Article

Lump Waves in a Spatial Symmetric Nonlinear Dispersive Wave Model in (2+1)-Dimensions

Wen-Xiu Ma [1,2,3,4]

1   Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China; mawx@cas.usf.edu
2   Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
3   Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
4   Material Science Innovation and Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa

Abstract: This paper aims to search for lump waves in a spatial symmetric (2+1)-dimensional dispersive wave model. Through an ansatz on positive quadratic functions, we conduct symbolic computations with Maple to generate lump waves for the proposed nonlinear model. A line of critical points of the lump waves is computed, whose two spatial coordinates travel at constant speeds. The corresponding maximum and minimum values are evaluated in terms of the wave numbers, and interestingly, all those extreme values do not change with time, either. The last section is the conclusion.

1. Introduction

Applied sciences rely heavily on closed-form mathematical theories. Much of such studies represent mathematical intuitions and skills of high order, challenging even for advanced researchers of today. A kind of multiple wave solution, called soliton solutions (first observed in water waves [1,2] and then in optical fibers [3]), are examples of closed-form solutions to integrable models of nonlinear dispersive waves. The nonlinearity and the dispersion play together in generating such nonlinear dispersive wave solutions.

In soliton theory, there are two powerful techniques, the inverse scattering transform [4] and the Hirota bilinear method [5], to soliton solutions. The inverse scattering transform was developed initially for solving Cauchy problems of nonlinear model equations, generated from Lax pairs of matrix spectral problems [6,7], It is a nonlinear version of the Fourier transform.

The Hirota bilinear method is the other direct but powerful technique to soliton waves. Hirota bilinear forms are the starting point to generate closed-form solutions [8,9]. In the (2+1)-dimensional case, take a polynomial R in time t and two space variables x, y. A (2+1)-dimensional Hirota bilinear differential equation is defined by

$$R(D_t, D_x, D_y)f \cdot f = 0, \tag{1}$$

where D_t, D_x and D_y are three Hirota bilinear derivatives given as follows [5]:

$$D_t^m D_x^n D_y^k f \cdot f = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^k f(t, x, y)f(t', x', y')\Big|_{t'=t, x'=x, y'=y}, \tag{2}$$

in which m, n, k are nonnegative integers. Associated with a Hirota bilinear equation, a nonlinear partial differential equation

$$X(u, u_t, u_x, u_y, \cdots) = 0 \tag{3}$$

with a dependent variable u is usually presented by the logarithmic derivative transformations

$$u = 2(\ln f)_{xx}, \ u = 2(\ln f)_{yy} \text{ or } u = 2(\ln f)_{xy}. \tag{4}$$

For multi-component integrable models (see, e.g., [10,11]), other kinds of transformations need to be introduced and implemented, and for non-integrable equations, the multiple exp-function method plays a similar role as the bilinear algorithm in exploring dispersive wave solutions (see, e.g., [12,13]). Within the Hirota bilinear theory, an N-soliton solution to a nonlinear equation can be presented by solving its corresponding Hirota bilinear equation (see, e.g., [8]).

Remarkably similar to solitons, lump waves (and rogue waves) are another kind of closed-form solution to nonlinear integrable models [14]. Lump waves are expressed in terms of analytic rational functions, which are localized in all directions in the spatial space (see, e.g., [14,15]):

$$\lim_{x^2+y^2 \to \infty, \ ax+by+c=0} u(x, y, t) = 0, \ t \in \mathbb{R}, \tag{5}$$

where a, b, c are arbitrary constants and $a^2 + b^2 \neq 0$. The KPI equation possesses abundant lump waves (see, e.g., [9]), and taking long wave limits of its soliton solutions can yield particular lump waves [16,17]. Many other integrable models possess lump waves, include the three-wave resonant interaction [18], the BKP equation [19], and the Ishimori-I equation [20]. Lump waves can exist in nonlinear nonintegrable models as well, and illustrative examples include generalized KP, BKP, Jimbo–Miwa and Bogoyavlensky–Konopelchenko equations [21–25]. There also exist lump waves in linear models in higher dimensions.

Quadratic functions are used to present exact solutions to Hirota bilinear equations and formulate lump wave solutions to nonlinear model equations [9,14]. The logarithmic derivative transformations are taken to link nonlinear model equations to bilinear equations. In this paper, we would like to search for lump waves in a spatial symmetric (2+1)-dimensional nonlinear dispersive wave model via such an ansatz using quadratic functions. The novelty is the consideration of a spatial symmetric nonlinearity with three nonlinear terms, which need to be balanced to create lump waves. We will conduct symbolic computations with Maple to determine its lump waves. Characteristic properties, such as critical points and extreme values, will be analyzed for the resulting lump waves. Concluding remarks will be given in the last section.

