrix eigenproblems described in (17) simplify to the case of multiple components. Specifically, when a set of potentials r_{1l} and s_{l1} are nonzero for $1 \leq l \leq n$, the matrix eigenproblems in (17) transform into the standard AKNS eigenproblems outlined in [21].

For any given pair of natural numbers m and n, the consistency conditions of the matrix eigenproblems defined in (17), represented by the flatness conditions

$$\mathcal{E}_t - \mathcal{F}_x^{[k]} + i[\mathcal{E}, \mathcal{F}^{[k]}] = 0,\ k \geq 0, \tag{26}$$

generate a local hierarchy of integrable matrix AKNS equations

$$r_t = i\gamma f^{[k+1]},\ s_t = -i\gamma g^{[k+1]},\ k \geq 0. \tag{27}$$

Based on the trace variational identity [22] and the Lax operator algebra formulation, it can be readily demonstrated that (27) forms a series of local commuting models. Every member in the hierarchy possesses a bi-Hamiltonian formulation, indicating the presence of infinitely many commuting conserved quantities.

2.2. AKNS Integrable Matrix mKdV Models

The AKNS integrable matrix mKdV equation models can be expressed as

$$r_t = -\frac{\delta}{\gamma^3}(r_{xxx} + 3rsr_x + 3r_x sr),\ s_t = -\frac{\delta}{\gamma^3}(s_{xxx} + 3s_x rs + 3srs_x), \tag{28}$$

in which the two matrix potentials, r and s, are given by (16). These equations correspond to the following spectral matrices:

$$\mathcal{E} = z\Sigma + A, \ \mathcal{F}^{[3]} = z^3\Delta + B^{[3]}, \tag{29}$$

where Σ and Δ are given by (19) and (20), A is defined by (21), and $B^{[3]}$ is expressed in terms of A as

$$B^{[3]} = B^{[3]}(p,z) = \frac{\delta}{\gamma}z^2A - \frac{\delta}{\gamma^2}zI_{m,n}(A^2 + iA_x) - \frac{\delta}{\gamma^3}(i[A, A_x] + A_{xx} + 2A^3), \tag{30}$$

with $I_{m,n} = \text{diag}(I_m, -I_n)$.

If $m = 1$ and $n = 1$, the AKNS integrable matrix mKdV equations reduce to

$$\begin{cases} r_{11,t} = r_{11,xxx} + 6r_{11}s_{11}r_{11,x}, \\ s_{11,t} = s_{11,xxx} + 6r_{11}s_{11}s_{11,x}. \end{cases} \tag{31}$$

For $m = 2$ and $n = 1$, the equations become

$$\begin{cases} r_{j1,t} = r_{j1,xxx} + 3(r_{11,x}s_{11} + r_{21,x}s_{12})r_{j1} + 3(r_{11}s_{11} + r_{21}s_{12})r_{j1,x}, \\ s_{1j,t} = s_{1j,xxx} + 3(r_{11}s_{11,x} + r_{21}s_{12,x})s_{1j} + 3(r_{11}s_{11} + r_{21}s_{12})s_{1j,x}, \end{cases} \tag{32}$$

with $1 \le j \le 2$. And for $m = 2$ and $n = 2$, the equations are

$$\begin{cases} r_{jl,t} = r_{jl,xxx} + 3\sum_{p,q=1}^{2} r_{jp}s_{pq}r_{ql,x} + 3\sum_{p,q=1}^{2} r_{jp,x}s_{pq}r_{ql}, \\ s_{lj,t} = s_{lj,xxx} + 3\sum_{p,q=1}^{2} s_{lp,x}r_{pq}s_{qj} + 3\sum_{p,q=1}^{2} s_{lp}r_{pq}s_{qj,x}, \end{cases} \tag{33}$$

with $1 \le j, l \le 2$. These equations are all Liouville integrable and exhibit symmetry under the reflection about the origin $(x \to -x, \ t \to -t)$.

2.3. SS-Type Integrable Matrix AKNS Equations

Let us now construct integrable reductions from the general integrable matrix AKNS Equation (27).

We choose two constant non-singular Hermitian matrices Θ_1, Θ_2 and another two constant non-singular symmetric matrices Ξ_1, Ξ_2, and we then define two local reductions for the initial spectral matrix $\mathcal{E}$ given in (29) as follows:

$$(\mathcal{E}(t,x,z^*))^\dagger = \Theta\mathcal{E}(t,x,z)\Theta^{-1}, \tag{34}$$

and

$$(\mathcal{E}(t,x,-z))^T = -\Xi\mathcal{E}(t,x,z)\Xi^{-1}, \tag{35}$$

in which two constant square matrices, Θ and Ξ, are determined by

$$\Theta = \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix}, \text{ where } \Theta_j^\dagger = \Theta_j, \ j = 1,2, \tag{36}$$

and

$$\Xi = \begin{bmatrix} \Xi_1 & 0 \\ 0 & \Xi_2 \end{bmatrix}, \text{ where } \Xi_j^T = \Xi_j, \ j = 1,2. \tag{37}$$

These two group constraints precisely require

$$A(t,x) = \Theta^{-1}A^\dagger(t,x)\Theta, \tag{38}$$

and

$$A(t,x) = -\Xi^{-1}A^T(t,x)\Xi, \tag{39}$$

which lead equivalently to the following potential reductions:

$$r(t,x) = \Theta_1^{-1}s^\dagger(t,x)\Theta_2, \tag{40}$$

and

$$r(t,x) = -\Xi_1^{-1}s^T(t,x)\Xi_2, \tag{41}$$

respectively. Therefore, we need a restriction for r as follows:

$$\Theta_2^{-1}r^\dagger(t,x)\Theta_1 = -\Xi_2^{-1}r^T(t,x)\Xi_1, \tag{42}$$

due to taking a pair of group constraints simultaneously.

