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An application of dual group reductions to the AKNS integrable mKdV model

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This work aims to explore coupled integrable modified Kortewegde Vries (mKdV) models through the framework of the Ablowitz-Kaup-Newell-Segur (AKNS) matrix spectral problem. By proposing and applying a set of dual group reductions, we derive reduced matrix integrable mKdV models. The results yield new examples of integrable models and offer valuable insights into the classification of mKdV type integrable models.

Keywords: Lax pair; zero-curvature equation; group reduction; integrable mKdV equations.

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1. Introduction

A central challenge in soliton theory is the construction and classification of integrable models. These models are characterized by rich mathematical structures,[1,2] including Lax pairs, infinite hierarchies of symmetries, conserved quantities, and exact solutions such as solitons, breathers, and rogue waves. The development of integrable models deepens our understanding of nonlinear wave phenomena and provides powerful tools for solving physically relevant partial differential equations. Techniques such as inverse scattering, bilinear transformations, and group reductions are frequently employed to uncover or construct integrable structures.[3–11] In particular, matrix generalizations and nonlocal extensions of classical integrable models have recently attracted significant attention (see, e.g. Refs. 12–15), opening new avenues for both theoretical exploration and practical application in fields such as optics, fluid dynamics, and quantum physics. Within the Lax pair formulation, integrable models arise from the compatibility condition of two matrix spectral problems:

$$-i\phi_x = U(u,\lambda)\phi, \quad -i\phi_t = V(u,\lambda)\phi, \tag{1.1}$$

where u denotes the dependent variable and λ is the spectral parameter. The associated zero curvature equation,

$$U_t - V_x + i[U,V] = 0, \tag{1.2}$$

where $[A,B]$ denotes the commutator, leads to an integrable model.[16] To construct new reduced integrable models, we consider a dual group reduction imposed on the Lax pair:

$$\Sigma U(\lambda)\Sigma^{-1} = -U^T(-\lambda), \quad \Delta U(\lambda)\Delta^{-1} = U(\lambda), \tag{1.3}$$

and

$$\Sigma V(\lambda)\Sigma^{-1} = -V^T(-\lambda), \quad \Delta V(\lambda)\Delta^{-1} = V(\lambda), \tag{1.4}$$

where A^{-1} and A^T denote the inverse and transpose of the matrix A, respectively. It is notably challenging to formulate such a dual group reduction for a given Lax pair.[17–19] These reductions preserve the form of the zero-curvature condition:

$$\Sigma(U_t - V_x + i[U,V])(\lambda)\Sigma^{-1} = -(U_t - V_x + i[U,V])^T(-\lambda), \tag{1.5}$$

and

$$\Delta(U_t - V_x + i[U,V])(\lambda)\Delta^{-1} = (U_t - V_x + i[U,V])(\lambda). \tag{1.6}$$

Thus, we obtain a reduced integrable model governed by the same zero-curvature equation but subject to the symmetry constraints defined by (1.3) and (1.4). Once the constant matrices Σ and Δ are specified, the resulting reductions generate novel integrable models. While the general AKNS matrix spectral problems with free potentials are standard and widely applicable, the reduced spectral problems are more constrained and technically more demanding. Nonetheless, dual group reductions offer a systematic route to discovering diverse integrable structures.

As a concrete example, we derive the matrix spectral problem given by

$$U = U = U(u,\lambda) = \begin{bmatrix} \frac{1}{2}\lambda & 0 & p_1 & p_2 & p_3 \\ 0 & \frac{1}{2}\lambda & p_3 & p_2 & p_1 \\ p_3 & p_1 & -\frac{1}{2}\lambda & 0 & 0 \\ p_2 & p_2 & 0 & -\frac{1}{2}\lambda & 0 \\ p_1 & p_3 & 0 & 0 & -\frac{1}{2}\lambda \end{bmatrix}, \tag{1.7}$$

where $u = (p_1, p_2, p_3)^T$. The associated coupled integrable mKdV model takes the form:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 3[(4p_1p_3 + p_2^2)p_{1,x} + p_2(p_1 + p_3)p_{2,x} + (2p_1^2 + p_2^2 + 2p_3^2)p_{3,x}], \\ p_{2,t} = p_{2,xxx} + 3\{p_2(p_1 + p_3)(p_{1,x} + p_{3,x}) + [(p_1 + p_3)^2 + 4p_2^2]p_{2,x}\}, \\ p_{3,t} = p_{3,xxx} + 3[(2p_1^2 + p_2^2 + 2p_3^2)p_{1,x} + p_2(p_1 + p_3)p_{2,x} + (4p_1p_3 + p_2^2)p_{3,x}]. \end{cases} \tag{1.8}$$