2. A Spatial Symmetric Nonlinear Model and Its Hirota Bilinear Form

Let α and β be real constants. We introduce a spatial symmetric (2+1)-dimensional nonlinear dispersive wave model equation:

$$\begin{aligned}
X(u) = {} & \alpha(3u_{xx}p_y + 3u_x p_{xy} + 3u_{xy}v + 3u_y v_x + u_{xxxy} + u_{tx} - u_{yy} \\
& + 3u_{yy}q_x + 3u_y q_{xy} + 3u_{xy}w + 3u_x w_y + u_{xyyy} + u_{ty} - u_{xx}) \\
& + \beta(4uu_{xy} + 5u_x u_y + u_{yy}v + u_{xx}w + v_x w_y + u_{xxyy}) = 0,
\end{aligned} \tag{6}$$

with $v_y = u_x, w_x = u_y, p_x = v, q_y = w$, and search for its lump waves via the indicated ansatz using quadratic functions. The example with $\alpha = 1$ and $\beta = 0$ of this nonlinear model gives the special spatial symmetric (2+1)-dimensional model equation

$$\begin{aligned}
& 3u_{xx}p_y + 3u_x p_{xy} + 3u_{xy}v + 3u_y v_x + u_{xxxy} + u_{tx} - u_{yy} \\
& + 3u_{yy}q_x + 3u_y q_{xy} + 3u_{xy}w + 3u_x w_y + u_{xyyy} + u_{ty} - u_{xx} = 0.
\end{aligned} \tag{7}$$

With the help of Maple, through the logarithmic derivative transformations

$$u = 2(\ln f)_{xy}, \ v = 2(\ln f)_{xx}, \ w = 2(\ln f)_{yy}, \ p = 2(\ln f)_x, \ q = 2(\ln f)_y, \tag{8}$$

the above spatial symmetric (2+1)-dimensional model Equation (6) is put into the Hirota bilinear form:

$$
\begin{aligned}
R(f) &= [\alpha(D_x^3 D_y + D_y^3 D_x + D_t D_x + D_t D_y - D_x^2 - D_y^2) + \beta D_x^2 D_y^2] f \cdot f \\
&= 2\alpha(f_{xxxy}f - 3f_{xxy}f_x + 3f_{xy}f_{xx} - f_y f_{xxx} + f_{tx}f - f_t f_x - f_{yy}f + f_y^2 \\
&\quad + f_{xyyy}f - 3f_{xyy}f_y + 3f_{xy}f_{yy} - f_x f_{yyy} + f_{ty}f - f_t f_y - f_{xx}f + f_x^2) \\
&\quad + 2\beta(f_{xxyy}f - 2f_{xxy}f_y - 2f_{xyy}f_x + f_{xx}f_{yy} + 2f_{xy}^2) = 0,
\end{aligned}
\tag{9}
$$

where D_t, D_x and D_y are the standard Hirota bilinear derivatives [5] (see also, (2)). By symbolic computation, a precise relation between the nonlinear model equation and the bilinear model equation can be explored to be

$$X(u) = \Big[\frac{R(f)}{f^2}\Big]_{xy}, \tag{10}$$

where the involved functions u, v, w, p, q are determined through the logarithmic derivative transformations of f in (8).

The same link also exists in a spatial symmetric KP model [26] and a spatial symmetric HSI model [27]. It is now evident that if f is a solution to the bilinear model Equation (9), then u, v, w, p, q determined by (8) solve the spatial symmetric (2+1)-dimensional dispersive wave model Equation (6). In the following section, we would like to look for a class of lump waves in this spatial symmetric nonlinear dispersive wave model.

3. Lump Wave Solutions

We would now like to compute lump wave solutions to the spatial symmetric (2+1)-dimensional dispersive wave model Equation (6), through conducting symbolic computations. A direct computation can show that the above general nonlinear model equation does not pass the three-soliton test and thus it doesn't possess an N-soliton solution.

Applying a general ansatz on lump waves in (2+1)-dimensions [9], we start looking for positive quadratic function solutions

$$f = \zeta_1^2 + \eta_2^2 + a_9, \ \zeta_1 = a_1 x + a_2 y + a_3 t + a_4, \ \zeta_2 = a_5 x + a_6 y + a_7 t + a_8, \tag{11}$$

to the corresponding Hirota bilinear Equation (9), and the task will be to determine the real constant parameters a_i, $1 \le i \le 9$ (see, e.g., [9,14] for illustrative examples). It is known that this is a general ansatz for lump wave solutions of lower order in (2+1)-dimensions [14].