We observe that the reductions in Equations (34) and (35) lead to the following transformations:

$$\begin{cases} (\mathcal{F}^{[2l+1]}(t,x,z^*))^\dagger = \Theta\mathcal{F}^{[2l+1]}(t,x,z)\Theta^{-1}, \\ (\mathcal{F}^{[2l+1]}(t,x,-z))^T = -\Xi\mathcal{F}^{[2l+1]}(t,x,z)\Xi^{-1}, \end{cases} \tag{43}$$

and

$$\begin{cases} (B^{[2l+1]}(t,x,z^*))^\dagger = \Theta B^{[2l+1]}(t,x,z)\Theta^{-1}, \\ (B^{[2l+1]}(t,x,-z))^T = -\Xi B^{[2l+1]}(t,x,z)\Xi^{-1}, \end{cases} \tag{44}$$

where $l \geq 0$. Here, $\mathcal{F}^{[2l+1]}$ is defined as in (18) and $B^{[2l+1]}$ is determined through (22).

As a result of reductions (40) and (41), the integrable matrix AKNS equations described in (27) with $k = 2l+1$, $l \geq 0$, lead to a series of SS-type integrable matrix AKNS equations:

$$r_t = i\gamma f^{[2l+2]}|_{s=\Theta_2^{-1}r^\dagger\Theta_1=-\Xi_2^{-1}r^T\Xi_1}, \; l \geq 0, \tag{45}$$

where $r = (r_{jl})_{m\times n}$ satisfies (42). Here, Θ_1 and Θ_2 are arbitrarily non-singular Hermitian matrices of sizes m and n, respectively, and Ξ_1 and Ξ_2 are arbitrarily non-singular symmetric matrices of sizes m and n. Each member in the hierarchy (45) possesses a Lax pair derived from the reduced matrix eigenproblems defined in (17) with $k = 2l+1$ $(l \geq 0)$. Furthermore, these equations exhibit infinitely many local commuting conserved quantities and symmetries, inherited from those of the integrable matrix AKNS equations defined in (27) with $k = 2l+1$ $(l \geq 0)$ under the two reductions.

2.4. SS-Type Integrable mKdV Equations

Let us focus on the case where $k = 3$, which corresponds to $l = 1$. In this context, the reduced integrable matrix AKNS equation from (45) specifically becomes

$$\begin{aligned} r_t &= -\frac{\delta}{\gamma^3}(r_{xxx} + 3r\Theta_2^{-1}r^\dagger\Theta_1 r_x + 3r_x\Theta_2^{-1}r^\dagger\Theta_1 r) \\ &= -\frac{\delta}{\gamma^3}(r_{xxx} - 3r\Xi_2^{-1}r^T\Xi_1 r_x - 3r_x\Theta_2^{-1}r^T\Xi_1 r), \end{aligned} \tag{46}$$

where the $m \times n$ matrix potential r needs to satisfy (42). This equation is known as the SS-type integrable matrix mKdV equation.

Based on the provided choices and calculations, let us illustrate the derivation and the resulting SS-type integrable mKdV equation.

If we take $m = 1$ and $n = 2$, and choose

$$\Theta_1 = 1, \; \Theta_2^{-1} = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, \; \Xi_1 = 1, \; \Xi_2^{-1} = \begin{bmatrix} 0 & \xi \\ \xi & 0 \end{bmatrix}, \tag{47}$$

with two real constants θ and ξ subject to $\theta^2 = \xi^2 = 1$. Then, the potential restriction (42) yields

$$r_2 = -\theta\xi r_1^*, \tag{48}$$

and, furthermore, the potential matrix A satisfying (42) becomes

$$A = \begin{bmatrix} 0 & r_1 & -\theta\xi r_1^* \\ \theta r_1^* & 0 & 0 \\ -\xi r_1 & 0 & 0 \end{bmatrix}. \tag{49}$$

Thus, the corresponding SS-type integrable mKdV equation reads

$$r_{1,t} = -\frac{\delta}{\gamma^3}[r_{1,xxx} + 6\theta|r_1|^2 r_{1,x} + 3\theta r_1(|r_1|^2)_x], \tag{50}$$

with $\theta = \pm 1$. Obviously, for $\theta = 1$, this equation precisely matches the SS mKdV equation, as referenced in the literature [23]. The equation allows for higher-order soliton solutions through its generalized Darboux transformation, as discussed in further studies [24].

If we still take $m = 1$ and $n = 2$, but choose

$$\Theta_1 = 1,\ \Theta_2^{-1} = \begin{bmatrix} 0 & \theta \\ \theta & 0 \end{bmatrix},\ \Xi_1 = 1,\ \Xi_2^{-1} = \begin{bmatrix} \xi & 0 \\ 0 & \xi \end{bmatrix}, \tag{51}$$

in which θ and ξ are two real constanst subject to the condition $\theta^2 = \xi^2 = 1$, then, the potential restriction (42) still leads to the relation (48), but the potential matrix A satisfying (42) becomes

$$A = \begin{bmatrix} 0 & r_1 & -\theta\xi r_1^* \\ -\xi r_1 & 0 & 0 \\ \theta r_1^* & 0 & 0 \end{bmatrix}. \tag{52}$$

Therefore, the corresponding SS-type integrable mKdV equation gives

$$r_{1,t} = -\frac{\delta}{\gamma^3}[r_{1,xxx} - 6\xi r_1^2 r_{1,x} - 3\xi r_1^*(|r_1|^2)_x], \tag{53}$$

with $\xi = \pm 1$. Clearly, these mathematical forms are different from the ones previously presented in (50).

Now, if we take $m = 1$ and $n = 4$, and choose

$$\Theta_1 = 1,\ \Theta_2^{-1} = \begin{bmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & \theta_1 & 0 & 0 \\ 0 & 0 & \theta_2 & 0 \\ 0 & 0 & 0 & \theta_2 \end{bmatrix},\ \Xi_1 = 1,\ \Xi_2^{-1} = \begin{bmatrix} 0 & \xi_1 & 0 & 0 \\ \xi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_2 \\ 0 & 0 & \xi_2 & 0 \end{bmatrix}, \tag{54}$$

in which θ_j and ξ_j are real constants subject to the condition $\theta_j^2 = \xi_j^2 = 1,\ j = 1,2$, then the potential restriction (42) engenders

$$r_2 = -\theta_1\xi_1 r_1^*,\ r_4 = -\theta_2\xi_2 r_3^*, \tag{55}$$

and, furthermore, the potential matrix A satisfying (42) becomes

$$A = \begin{bmatrix} 0 & r_1 & -\theta_1\xi_1 r_1^* & r_3 & -\theta_2\xi_2 r_3^* \\ \theta_1 r_1^* & 0 & 0 & 0 & 0 \\ -\xi_1 r_1 & 0 & 0 & 0 & 0 \\ \theta_2 r_3^* & 0 & 0 & 0 & 0 \\ -\xi_2 r_3 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (56)$$