In this paper, we analyze a general framework for generating reduced matrix integrable mKdV models using dual group reductions applied to the AKNS matrix integrable mKdV models. The structure of the remainder of the paper is as follows. In Sec. 2, we review the AKNS matrix spectral problems and their associated integrable hierarchies, and in Sec. 3, outline the procedure of performing dual group reductions. In Sec. 4, we present a concrete example of dual group reductions and derive the corresponding reduced matrix integrable mKdV models, including two specific examples of reduced spectral matrices and the resulting integrable models. Finally, in the concluding section, we summarize the results and provide further remarks.

2. Revisiting the Standard AKNS Matrix Integrable Hierarchies

Within the AKNS framework for integrable models (see, e.g. Ref. 20), we denote the dependent variable as $u = u(p, q)$, where $p = p(x, t) = (p_{jk})_{m\times n}$ and $q = q(x, t) = (q_{kj})_{n\times m}$ are two matrix-valued potentials. The AKNS matrix spectral problems involve the Lax pair:

$$-i\phi_x = U\phi, \quad -i\phi_t = V^{[r]}\phi, \tag{2.1}$$

where the spectral matrix and the Lax matrix are defined as

$$U = U(u, \lambda) = \lambda\Lambda + P, \quad V^{[r]} = V^{[r]}(u, \lambda) = \lambda^r\Omega + Q^{[r]}, \tag{2.2}$$

respectively. Here, the matrices Λ, P, Ω, and $Q^{[r]}$ are given by

$$\Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix}, \quad P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.3}$$

and

$$\Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix}, \quad Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \tag{2.4}$$

where λ is the spectral parameter, I_k denotes the $k \times k$ identity matrix, and $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$ are constants. By convention, $Q^{[0]}$ is the zero matrix. The stationary zero-curvature equation,

$$W_x = i[U, W], \tag{2.5}$$

has a unique solution given by the Laurent expansion

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]} = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}, \tag{2.6}$$

with the initial condition $W^{[0]} = \Omega$. This expansion generates a hierarchy of commuting integrable systems (see, e.g. Refs. 21–24).

The compatibility condition of the Lax pair (2.1), i.e. the zero-curvature equation

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \tag{2.7}$$

leads to the matrix AKNS integrable hierarchy:

$$p_t = i\alpha b^{[r+1]}, \quad q_t = -i\alpha c^{[r+1]}, \quad \alpha = \alpha_1 - \alpha_2. \tag{2.8}$$

When $m = n = 1$, this reduces to the scalar AKNS system.[25] Every model in this hierarchy possesses a bi-Hamiltonian structure along with infinite symmetries and conserved quantities (see, e.g. Refs. 26–28).

For $r = 2s + 1,\ s \geq 1$, the hierarchy (2.8) yields a matrix mKdV integrable hierarchy. In particular, for $s = 1$, the Lax matrix $V^{[3]}$ takes the form

$$V^{[3]} = \lambda^3 \Omega + \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3), \tag{2.9}$$

where $I_{m,n} = \mathrm{diag}(I_m, -I_n)$ and $\beta = \beta_1 - \beta_2$. This leads to the matrix mKdV integrable model:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_x qp), \quad q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_x pq + 3qpq_x), \tag{2.10}$$

which serve as a foundational example in the study of matrix mKdV integrable models. Higher-order generalizations can be systematically derived (see, e.g. Ref. 29).

3. The Procedure of Dual Group Reductions

3.1. *Dual group reductions of the AKNS matrix spectral problems*

Let Σ_1 and Σ_2 be two constant, invertible, symmetric matrices of orders m and n, respectively. Similarly, let Δ_1 and Δ_2 be constant invertible matrices of the same respective orders. Using these, define two block-diagonal matrices of order $m + n$ by

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{3.1}$$

We consider two simultaneous reductions applied to the AKNS spectral matrix U defined in (2.2):

$$\Sigma U(\lambda)\Sigma^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T, \quad \Delta U(\lambda)\Delta^{-1} = U(\lambda), \tag{3.2}$$

where A^{-1} and A^T again denote the inverse and transpose of the matrix A, respectively. The first condition in (3.2) corresponds to a transpose reduction, which is shown in Ref. 30 to preserve the zero-curvature condition. The second condition is a similarity constraint that maintains the structure of the AKNS problem.