We substitute f by (11) into the Hirota bilinear Equation (9) and obtain a system of algebraic equations on the involved parameters. A direct Maple computation to solve this system for a_3, a_7 and a_9 yields a set of solutions for the parameters:

$$
\begin{cases}
a_3 = \dfrac{(a_1 + a_2)[a_1^2 + a_2^2 + (a_5 + a_6)^2] - 2a_1 a_6^2 - 2a_2 a_5^2}{(a_1 + a_2)^2 + (a_5 + a_6)^2}, \\[2ex]
a_7 = \dfrac{(a_5 + a_6)[(a_1 + a_2)^2 + a_5^2 + a_6^2] - 2a_2^2 a_5 - 2a_1^2 a_6}{(a_1 + a_2)^2 + (a_5 + a_6)^2}, \\[2ex]
a_9 = \dfrac{3(a_1 a_2 + a_5 a_6)(a_1^2 + a_2^2 + a_5^2 + a_6^2)[(a_1 + a_2)^2 + (a_5 + a_6)^2]}{2(a_1 a_6 - a_2 a_5)^2} \\[2ex]
\quad + \dfrac{\beta[(a_1 a_2 - a_5 a_6)^2 + 2(a_1 a_2 + a_5 a_6)^2 + (a_1 a_6 + a_2 a_5)^2][(a_1 + a_2)^2 + (a_5 + a_6)^2]}{2\alpha(a_1 a_6 - a_2 a_5)^2},
\end{cases}
\tag{12}
$$

and all other parameters are arbitrary.

The above two frequency parameters, a_3 and a_7, exhibit a class of dispersion relations in (2+1)-dimensional nonlinear dispersive waves, and the constant term parameter, a_9, tells a complicated expression of the wave numbers, which is crucial in formulating lump waves within the Hirota bilinear theory. Interestingly, there also exists a kind of higher-order dispersion relations appearing in lump waves of the second model equation of the integrable KP hierarchy.

Let us analyze the analyticity of the lump waves by observing the above simplified expressions for the wave frequencies and the constant term in (12). Obviously, if

$$a_1 + a_2 = a_5 + a_6 = 0 \tag{13}$$

then

$$a_1 a_6 - a_2 a_5 = 0. \tag{14}$$

This implies that if $a_1 a_6 - a_2 a_5 \neq 0$, then we have

$$(a_1 + a_2)^2 + (a_5 + a_6)^2 > 0, \; a_1^2 + a_2^2 + a_5^2 + a_6^2 > 0.$$

Therefore, to generate lump wave solutions through the logarithmic derivative transformations, we require two basic conditions:

$$a_1 a_6 - a_2 a_5 \neq 0, \tag{15}$$

and

$$3(a_1 a_2 + a_5 a_6) + \frac{\beta[(a_1 a_2 - a_5 a_6)^2 + 2(a_1 a_2 + a_5 a_6)^2 + (a_1 a_6 + a_2 a_5)^2]}{\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)} > 0. \tag{16}$$

Those two necessary and sufficient conditions really guarantee the fundamental properties of lump waves. First, the resulting solutions of u, v, w are localized in all spatial directions, under (15). Second, they are analytic in the whole spatial and temporal space, under (15) and (16), which lead equivalently to that $a_9 > 0$. We will show in the next section that $a_9 > 0$ is also necessary for u, v, w to be analytic in $\mathbb{R}^3$.

The second condition defined by (16) contains the two coefficients, α and β. Clearly, if

$$a_1 a_2 + a_5 a_6 \geq 0, \; \alpha \beta \geq 0, \; (a_1 a_2 + a_5 a_6)^2 + (\alpha \beta)^2 > 0, \tag{17}$$

then we have $a_9 > 0$. Therefore, the nonlinearity affects the analyticity of the lump waves in the model Equation (6), but it does not affect the speeds of the two single waves in the lumps, in view of (12).

One reduced case can be worked out. When $\alpha = 1$ and $\beta = 0$, we obtain

$$a_9 = \frac{3(a_1 a_2 + a_5 a_6)(a_1^2 + a_2^2 + a_5^2 + a_6^2)[(a_1 + a_2)^2 + (a_5 + a_6)^2]}{2(a_1 a_6 - a_2 a_5)^2}. \tag{18}$$

Then, the conditions for the existence of lump waves in this reduced case simply become

$$(a_1 a_5 - a_2 a_5) \neq 0, \; a_1 a_2 + a_5 a_6 > 0. \tag{19}$$

We remark that the above ansatz on lump waves is increasingly being adopted in searching for exact and explicit solutions to nonlinear wave models [28–36].

4. Characteristics of the Lump Waves

In this section, we would like to consider the characteristic behaviors of the resultant lump waves presented previously.