This allows us to generate the following two-component SS-type integrable mKdV equation:

$$\begin{cases} r_{1,t} = -\dfrac{\delta}{\gamma^3}[r_{1,xxx} + 6(\theta_1|r_1|^2 + \theta_2|r_3|^2)r_{1,x} + 3(\theta_1|r_1|^2 + \theta_2|r_3|^2)_x r_1], \\ r_{3,t} = -\dfrac{\delta}{\gamma^3}[r_{3,xxx} + 6(\theta_1|r_1|^2 + \theta_2|r_3|^2)r_{3,x} + 3(\theta_1|r_1|^2 + \theta_2|r_3|^2)_x r_3], \end{cases} \quad (57)$$

with the real constants θ_j subject to $\theta_j^2 = 1,\ j = 1,2$. The long-time asymptotics of the equation associated with $\theta_j = 1,\ j = 1,2$ have been studied via the nonlinear Deift–Zhou steepest descent technique in [25,26].

Following the same approach as in the previous example, we can derive another two-component SS-type integrable mKdV equation as follows:

$$\begin{cases} r_{1,t} = -\dfrac{\delta}{\gamma^3}\{r_{1,xxx} - 3\xi_1[2r_1^2 r_{1,x} + r_1^*(|r_1|^2)_x] - 3\xi_2[r_3(r_1r_3)_x + r_3^*(r_1r_3^*)_x]\}, \\ r_{3,t} = -\dfrac{\delta}{\gamma^3}\{r_{3,xxx} - 3\xi_1[r_1(r_1r_3)_x + r_1^*(r_1^*r_3)_x] - 3\xi_2[2r_3^2 r_{3,x} + r_3^*(|r_3|^2)_x]\}, \end{cases} \quad (58)$$

with the real constants ξ_j subject to $\xi_j^2 = 1,\ j = 1,2$. This example is different from the previous one in (57).

Similarly, we can derive two N-component SS-type integrable mKdV equations. The equations for the three-component case are as follows:

$$\begin{cases} r_{1,t} = -\dfrac{\delta}{\gamma^3}[r_{1,xxx} + 6(\theta_1|r_1|^2 + \theta_2|r_3|^2 + \theta_3|r_5|^2)r_{1,x} + 3(\theta_1|r_1|^2 + \theta_2|r_3|^2 + \theta_3|r_5|^2)_x r_1], \\ r_{3,t} = -\dfrac{\delta}{\gamma^3}[r_{3,xxx} + 6(\theta_1|r_1|^2 + \theta_2|r_3|^2 + \theta_3|r_5|^2)r_{3,x} + 3(\theta_1|r_1|^2 + \theta_2|r_3|^2 + \theta_3|r_5|^2)_x r_3], \\ r_{5,t} = -\dfrac{\delta}{\gamma^3}[r_{5,xxx} + 6(\theta_1|r_1|^2 + \theta_2|r_3|^2 + \theta_3|r_5|^2)r_{5,x} + 3(\theta_1|r_1|^2 + \theta_2|r_3|^2 + \theta_3|r_5|^2)_x r_5], \end{cases} \quad (59)$$

with the real constants θ_j subject to $\theta_j^2 = 1,\ j = 1,2,3$, and

$$\begin{cases} r_{1,t} = -\dfrac{\delta}{\gamma^3}\{r_{1,xxx} - 3\xi_1[2r_1^2 r_{1,x} + r_1^*(r_1r_1^*)_x] - 3\xi_2[r_3(r_1r_3)_x + r_3^*(r_1r_3^*)_x] \\ \qquad -3\xi_3[r_5(r_1r_5)_x + r_5^*(r_1r_5^*)_x]\}, \\ r_{3,t} = -\dfrac{\delta}{\gamma^3}\{r_{3,xxx} - 3\xi_1[r_1(r_1r_3)_x + r_1^*(r_1^*r_3)_x] - 3\xi_2[2r_3^2 r_{3,x} + r_3^*(|r_3|^2)_x] \\ \qquad -3\xi_3[r_5(r_3r_5)_x + r_5^*(r_3r_5^*)_x]\}, \\ r_{5,t} = -\dfrac{\delta}{\gamma^3}\{r_{5,xxx} - 3\xi_1[r_1(r_1r_5)_x + r_1^*(r_1^*r_5)_x] - 3\xi_2[r_3(r_3r_5)_x + r_3^*(r_3^*r_5)_x] \\ \qquad -3\xi_3[2r_5^2 r_{5,x} + r_5^*(|r_5|^2)_x]\}, \end{cases} \quad (60)$$

with the real constants ξ_j subject to $\xi_j^2 = 1,\ j = 1,2,3,$ The equation defined by (59) with $\theta_j = 1,\ j = 1,2,3,$ has been investigated by the Riemann–Hilbert method in [27]. Additionally, the equations defined by (60) are entirely novel.

Some of the aforementioned examples have also been addressed and solved using corresponding Riemann–Hilbert problems in [28]. Certainly, there are numerous other intriguing examples, such as those discussed in [13,29], where different choices of non-singular Hermitian matrices Θ_1, Θ_2 and non-singular symmetric matrices Ξ_1, Ξ_2 were taken.

3. Binary Darboux Transformations

3.1. Distribution of Eigenvalues

Observe that the local reduction in (34) (or (35)) ensures that z is an eigenvalue of the matrix eigenproblems in (17) if $\hat{z} := z^*$ (or $\hat{z} := -z$) is a dual eigenvalue. Specifically, it solves the adjoint counterparts of the matrix eigenproblems:

$$i\tilde{\phi}_x = \mathcal{E}\tilde{\phi} = \mathcal{E}(p,\hat{z})\tilde{\phi},\ i\tilde{\phi}_t = \mathcal{F}^{[k]}\tilde{\phi} = \mathcal{F}^{[k]}(p,\hat{z})\tilde{\phi},$$

where $k = 2l+1$ with $l \geq 0$. Therefore, eigenvalues exhibit the following patterns: $z:\ \eta,\ -\eta^*,\ i\zeta,$ and dual eigenvalues have a corresponding pattern: $z^*:\ \eta^*,\ -\eta,\ -i\zeta$ (or $-z:\ -\eta,\ \eta^*,\ -i\zeta$), where $\eta \notin i\mathbb{R}$ and $\zeta \in \mathbb{R}$.