4.1. Line of Critical Points

Let us first compute critical points of f defined by (11) as a function of x and y. To this end, we need to determine solutions to the system

$$f_x(x(t), y(t), t) = 0, \ f_y(x(t), y(t), t) = 0. \tag{20}$$

Since f is a quadratic polynomial in x and y, this system just requires

$$a_1\zeta_1 + a_5\zeta_2 = 0, \ a_2\zeta_1 + a_6\zeta_2 = 0.$$

Accordingly, based on the condition (15), we have $\zeta_1 = \zeta_2 = 0$, i.e.,

$$a_1x + a_2y + a_3t + a_4 = 0, \ a_5x + a_6y + a_7t + a_8 = 0. \tag{21}$$

This is a linear system of x and y, and all solutions are critical points of the quadratic function f:

$$\begin{cases} x(t) = -\dfrac{(a_1 + a_2)^2 + (a_5 + a_6)^2 - 2a_2^2 - 2a_6^2}{(a_1 + a_2)^2 + (a_5 + a_6)^2} t + \dfrac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \\[4mm] y(t) = \dfrac{(a_1 - a_2)^2 + (a_5 - a_6)^2 - 2a_2^2 - 2a_6^2}{(a_1 + a_2)^2 + (a_5 + a_6)^2} t - \dfrac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}, \end{cases} \tag{22}$$

at an arbitrary time t.

Evidently, those critical points form a straight line, whose two spatial coordinates travel at constant speeds. Now, a further straightforward computation can verify that all those points $(x(t), y(t))$ determined above are also critical points of the three solution functions u, v and w defined by (8).

4.2. Analyticity Condition

Taking advantage of (21), we see that the sum of two squares, i.e., the function $f - a_9 = \zeta_1^2 + \zeta_2^2$ becomes zero at all critical points defined by (22). Accordingly, the quadratic function $f > 0$ in $\mathbb{R}^3$ if and only if the constant term $a_9 > 0$. The sufficiency is clear, as analyzed earlier. The necessity is true, because we have that f vanishes at the critical points if $a_9 = 0$, and f vanishes at all points on the circle $\zeta_1^2 + \zeta_2^2 = -a_9$ if $a_9 < 0$.

Consequently, the three solutions u, v, w defined by (8) are analytic in $\mathbb{R}^3$ if and only if the constant parameter a_9 must be positive. Further, in view of the analysis of the positiveness of a_9 made in the previous section, the necessary and sufficient conditions for u, v, w to be analytic are the two conditions in (15) and (16) on the wave numbers a_1, a_2, a_5, a_6 and the coefficients α and β.

4.3. Extreme Values

Applying the second partial derivative test, we can see that the both lump waves, v and w, have a peak at the critical points $(x(t), y(t))$. This is because we have

$$\begin{cases} v_{xx} = -\dfrac{32\alpha^2(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)^4}{3[(a_1 + a_2)^2 + (a_5 + a_6)^2]^2 a_{10}^2} < 0, \\[4mm] v_{xx}v_{yy} - v_{xy}^2 = \dfrac{1024\alpha^4(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)^{10}}{27[(a_1 + a_2)^2 + (a_5 + a_6)^2]^4 a_{10}^4} > 0, \end{cases} \tag{23}$$

and

$$\begin{cases} w_{yy} = -\dfrac{32\alpha^2(a_2^2 + a_6^2)^2(a_1a_6 - a_2a_5)^4}{3[(a_1 + a_2)^2 + (a_5 + a_6)^2]^2 a_{10}^2} < 0, \\[4mm] w_{xx}w_{yy} - w_{xy}^2 = \dfrac{1024\alpha^4(a_2^2 + a_6^2)^2(a_1a_6 - a_2a_5)^{10}}{27[(a_1 + a_2)^2 + (a_5 + a_6)^2]^4 a_{10}^4} > 0, \end{cases} \tag{24}$$

where a_{10} is defined by

$$
\begin{aligned}
a_{10} = {} & \alpha(a_1 a_2 + a_5 a_6)(a_1^2 + a_2^2 + a_5^2 + a_6^2) \\
& + \frac{1}{3}\beta[(a_1 a_2 + a_5 a_6)^2 + (a_1 a_6 + a_2 a_5)^2 + 2a_1^2 a_2^2 + 2a_5^2 a_6^2].
\end{aligned} \tag{25}
$$

In a similar way, we can work out that

$$
\begin{cases}
u_{xx} = -\dfrac{32\alpha^2(a_1 a_2 + a_5 a_6)(a_1^2 + a_5^2)(a_1 a_6 - a_2 a_5)^4}{3[(a_1 + a_2)^2 + (a_5 + a_6)^2]^2 a_{10}^2}, \\[3ex]
u_{xx} u_{yy} - u_{xy}^2 \\[1ex]
\quad = \dfrac{1024\alpha^4 [3(a_1 a_2 + a_5 a_6)^2 - (a_1 a_6 - a_2 a_5)^2](a_1 a_6 - a_2 a_5)^{10}}{81[(a_1 + a_2)^2 + (a_5 + a_6)^2]^4 a_{10}^4},
\end{cases} \tag{26}
$$

where a_{10} is given by (25). Accordingly, the lump wave u has the maximum (or minimum) points $(x(t), y(t))$, when $a_1 a_2 + a_5 a_6 > 0$ (or $a_1 a_2 + a_5 a_6 < 0$) and

$$
3(a_1 a_2 + a_5 a_6)^2 - (a_1 a_6 - a_2 a_5)^2 > 0;
$$

the lump wave u has the saddle points $(x(t), y(t))$, when

$$
3(a_1 a_2 + a_5 a_6)^2 - (a_1 a_6 - a_2 a_5)^2 < 0;
$$

and the second partial derivative test is inconclusive, when

$$
3(a_1 a_2 + a_5 a_6)^2 - (a_1 a_6 - a_2 a_5)^2 = 0.
$$

A direct computation can generate the extreme values of v, w and u, achieved at the critical points $(x(t), y(t))$, as follows:

$$
v_{maximum} = \frac{8\alpha(a_1^2 + a_5^2)(a_1 a_6 - a_2 a_5)^2}{3[(a_1 + a_2)^2 + (a_5 + a_6)^2]a_{10}}, \tag{27}
$$

$$
w_{maximum} = \frac{8\alpha(a_2^2 + a_6^2)(a_1 a_6 - a_2 a_5)^2}{3[(a_1 + a_2)^2 + (a_5 + a_6)^2]a_{10}}, \tag{28}
$$

$$
u_{extremum} = \frac{8\alpha(a_1 a_2 + a_5 a_6)(a_1 a_6 - a_2 a_5)^2}{3[(a_1 + a_2)^2 + (a_5 + a_6)^2]a_{10}}, \tag{29}
$$

where a_{10} is defined by (25). Upon observing those expressions for the extreme values, we find that all extreme values do not depend on time t; they are all constants on the characteristic line of critical points (see also, [26,27] for other examples). Furthermore, when $a_1 a_6 - a_2 a_5$ goes to zero, i.e., the two spatial directions (a_1, a_2) and (a_5, a_6) tends to be parallel to each other, the lump waves of u, v, w may not decay in all cases of the wave numbers a_1, a_2, a_5 and a_6.

5. Conclusions

Through conducting symbolic computations with Maple, we have explored lump waves in a spatial symmetric (2+1)-dimensional dispersive wave model. The resulting lump waves have a line of critical points, whose spatial coordinates travel with constant velocities. The frequencies a_3, a_7 and the constant term a_9 of the lump waves were computed in terms of the wave numbers in the quadratic function f. Characteristic properties of the lump waves, such as critical points and extreme values, were worked out, and the effects of the nonlinear terms and the wave numbers were analyzed.

Interestingly, abundant lump waves also exist in linear wave model equations, besides nonlinear (2+1)-dimensional models (see, e.g., [37–39]) and (3+1)-dimension models

(see, e.g., [40–42]). The Hirota bilinear forms and the generalized bilinear forms are the starting points exhibiting a great convenience in determining lump waves. Interaction solutions between lump waves and other interesting waves, including homoclinic and heteroclinic solutions, can be explored for integrable and nonintegrable model equations (see, e.g., [43–48]). Connections with other solution approaches in soliton theory should deserve further investigation, which include Darboux transformations [49,50], the Wronskian technique [51], auto-Bäcklund transformations [52,53], the Riemann–Hilbert technique [54–56], symmetry reductions [57,58] and symmetry constraints [59].

It is also known that *N*-soliton solutions have been systematically studied by the Riemann–Hilbert technique for local and nonlocal integrable equations generated from group reductions of matrix spectral problems (see, e.g., [60–63]). It is intriguing to analyze the existence of lump waves in reduced integrable equations (see, e.g., [64–66]), both local and nonlocal. It is expected that studies of lump waves could advance our understanding of nonlinear wave phenomena and their integrability theory.