3.2. Darboux Matrices

We aim to establish a general framework for Darboux matrices, where the chosen eigenvalues and their adjoints are potentially equal to each other.

Assume that $N_1, N_2 \geq 0$ are two natural numbers such that $N = 2N_1 + N_2 \geq 1$. First, we take a set of eigenvalues as follows:

$$\{z_k \,|\, 1 \leq k \leq N\} = \{\eta_k,\ -\eta_k^*,\ 1 \leq k \leq N_1;\ i\zeta_k,\ 1 \leq k \leq N_2\} \tag{61}$$

and another set of dual eigenvalues as follows:

$$\{\hat{z}_k \,|\, 1 \leq k \leq N\} = \{\eta_k^*,\ -\eta_k,\ 1 \leq k \leq N_1;\ -i\zeta_k,\ 1 \leq k \leq N_2\}, \tag{62}$$

where $\eta_k \notin i\mathbb{R},\ 1 \leq k \leq N_1$ and $\zeta_k \in \mathbb{R},\ 1 \leq k \leq N_2$. Two groups of the corresponding eigenfunctions and dual eigenfunctions are determined by

$$-i\varphi_{k,x} = \mathcal{E}(r,s;z_k)\varphi_k,\ -i\varphi_{k,t} = \mathcal{F}^{[3]}(r,s;z_k)\varphi_k,\ 1 \leq k \leq N, \tag{63}$$

and

$$i\hat{\varphi}_{k,x} = \hat{\varphi}_k\mathcal{E}(r,s;\hat{z}_k),\ i\hat{\varphi}_{k,t} = \hat{\varphi}_k\mathcal{F}^{[3]}(r,s;\hat{z}_k),\ 1 \leq k \leq N. \tag{64}$$

For simplicity, we introduce

$$\varphi = (\varphi_1, \cdots, \varphi_N),\ \hat{\varphi} = (\hat{\varphi}_1^T, \cdots, \hat{\varphi}_N^T)^T, \tag{65}$$

and

$$\Lambda = \mathrm{diag}(z_1, \cdots, z_N),\ \hat{\Lambda} = \mathrm{diag}(\hat{z}_1, \cdots, \hat{z}_N). \tag{66}$$

Then, the equations for the eigenfunctions read

$$-i\varphi_x = \Sigma\varphi\Lambda + A\varphi,\ i\hat{\varphi}_x = \hat{\Lambda}\hat{\varphi}\Sigma + \hat{\varphi}A, \tag{67}$$

and

$$\begin{cases} -i\varphi_t = \Delta\varphi\Lambda^3 + (B^{[3]}(z_1)\varphi_1, \cdots, B^{[3]}(z_N)\varphi_N), \\ i\hat{\varphi}_t = \hat{\Lambda}^3\hat{\varphi}\Delta + (\hat{\varphi}_1 B^{[3]}(\hat{z}_1), \cdots, \hat{\varphi}_N B^{[3]}(\hat{z}_N)), \end{cases} \tag{68}$$

where the four square matrices $\Sigma, \Delta, A,$ and $B^{[3]}$ are given by (19), (20), (21), and (30), respectively.

To establish a general framework for Darboux matrices, where the chosen eigenvalues and their adjoints may coincide, we introduce an $N \times N$ matrix $\Omega = (\omega_{kl})_{N\times N}$, whose elements are given by

$$\omega_{kl} = \begin{cases} \dfrac{\hat{\varphi}_k \varphi_l}{z_l - \hat{z}_k}, & \text{if } z_l \neq \hat{z}_k, \\ \omega^c_{kl}(t,x), & \text{if } z_l = \hat{z}_k, \end{cases} \quad \text{where } 1 \le k,l \le N. \tag{69}$$

Here, the matrix Ω includes a novel type of elements ω^c_{kl} in the case of $z_l = \hat{z}_k$ for a pair $1 \le k,l \le N$, which will be specified later. This generalization extends beyond the traditional cases found in the literature (see, e.g., [3,30,31]). Such novel matrices Ω arise particularly in the formulation of soliton solutions for nonlocal integrable equations.

If Ω is non-singular, let us define the following two Darboux matrices:

$$\begin{cases} D^+ = D^+(z) = I_{m+n} - \displaystyle\sum_{k,l=1}^{N} \frac{\varphi_k (\Omega^{-1})_{kl} \hat{\varphi}_l}{z - \hat{z}_l}, \\ D^- = D^-(z) = I_{m+n} + \displaystyle\sum_{k,l=1}^{N} \frac{\varphi_k (\Omega^{-1})_{kl} \hat{\varphi}_l}{z - z_k}. \end{cases} \tag{70}$$

Through partial fractional decomposition, these two Darboux matrices can be expressed concisely. It is important to note that the partial fractional decomposition yields

$$D^+ = I_{m+n} - \sum_{l=1}^{N} \frac{\varphi^\Omega_l \hat{\varphi}_l}{z - \hat{z}_l}, \; D^- = I_{m+n} + \sum_{k=1}^{N} \frac{\varphi_k \hat{\varphi}^\Omega_k}{z - z_k}, \tag{71}$$

where one assumes

$$(\varphi^\Omega_1, \cdots, \varphi^\Omega_N) = (\varphi_1, \cdots, \varphi_N)\Omega^{-1}, \tag{72}$$

and

$$((\hat{\varphi}^\Omega_1)^T, \cdots, (\hat{\varphi}^\Omega_N)^T)^T = \Omega^{-1}(\hat{\varphi}^T_1, \cdots, \hat{\varphi}^T_N)^T. \tag{73}$$

Therefore, the two Darboux matrices can be expressed concisely as follows:

$$D^+ = I_{m+n} - \varphi\Omega^{-1}\hat{R}\hat{\varphi}, \; D^- = I_{m+n} + \varphi R \Omega^{-1}\hat{\varphi}, \tag{74}$$

where we denote

$$R = \begin{bmatrix} \frac{1}{z-z_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{z-z_N} \end{bmatrix}, \; \hat{R} = \begin{bmatrix} \frac{1}{z-\hat{z}_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{z-\hat{z}_N} \end{bmatrix}. \tag{75}$$

Now, we introduce

$$D^\pm_1 = \lim_{z\to\infty} [z(D^\pm(z) - I_{m+n})]. \tag{76}$$

It then follows that

$$D^+_1 = -\varphi\Omega^{-1}\hat{\varphi}, \; D^-_1 = \varphi\Omega^{-1}\hat{\varphi}, \tag{77}$$

which implies that

$$D^+_1 = -D^-_1.$$

Finally, we can derive two fundamental properties for the resulting Darboux matrices D^+ and D^- as follows:

(i) A spectral characteristic identity

$$\Big[\prod_{l=1}^{N}(z - \hat{z}_l) D^+(z)\Big]\Big|_{z=z_k} \varphi_k = 0, \; \hat{\varphi}_k \Big[\prod_{l=1}^{N}(z - z_l) D^-(z)\Big]\Big|_{z=\hat{z}_k} = 0, \; 1 \le k \le N, \tag{78}$$

is satisfied.

(ii) When an orthogonality

$$\hat{\varphi}_k \varphi_l = 0 \text{ when } z_l = \hat{z}_k, \text{ where } 1 \leq k, l \leq N, \tag{79}$$

holds, we achieve $\hat{R}\hat{\varphi}\varphi R = \Omega R - \hat{R}\Omega$, and so, D^+ and D^- solve

$$D^+(z)D^-(z) = I_{m+n}. \tag{80}$$

This implies that when all eigenvalues z_k, $1 \leq k \leq N$, are situated in the upper right quadrant of the complex plane, D^+ and D^- yield a solution to a reflectionless Riemann–Hilbert problem.

3.3. Binary Darboux Transformations

To construct binary DTs, it is essential to examine the dependence of the Ω-matrix on t and x. Note that the condition

$$\omega^c_{kl,x} = i\hat{\varphi}_k \frac{\mathcal{E}(r,s;z_l) - \mathcal{E}(r,s;\hat{z}_k)}{z_l - \hat{z}_k} \varphi_l = i\hat{\varphi}_k \Sigma \varphi_l \text{ when } z_l = \hat{z}_k, \tag{81}$$

with $1 \leq k, l \leq N$, guarantees the following spatial derivative formula:

$$\Omega_x = i\hat{\varphi}\Sigma\varphi; \tag{82}$$

and, moreover, the condition

$$\omega^c_{kl,t} = i\hat{\varphi}_k \frac{\mathcal{F}^{[3]}(r,s;z_l) - \mathcal{F}^{[3]}(r,s;\hat{z}_k)}{z_l - \hat{z}_k} \varphi_l = i\hat{\varphi}_k \Delta_{[k,l]} \varphi_l \text{ when } z_l = \hat{z}_k, \tag{83}$$

with $1 \leq k, l \leq N$, and $\Delta_{[k,l]}$ being worked out as follows:

$$\Delta_{[k,l]} = (\hat{z}_k^2 + \hat{z}_k z_l + z_l^2)\Delta + \frac{\delta}{\gamma}(\hat{z}_k + z_l)A - \frac{\delta}{\gamma^2} I_{m,n}(A^2 + iA_x), \ 1 \leq k, l \leq N, \tag{84}$$

which guarantees the following temporal derivative formula:

$$\Omega_t = i[\hat{\varphi}\hat{\Lambda}^2\Delta\varphi + \hat{\varphi}\hat{\Lambda}\Delta\Lambda\varphi + \hat{\varphi}\Delta\Lambda^2\varphi + \frac{\delta}{\gamma}(\hat{\varphi}\hat{\Lambda}A\varphi + \hat{\varphi}A\Lambda\varphi) - \frac{\delta}{\gamma^2}\hat{\varphi}I_{m,n}(A^2 + iA_x)\varphi]. \tag{85}$$

Given the defined pattern of eigenvalues and dual eigenvalues in (61) and (62), let us consider

$$\hat{\varphi}_k = \varphi_k^\dagger \Theta = \varphi^T_{N_1+k}\Xi, \ \hat{\varphi}_{N_1+k} = \varphi^\dagger_{N_1+k}\Theta = \varphi_k^T \Xi, \ 1 \leq k \leq N_1, \tag{86}$$

and

$$\hat{\varphi}_k = \varphi_k^\dagger \Theta = \varphi_k^T \Xi, \ 2N_1 + 1 \leq k \leq N. \tag{87}$$

In this way, it becomes evident that D_1^+ satisfies the required involution properties as follows:

$$(D_1^+(t,x))^\dagger = -\Theta D_1^+(t,x)\Theta^{-1}, \ (D_1^+(t,x))^T = \Xi D_1^+(t,x)\Xi^{-1}, \tag{88}$$

where Θ and Ξ are defined by (36) and (37), and $A' = [D_1^+, \Sigma]$ satisfies the two group constraints in (34) and (35).

All of the analyses above enable us to formulate a comprehensive framework for binary DTs as outlined below.

Theorem 1. *Let Σ, Δ, and $\Delta_{[k,l]}$ be given by (19), (20), and (84), and let the dual eigenvalues $\{\hat{z}_k | 1 \leq k \leq N\}$ be chosen as in (62) and the corresponding dual eigenfunctions $\{\hat{\varphi}_k | 1 \leq k \leq N\}$*

be specified through (86) and (87). Assume that $D^{\pm}$ and $D_1^{\pm}$ are presented by (70) and (76). Then, the conditions in (79), (81), and (83) guarantee a binary DT as follows:

$$\phi' = D^{+}\phi,\ \tilde{\phi}' = \tilde{\phi}D^{-}, \tag{89}$$

with a Bäcklund transformation

$$A' = A + [D_1^{+}, \Sigma], \tag{90}$$

for the SS-type integrable matrix mKdV equation (46).