Data Availability Statement: All data generated or analyzed during this study are included in this published article.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Russell, J.S. Report on waves (Liverpool, 1837). In *Report of the 7th Meeting of the British Association for the Advancement of Science*; John Murray: London, UK, 1838; pp. 417–496.
2. Russell, J.S. Report on waves (York, 1844). In *Report of the 14th Meeting of the British Association for the Advancement of Science*; John Murray: London, UK, 1845; pp. 311–390.
3. Mollenauer, L.F.; Stolen, R.H.; Gordon, J.P. Experimental observation of picosecond pulse narrowing and solitons in optical fibers. *Phys. Rev. Lett.* **1980**, *45*, 1095–1098. [CrossRef]
4. Ablowitz, M.J.; Segur, H. *Solitons and the Inverse Scattering Transform*; SIAM: Philadelphia, PA, USA, 1981.
5. Hirota, R. *The Direct Method in Soliton Theory*; Cambridge University Press: New York, NY, USA, 2004.
6. Ablowitz, M.J.; Newell, A.C. The decay of the continuous spectrum for solutions of the Korteweg-de Vries equation. *J. Math. Phys.* **1973**, *14*, 1277–1284. [CrossRef]
7. Zakharov, V.E.; Shabat, A.B. A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I. *Funct. Anal. Appl.* **1974**, *8*, 226–235. [CrossRef]
8. Hirota, R. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* **1971**, *27*, 1192–1194. [CrossRef]
9. Ma, W.X. Lump solutions to the Kadomtsev-Petviashvili equation. *Phys. Lett. A* **2015**, *379*, 1975–1978. [CrossRef]
10. Ma, W.X. A Liouville integrable hierarchy with four potentials and its bi-Hamiltonian structure. *Rom. Rep. Phys.* **2023**, *75*, 115.
11. Ma, W.X. A six-component integrable hierarchy and its Hamiltonian formulation. *Mod. Phys. Lett. B* **2023**, *37*, 2350143. [CrossRef]
12. Ma, W.X.; Huang, T.W.; Zhang, Y. A multiple exp-function method for nonlinear differential equations and its application. *Phys. Scr.* **2010**, *82*, 065003. [CrossRef]
13. Song, Y.J.; Wang, Z.G.; Ma, Y.Z.; Yang, B. Solitary, kink and periodic wave solutions of the (3+1)-dimensional Hirota–Satsuma–Ito-like equation. *Results Phys.* **2022**, *42*, 106013. [CrossRef]
14. Ma, W.X.; Zhou, Y. Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. *J. Differ. Equ.* **2018**, *264*, 2633–2659. [CrossRef]
15. Tan, W.; Dai, H.P.; Dai, Z.D.; Zhong, W.Y. Emergence and space-time structure of lump solution to the (2+1)-dimensional generalized KP equation. *Pramana J. Phys.* **2017**, *89*, 77. [CrossRef]
16. Manakov, S.V.; Zakharov, V.E.; Bordag, L.A.; Matveev, V.B. Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction. *Phys. Lett. A* **1977**, *63*, 205–206. [CrossRef]
17. Satsuma, J.; Ablowitz, M.J. Two-dimensional lumps in nonlinear dispersive systems. *J. Math. Phys.* **1979**, *20*, 1496–1503. [CrossRef]
18. Kaup, D.J. The lump solutions and the Bäcklund transformation for the three-dimensional three-wave resonant interaction. *J. Math. Phys.* **1981**, *22*, 1176–1181. [CrossRef]
19. Gilson, C.R.; Nimmo, J.J.C. Lump solutions of the BKP equation. *Phys. Lett. A* **1990**, *147*, 472–476. [CrossRef]
20. Imai, K. Dromion and lump solutions of the Ishimori-I equation. *Prog. Theor. Phys.* **1997**, *98*, 1013–1023. [CrossRef]
21. Zhang, R.F.; Bilige, S.D. Bilinear neural network method to obtain the exact analytical solutions of nonlinear partial differential equations and its application to p-gBKP equation. *Nonlinear Dyn.* **2019**, *95*, 3041–3048. [CrossRef]

22. Sulaiman, T.A.; Yusuf, A.; Atangana, A. New lump, lump-kink, breather waves and other interaction solutions to the (3+1)-dimensional soliton equation. *Commun. Theor. Phys.* **2020**, *72*, 085004. [CrossRef]

23. Wang, X.M.; Bilige, S.D. Novel interaction phenomena of the (3+1)-dimensional Jimbo-Miwa equation. *Commun. Theor. Phys.* **2020**, *72*, 045001. [CrossRef]

24. Ma, H.C.; Yue, S.P.; Gao, Y.D.; Deng, A.P. Lump solution, breather soliton and more soliton solutions for a (2+1)-dimensional generalized Benjamin–Ono equation. *Qual. Theory Dyn. Syst. Qual. Theory Dyn. Syst.* **2023**, *22*, 72. [CrossRef]

25. Chen, S.J.; Lü, X.; Yin, Y.H. Dynamic behaviors of the lump solutions and mixed solutions to a (2+1)-dimensional nonlinear model. *Commun. Theor. Phys.* **2023**, *75*, 055005. [CrossRef]

26. Ma, W.X.; Batwa, S.; Manukure, S. Dispersion-managed lump waves in a spatial symmetric KP model. *East Asian J. Appl. Math.* **2023**, *13*, 246–256. [CrossRef]

27. Ma, W.X.; Bai, Y.S.; Adjiri, A. Nonlinearity-managed lump waves in a spatial symmetric HSI model. *Eur. Phys. J. Plus* **2021**, *136*, 240. [CrossRef]