We note that if we choose

$$\omega_{kl}^{c} = 0 \text{ when } z_l = \hat{z}_k, \tag{91}$$

the three conditions in (79), (81), and (83) can be satisfied by requiring

$$\varphi_k^{\dagger}\Theta\varphi_l = \varphi_k^{\dagger}\Theta\Sigma\varphi_l = \varphi_k^{\dagger}\Theta\Delta_{[k,l]}\varphi_l = 0 \text{ when } z_l = \hat{z}_k, \tag{92}$$

with $1 \leq k, l \leq N$. By utilizing (86) and (87), those orthogonal conditions can be expressed equivalently as

$$\varphi_{N_1+k}^{T}\Xi\varphi_l = \varphi_{N_1+k}^{T}\Xi\Sigma\varphi_l = \varphi_{N_1+k}^{T}\Xi\Delta_{[k,l]}\varphi_l = 0,\ \text{ when } z_l = \hat{z}_k,\ 1 \leq k \leq N_1, \tag{93}$$

$$\varphi_{k-N_1}^{T}\Xi\varphi_l = \varphi_{k-N_1}^{T}\Xi\Sigma\varphi_l = \varphi_{k-N_1}^{T}\Xi\Delta_{[k,l]}\varphi_l = 0,\ \text{ when } z_l = \hat{z}_k,\ N_1+1 \leq k \leq 2N_1, \tag{94}$$

and

$$\varphi_k^{T}\Xi\varphi_l = \varphi_k^{T}\Xi\Sigma\varphi_l = \varphi_k^{T}\Xi\Delta_{[k,l]}\varphi_l = 0,\ \text{ when } z_l = \hat{z}_k,\ 2N_1+1 \leq k \leq N, \tag{95}$$

with $1 \leq l \leq N$.

3.4. N-fold Decomposition Feature

Next, we would like to demonstrate an N-fold decomposition property for the established binary DT in the traditional case, focusing from the fact that the intersection of $\{z_k | 1 \leq k \leq N\}$ and $\{\hat{z}_k | 1 \leq k \leq N\}$ is empty.

To proceed, let us establish two groups of novel binary Darboux matrices using a single pair of eigenvalues and dual eigenvalues recursively as follows:

$$\begin{cases} D^{+}\{k\} = D^{+}\{k\}(z) = I_{m+n} - \dfrac{z_k - \hat{z}_k}{z - \hat{z}_k}\dfrac{\varphi_k'\hat{\varphi}_k'}{\hat{\varphi}_k'\varphi_k'},\ 1 \leq k \leq N, \\ D^{-}\{k\} = D^{-}\{k\}(z) = I_{m+n} + \dfrac{z_k - \hat{z}_k}{z - z_k}\dfrac{\varphi_k'\hat{\varphi}_k'}{\hat{\varphi}_k'\varphi_k'},\ 1 \leq k \leq N. \end{cases} \tag{96}$$

Here, N pairs of new eigenfunctions and dual eigenfunctions are given by

$$\varphi_k' = D^{+}[\![k-1]\!](z_k)\varphi_k,\ \hat{\varphi}_k' = \hat{\varphi}_k D^{-}[\![k-1]\!](\hat{z}_k),\ 1 \leq k \leq N, \tag{97}$$

with

$$\begin{cases} D^{+}[\![0]\!] = D^{-}[\![0]\!] = I_{m+n}, \\ D^{+}[\![k]\!] = D^{+}\{k\} \cdots D^{+}\{2\}D^{+}\{1\},\ 1 \leq k \leq N, \\ D^{-}[\![k]\!] = D^{-}\{1\}D^{-}\{2\} \cdots D^{-}\{k\},\ 1 \leq k \leq N. \end{cases} \tag{98}$$

At this moment, a straightforward computation can confirm the following N-fold decomposition:

$$D^{+} = D^{+}\{N\}D^{+}\{N-1\} \cdots D^{+}\{1\},\ D^{-} = D^{-}\{1\} \cdots D^{-}\{N-1\}D^{-}\{N\}, \tag{99}$$

where $D^{+}\{k\}$ and $D^{-}\{k\}$, $1 \leq k \leq N$, are defined by (96).

4. Soliton Solutions

We focus on the dual eigenvalues $\{\hat{z}_k | 1 \leq k \leq N\}$ defined previously as in (62). By considering the zero seed solution $r = 0$, which implies the zero potential matrix $A = 0$ due to the two group constraints, we can readily determine the corresponding eigenfunctions and dual eigenfunctions

$$\varphi_k(t,x) = \mathrm{e}^{iz_k\Sigma x + iz_k^3\Delta t}\chi_k,\ 1 \leq k \leq N, \tag{100}$$
$$\hat{\varphi}_k(t,x) = \chi_k^\dagger \mathrm{e}^{-i\hat{z}_k\Sigma x - i\hat{z}_k^3\Delta t}\Theta,\ 1 \leq k \leq N, \tag{101}$$

where χ_k, $1 \leq k \leq N$, denote constant column vectors. Those choices in (86) and (87) result in the conditions on the constant vectors χ_k, $1 \leq k \leq N$ as follows:

$$\begin{cases} \chi_k^T(\Xi\Theta^{-1} - \Theta^*\Xi^{*-1}) = 0,\ 1 \leq k \leq N_1, \\ \chi_k = \Xi^{-1}\Theta^*\chi_{k-N_1}^*,\ N_1 + 1 \leq k \leq 2N_1, \\ \chi_k^\dagger\Theta = \chi_k^T\Xi,\ 2N_1 + 1 \leq k \leq N, \end{cases} \tag{102}$$

where $*$ stands for the complex conjugate of a matrix. t is important to note that the purpose of all these conditions is to satisfy the group constraints in (38) and (39).

The three orthogonal conditions in (92) can be expressed as follows:

$$\chi_k^\dagger\Theta\chi_l = \chi_k^\dagger\Theta\Sigma\chi_l = (\hat{z}_k^2 + \hat{z}_k z_l + z_l^2)\chi_k^\dagger\Theta\Delta\chi_l = 0 \text{ when } z_l = \hat{z}_k, \tag{103}$$

with $1 \leq k, l \leq N$, and Σ, Δ, and Θ being defined by (19), (20), and (36), respectively.

It is worth noting that the case where $z_k = \hat{z}_k$ occurs only when $z_k = 0$ for $2N_1 + 1 \leq k \leq N$. Given that γ_1 and δ_1 are different from γ_2 and δ_2, respectively, the three conditions outlined in (103) equivalently yield

$$(\chi_k^1)^\dagger\Theta_1\chi_l^1 = 0,\ (\chi_k^2)^\dagger\Theta_2\chi_l^2 = 0 \text{ when } z_l = \hat{z}_k, \text{ where } 1 \leq k, l \leq N, \tag{104}$$

where we assume $\chi_k = ((\chi_k^1)^T, (\chi_k^2)^T)^T$. Here, χ_k^1 and χ_k^2 denote column vectors of dimensions m and n, respectively. These conditions ensure the orthogonality restrictions, which can also be represented using the non-singular matrix Ξ.

Now, following the binary DT theory presented in the previous theorem, we define a new potential matrix as follows:

$$A' = [D_1^+, \Sigma],\ D_1^+ = -\varphi\Omega^{-1}\hat{\varphi} = -\sum_{k,l=1}^{N} \varphi_k(\Omega^{-1})_{kl}\hat{\varphi}_l. \tag{105}$$

This formula leads to a class of soliton solutions for the following SS-type integrable matrix mKdV Equation (46):

$$r = \gamma \sum_{k,l=1}^{N} \varphi_k^1(\Omega^{-1})_{kl}\hat{\varphi}_l^2, \tag{106}$$

where we split $\varphi_k = ((\varphi_k^1)^T, (\varphi_k^2)^T)^T$, as we did for χ_k before, and $\hat{\varphi}_k = (\hat{\varphi}_k^1, \hat{\varphi}_k^2)$. Here, $\hat{\varphi}_k^1$ and $\hat{\varphi}_k^2$ denote row vectors of dimensions m and n, respectively.

Finally, we conclude that by selecting χ_k as in (102) and satisfying the conditions in (104), Formula (106), along with (69), (100), and (101), yields a matrix potential r satisfying (42). Consequently, these provide the required soliton solutions for the SS-type integrable matrix mKdV equations (46). Such soliton solutions complement those obtained in [32].

5. Concluding Remarks

The paper explores Sasa–Satsuma (SS)-type integrable matrix mKdV equations through two local group constraints applied to the matrix AKNS eigenvalue problem of general order. It establishes a general framework for binary Darboux transformations (DTs) applicable to the derived SS-type integrable matrix mKdV equations, leveraging the associated Lax pair and dual Lax pair of matrix eigenproblems. These binary DTs are then employed to formuate soliton solutions for the SS-type integrable matrix mKdV equations, expanding upon the binary DT theory developed for reduced integrable mKdV equations [32,33].

The crucial aspect of our analysis involves applying both local group constraints simultaneously to derive reduced integrable equations, which forms the foundation for the SS mKdV equation. In constructing binary DTs, we utilize a generalized Ω-matrix where adopted eigenvalues and dual eigenvalues can coincide. This extension of the Ω-matrix is inspired by a comprehensive exploration of Riemann–Hilbert problems in the context of nonlocal integrable equations. The framework for binary DTs presented here is applicable to both local and nonlocal integrable equations (see, for example, [34–37] for nonlocal cases). It is noteworthy that Darboux matrices involving higher order singularities can be generated by introducing repeated eigenvalues or dual eigenvalues, while generalized DTs can be constructed by differentiating with respect to eigenvalues or dual eigenvalues.

We emphasize that the discussed group symmetric reductions for matrix AKNS eigenproblems, involving transformations $z \to z^*$ and $z \to -z$, constitute two fundamental classes producing reduced local integrable equations. Exploring the simultaneous adoption of these two reductions for other matrix eigenproblems could yield diverse forms of reduced local integrable equations, which presents an intriguing avenue for future research. In the realm of DTs, there are numerous intriguing challenges. For instance, how can DTs be effectively employed to generate additional types of exact and explicit solutions such as for instance, breather and rogue wave solutions and lump wave solutions? Another important question is the formulation of binary DTs for reduced integrable couplings linked to non-semisimple Lie algebras. Furthermore, it is crucial to explore the connections between binary DT theories and other robust solution techniques, including the inverse scattering approach, the Hirota direct method, and the Riemann–Hilbert technique (see, e.g., [38–40]). Understanding these connections could lead to deeper insights and broader applications in the field of integrability research, particularly on multi-compoenent integrable models (see, e.g., [41,42]).

Funding: The work was supported in part by the Ministry of Science and Technology of China (G2021016032L and G2023016011L), and NSFC under the grants 12271488 and 11975145, and the Natural Science Foundation for Colleges and Universities in Jiangsu Province (17 KJB 110020).

Data Availability Statement: All data generated or analyzed during this study are included in this published article.

Acknowledgments: The author would also like to thank Alle Adjiri, Yushan Bai, Li Cheng, Jingwei He, Solomon Manukure, Morgan McAnally, and Yi Zhang for their valuable discussions.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Ablowitz, M.J.; Segur, H. *Solitons and the Inverse Scattering Transform*; SIAM: Philadelphia, PA, USA, 1981.
2. Calogero, F.; Degasperis, A. *Solitons and Spectral Transform I*; North-Holland: Amsterdam, The Netherlands, 1982.
3. Novikov, S.P.; Manakov, S.V.; Pitaevskii, L.P.; Zakharov, V.E. *Theory of Solitons: The Inverse Scattering Method*; Consultants Bureau: New York, NY, USA, 1984.
4. Matveev, V.B.; Salle, M.A. *Darboux Transformations and Solitons*; Springer: Berlin/Heidelberg, Germany, 1991.
5. Gu, C.H.; Hu, H.S.; Zhou, Z.X. *Darboux Transformations in Integrable Systems*; Springer: Dordrecht, The Netherlands, 2005.
6. Ma, W.X. An integrated integrable hierarchy arising from a broadened Ablowitz–Kaup–Newell–Segur scenario. *Axioms* **2024**, *13*, 563.
7. Mañas, M. Darboux transformations for the nonlinear Schrödinger equations. *J. Phys. A Math. Gen.* **1996**, *29*, 7721–7737.
8. Zeng, Y.B.; Ma, W.X.; Shao, Y.J. Two binary Darboux transformations for the KdV hierarchy with self-consistent sources. *J. Math. Phys.* **2001**, *42*, 2113–2128.
9. Doktorov, E.V.; Leble, S.B. *A Dressing Method in Mathematical Physics*; Springer: Dordrecht, The Netherlands, 2007.