28. Tang, Y.N.; Tao, S.Q.; Guan, Q. Lump solitons and the interaction phenomena of them for two classes of nonlinear evolution equations. *Comput. Math. Appl.* **2016**, *72*, 2334–2342. [CrossRef]

29. Wang, C.J. Lump solution and integrability for the associated Hirota bilinear equation. *Nonlinear Dyn.* **2017**, *87*, 2635–2642. [CrossRef]

30. Lü, J.Q.; Bilige, S.D. The study of lump solution and interaction phenomenon to (2+1)-dimensional potential Kadomste–Petviashvili equation. *Anal. Math. Phys.* **2019**, *9*, 1497–1509. [CrossRef]

31. Xu, H.N.; Ruan, W.R.; Zhang, Y.; Lü, X. Multi-exponential wave solutions to two extended Jimbo-Miwa equations and the resonance behavior. *Appl. Math. Lett.* **2020**, *99*, 105976. [CrossRef]

32. Zhou, X.J.; Ilhan, O.A.; Zhou, F.Y.; Sutarto, S.; Manafian, J.; Abotaleb, M. Lump and interaction solutions to the (3+1)-dimensional variable-coefficient nonlinear wave equation with multidimensional binary Bell polynomials. *J. Funct. Spaces* **2021**, *2021*, 4550582. [CrossRef]

33. Guan, X.; Liu, W.J. Multiple-soliton and lump-kink solutions for a generalized (3+1)-dimensional Kadomtsev-Petviashvili equation. *Results Phys.* **2020**, *17*, 103149. [CrossRef]

34. Rizvi, S.T.R.; Seadawy, A.R.; Ahmeda, S.; Younis, M.; Ali, K. Study of multiple lump and rogue waves to the generalized unstable space time fractional nonlinear Schrödinger equation. *Chaos Solitons Fractals* **2021**, *151*, 111251. [CrossRef]

35. Wang, Y.H. Nonautonomous lump solutions for a variable–coefficient Kadomtsev–Petviashvili equation. *Appl. Math. Lett.* **2021**, *119*, 107201. [CrossRef]

36. Yusuf, A.; Sulaiman, T.A.; Abdeljabbar, A.; Alquran, M. Breather waves, analytical solutions and conservation laws using Lie–Bäcklund symmetries to the (2+1)-dimensional Chaffee–Infante equation. *J. Ocean Eng. Sci.* **2023**, *8*, 145–151. [CrossRef]

37. Wang, H. Lump and interaction solutions to the (2+1)-dimensional Burgers equation. *Appl. Math. Lett.* **2018**, *85*, 27–34. [CrossRef]

38. Yu, J.P.; Sun, Y.L. Study of lump solutions to dimensionally reduced generalized KP equations. *Nonlinear Dyn.* **2017**, *87*, 2755–2763. [CrossRef]

39. Zhao, Z.L.; He, L.C.; Gao, Y.B. Rogue wave and multiple lump solutions of the (2+1)-dimensional Benjamin-Ono equation in fluid mechanics. *Complexity* **2019**, *2019*, 8249635. [CrossRef]

40. Roshid, H.O. Lump solutions to a (3+1)-dimensional potential-Yu–Toda–Sasa–Fukuyama (YTSF) like equation. *Int. J. Appl. Comput. Math.* **2017**, *3* (Suppl. 1), S1455–S1461. [CrossRef]

41. Sun, Y.; Tian, B.; Xie, X.Y.; Chai, J.; Yin, H.M. Rogue waves and lump solitons for a (3+1)-dimensional B-type Kadomtsev-Petviashvili equation in fluid dynamics. *Wave Random Complex.* **2018**, *28*, 544–552. [CrossRef]

42. Zhang, Y.; Liu, Y.P.; Tang, X.Y. *M*-lump solutions to a (3+1)-dimensional nonlinear evolution equation. *Comput. Math. Appl.* **2018**, *76*, 592–601. [CrossRef]

43. Kofane, T.C.; Fokou, M.; Mohamadou, A.; Yomba, E. Lump solutions and interaction phenomenon to the third-order nonlinear evolution equation. *Eur. Phys. J. Plus* **2017**, *132*, 465. [CrossRef]

44. Wang, H.; Tian, S.F.; Zhang, T.T.; Chen, Y. The breather wave solutions, *M*-lump solutions and semi-rational solutions to a (2+1)-dimensional generalized Korteweg-de Vries equation. *J. Appl. Anal. Comput.* **2020**, *10*, 118–130. [CrossRef]

45. Guo, Y.F.; Dai, Z.D.; Guo, C.X. Lump solutions and interaction solutions for (2+1)-dimensional KPI equation. *Front. Math. China* **2022**, *17*, 875–886. [CrossRef]

46. Chen, W.X.; Guan, R.; Dong, M.J.; Tian, L.X.; Ma, J.J. *N*-solitons, lump solution and interaction phenomenon to the Boussinesq equation. *Int. J. Comput. Math.* **2022**, *99*, 2237–2249. [CrossRef]

47. Batool, T.; Seadawy, A.R.; Rizvi, S.T.R. Multiple lump solutions and their interactionsfor an integrable nonlinear dispersionless PDE in vector fields. *Nonlinear Anal. Model. Control* **2023**, *28*, 264–287.