10. Zhang, H.Q.; Wang, Y.; Ma, W.X. Binary Darboux transformation for the coupled SS equations. *Chaos* **2017**, *27*, 073102.
11. Zhang, Y.; Ye, R.S.; Ma, W.X. Binary Darboux transformation and soliton solutions for the coupled complex modified Korteweg-de Vries equations. *Math. Methods Appl. Sci.* **2020**, *43*, 613–627.
12. Xu, L.; Wang, D.S.; Wen, X.Y.; Jiang, Y.L. Exotic localized vector waves in a two-component nonlinear wave system. *J. Nonlinear Sci.* **2020**, *30*, 537–564.
13. Geng, X.G.; Li, Y.H.; Wei, J.; Zhai, Y.Y. Darboux transformation of a two-component generalized SS equation and explicit. *Math. Methods Appl. Sci.* **2021**, *44*, 12727–12745.
14. Ma, W.X. A novel kind of reduced integrable matrix mKdV equations and their binary Darboux transformations. *Mod. Phys. Lett. B*. **2022**, *36*, 2250094.
15. Degasperis, A.; Lombardo, S. Multicomponent integrable wave equations: I. Darboux-dressing transformation. *J. Phys. A Math Theor.* **2007**, *40*, 961–977.
16. Degasperis, A.; Lombardo, S. Multicomponent integrable wave equations: II. Soliton solutions. *J. Phys. A Math Theor.* **2009**, *42*, 385206.
17. Li, R.M.; Geng, X.G. A matrix Yajima-Oikawa long-wave-short-wave resonance equation, Darboux transformations and rogue wave solutions. *Commun. Nonlinear Sci. Numer. Simul.* **2020**, *90*, 105408.
18. Fordy, A.P.; Kulish, P.P. Nonlinear Schrödinger equations and simple Lie algebras. *Commun. Math. Phys.* **1983**, *89*, 427–443.
19. Gerdjikov, V.S.; Kaup, D.J.; Kostov, N.A.; Valchev, T.I. On classification of soliton solutions of multicomponent nonlinear evolution equations. *J. Phys. A Math. Theor.* **2008**, *41*, 315213.
20. Ma, W.X. Riemann-Hilbert problems and soliton solutions of a multicomponent mKdV system and its reduction. *Math. Meth. Appl. Sci.* **2019**, *42*, 1099–113.
21. Ablowitz, M.J.; Kaup, D.J.; Newell, A.C.; Segur, H. The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud. Appl. Math.* **1974**, *53*, 249–315.
22. Tu, G.Z. On Liouville integrability of zero-curvature equations and the Yang hierarchy. *J. Phys. A Math. Gen.* **1989**, *22*, 2375–2392.
23. Sasa, N.; Satsuma, J. New-type of soliton solutions for a higher-order nonlinear Schrödinger equation. *J. Phys. Soc. Jpn.* **1991**, *60*, 409–417.
24. Ling, L.M. The algebraic representation for high order solution of SS equation. *Discrete Contin. Dyn. Syst. Ser. S* **2016**, *9*, 1975–2010.
25. Geng, X.G.; Chen, M.M.; Wang, K.D. Application of the nonlinear steepest descent method to the coupled SS equation. *East Asian J. Appl. Math.* **2020**, *11*, 181–206.
26. Wang, X.B.; Han, B. The nonlinear steepest descent approach for long time behavior of the two-component coiupled SS equation with a 5×5 Lax pair. *Taiwan J. Math.* **2020**, *25*, 381–407.
27. Xu, S.Q.; Li, R.M.; Geng, X.G. Riemann-Hilbert method for the three-component SS equation and its N-soliton solutions. *Rep. Math. Phys.* **2020**, *85*, 77–103.
28. Ma, W.X. SS type matrix integrable hierarchies and their Riemann-Hilbert problems and soliton solutions. *Phys. D* **2023**, *446*, 133672.
29. Du, Z.; Zhao, X.J. Vector localized and periodic waves for the matrix Hirota equation with sign-alternating nonlinearity via the binary Darboux transformation. *Phys. Fluids* **2023**, *35*, 075108.
30. Kawata, T. Riemann spectral method for the nonlinear evolution equation. In *Advances in Nonlinear Waves*; Debnath, L., Ed.; Pitman: Boston, MA, USA, 1984; Volume I, pp. 210–225.
31. Yang, J. *Nonlinear Waves in Integrable and Nonintegrable Systems*; SIAM: Philadelphia, PA, USA, 2010.
32. Wu, J.P. Spectral and soliton structures of the Sasa–Satsuma higher-order nonlinear Schrödinger equation. *Anal. Math. Phys.* **2021**, *11*, 97.
33. Ma, W.X.; Huang, Y.H; Wang, F.D.; Zhang, Y.; Ding, L.Y. Binary Darboux transformation of vector nonlocal reverse-space nonlinear Schrödinger equations. *Int. J. Geom. Methods Mod. Phys.* **2024**, *21*, 2450182.
34. Ablowitz, M.J.; Musslimani, Z.H. Integrable nonlocal nonlinear Schrödinger equation. *Phys. Rev. Lett.* **2013**, *110*, 064105.
35. Gürses, M.; Pekcan, A. Nonlocal nonlinear Schrödinger equations and their soliton solutions. *J. Math. Phys.* **2018**, *59*, 051501.
36. Fokas, A.S. Integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation. *Nonlinearity* **2016**, *29*, 319–324.
37. Song, C.Q.; Xiao, D.M.; Zhu, Z.N. Solitons and dynamics for a general integrable nonlocal coupled nonlinear Schrödinger equation. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *45*, 13–28.
38. Ablowitz, M.J.; Musslimani, Z.H. Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation. *Nonlinearity* **2016**, *29*, 915–946.
39. Bai, Y.S.; Zhang, L.N.; Ma, W.X.; Yun, Y.S. Hirota bilinear approach to multi-component nonlocal nonlinear Schrödinger equations. *Mathematics* **2024**, *12*, 2594.
40. Hu, B.B.; Zhang, L.; Shen, Z.Y. Nonlocal combined nonlinear Schrödinger-Gerdjikov-Ivanov model: integrability, Riemann-Hilbert problem with simple and double poles, Cauchy problem with step-like initial data. *J. Math. Phys.* **2024**, *65*, 103501.
41. Ma, W.X. A combined derivative nonlinear Schrödinger soliton hierarchy. *Rep. Math. Phys.* **2024**, *93*, 313–325.
42. Ma, W.X. A combined generalized Kaup-Newell soliton hierarchy and its hereditary recursion operator and bi-Hamiltonian structure. *Theor. Math. Phys.* **2024**, *221*, 1603–1614.