48. Raut, S.; Barman, R.; Sarkar, T. Integrability, breather, lump and quasi-periodic waves of non-autonomous Kadomtsev–Petviashvili equation based on Bell-polynomial approach. *Wave Motion* **2023**, *119*, 103125. [CrossRef]

49. Xu, X.X. A deformed reduced semi-discrete Kaup-Newell equation, the related integrable family and Darboux transformation. *Appl. Math. Comput.* **2015**, *251*, 275–283. [CrossRef]

50. Han, G.F.; Li, X.Y.; Zhao, Q.L.; Li, C.Z. Interaction structures of multi localized waves within the Kadomtsev–Petviashvili I equation. *Phys. D* **2023**, *446*, 133671. [CrossRef]

51. Wu, J.P.; Geng, X.G. Novel Wronskian condition and new exact solutions to a (3+1)-dimensional generalized KP equation. *Commun. Theor. Phys.* **2013**, *60*, 556–560. [CrossRef]

52. Kaplan, M.; Ozer, M.N. Auto-Bäcklund transformations and solitary wave solutions for the nonlinear evolution equation. *Opt. Quantum Electron.* **2018**, *50*, 33. [CrossRef]

53. Raza, N.; Kaplan, M.; Javid, A.; Inc, M. Complexiton and resonant multi-solitons of a (4+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. *Opt. Quantum Electron.* **2022**, *54*, 95. [CrossRef]

54. Novikov, S.; Manakov, S.V.; Pitaevskii, L.P.; Zakharov, V.E. *Theory of Solitons—The Inverse Scattering Method*; Consultants Bureau: New York, NY, USA, 1984.

55. Ablowitz, M.J.; Fokas, A.S. *Complex Variables: Introduction and Applications*; Cambridge University Press: Cambridge, UK, 1997.

56. Its, A.R. The Riemann-Hilbert problem and integrable systems. *Not. Am. Math. Soc.* **2003**, *50*, 1389–1400.

57. Dorizzi, B.; Grammaticos, B.; Ramani, A.; Winternitz, P. Are all the equations of the Kadomtsev-Petviashvili hierarchy integrable? *J. Math. Phys.* **1986**, *27*, 2848–2852. [CrossRef]

58. Wang, D.S.; Yin, Y.B. Symmetry analysis and reductions of the two-dimensional generalized Benney system via geometric approach. *Comput. Math. Appl.* **2016**, *71*, 748–757. [CrossRef]

59. Konopelchenko, B.; Strampp, W. The AKNS hierarchy as symmetry constraint of the KP hierarchy. *Inverse Probl.* **1991**, *7*, L17–L24. [CrossRef]

60. Ma, W.X. Soliton solutions to constrained nonlocal integrable nonlinear Schrödinger hierarchies of type $(-\lambda, \lambda)$. *Int. J. Geom. Methods Mod. Phys.* **2023**, *20*, 2350098. [CrossRef]

61. Ma, W.X. Integrable nonlocal nonlinear Schrödinger hierarchies of type $(-\lambda^*, \lambda)$ and soliton solutions. *Rep. Math. Phys.* **2023**, *92*, 19–36. [CrossRef]

62. Ye, R.S.; Zhang, Y. General soliton solutions to a reverse-time nonlocal nonlinear Schrödinger equation. *Stud. Appl. Math.* **2020**, *145*, 197–216. [CrossRef]

63. Li, J.; Xia, T.C.; Guo, H.D. Long-time asymptotics for the nonlocal Kundu-nonlinear-Schrödinger equation by the nonlinear steepest descent method. *Theor. Math. Phys.* **2022**, *213*, 1706–1726. [CrossRef]

64. Ma, W.X. AKNS type reduced integrable bi-Hamiltonian hierarchies with four potentials. *Appl. Math. Lett.* **2023**, *145*, 108775. [CrossRef]

65. Ma, W.X. Four-component integrable hierarchies of Hamiltonian equations with $(m + n + 2)$th-order Lax pairs. *Theor. Math. Phys.* **2023**, *216*, 1180–1188. [CrossRef]

66. Ma, W.X. Novel Liouville integrable Hamiltonian models with six components and three signs. *Chin. J. Phys.* **2023**, *86*, 292–299. [CrossRef]