Given the block structure of the spectral matrix U, these reductions imply the following constraints on the potential matrix P:

$$\Sigma P \Sigma^{-1} = -P^T, \quad \Delta P \Delta^{-1} = P. \tag{3.3}$$

From these, we derive corresponding conditions on the matrix potentials p and q:

$$p = -\Sigma_1^{-1} q^T \Sigma_2 \quad \text{or } q = -\Sigma_2^{-1} p^T \Sigma_1, \tag{3.4}$$

and

$$p = \Delta_1 p \Delta_2^{-1}, \quad q = \Delta_2 q \Delta_1^{-1}. \tag{3.5}$$

Combining these yields the following compatibility conditions. Either

$$\Delta_1 p = p \Delta_2, \quad \Sigma_1^{-1} \Delta_1^T \Sigma_1 p = p \Sigma_2^{-1} \Delta_2^T \Sigma_2, \tag{3.6}$$

or, equivalently, the matrix potential q must satisfy

$$q \Delta_1 = \Delta_2 q, \quad q \Sigma_1^{-1} \Delta_1^T \Sigma_1 = \Sigma_2^{-1} \Delta_2^T \Sigma_2 q. \tag{3.7}$$

Under these reductions, the original AKNS spectral problem reduces to either of the following forms. In terms of p, we have:

$$-i\phi_x = U\phi, \quad U = \begin{bmatrix} \alpha_1 \lambda I_m & p \\ -\Sigma_2^{-1} p^T \Sigma_1 & \alpha_2 \lambda I_n \end{bmatrix}, \tag{3.8}$$

where p satisfies the conditions in (3.6). Alternatively, in terms of q:

$$-i\phi_x = U\phi, \quad U = \begin{bmatrix} \alpha_1 \lambda I_m & -\Sigma_1^{-1} q^T \Sigma_2 \\ q & \alpha_2 \lambda I_n \end{bmatrix}, \tag{3.9}$$

where q satisfies the conditions in (3.7).

3.2. *Reductions of the matrix mKdV integrable hierarchies*

Due to the uniqueness of the Laurent series solution to the stationary zero-curvature equation (2.5), the reductions in (3.2) induce the following symmetry properties of the formal series $W(\lambda)$ defined in (2.6):

$$\Sigma W(\lambda) \Sigma^{-1} = W^T(-\lambda) = (W(-\lambda))^T, \quad \Delta W(\lambda) \Delta^{-1} = W(\lambda). \tag{3.10}$$

As a result, the Lax matrices $V^{[2s+1]} = (\lambda^{2s+1} W)_+$, for $s \geq 0$, inherit the following invariance:

$$\Sigma V^{[2s+1]}(\lambda) \Sigma^{-1} = -V^{[2s+1]T}(-\lambda) = -(V^{[2s+1]}(-\lambda))^T, \quad \Delta V^{[2s+1]}(\lambda) \Delta^{-1} = V^{[2s+1]}(\lambda). \tag{3.11}$$

Therefore, the zero-curvature condition

$$U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}] = 0, \tag{3.12}$$

is preserved under both group reductions:

$$\Sigma(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Sigma^{-1} = -(U_t^T + V_x^{[2s+1]T} + i[U^T, V^{[2s+1]T}])(-\lambda), \tag{3.13}$$

and

$$\Delta(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Delta^{-1} = (U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda). \tag{3.14}$$

This leads to the reduction of the matrix AKNS hierarchy (2.8) into the following reduced matrix mKdV integrable hierarchies. In terms of p, we have:

$$p_t = i\alpha b^{[2s+2]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}, \quad s \geq 0, \tag{3.15}$$

where p satisfies (3.6). Alternatively, in terms of q, we obtain:

$$q_t = -i\alpha c^{[2s+2]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}, \quad s \geq 0, \tag{3.16}$$

with q satisfying (3.7).

Each member of the reduced hierarchy is associated with a spatial spectral problem of the form (3.8) or (3.9), and admits an infinite set of commuting symmetries and conserved quantities inherited from the full AKNS hierarchy.

The corresponding temporal parts of the Lax pairs are given by:

$$-i\phi_t = V^{[2s+1]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi, \quad s \geq 0, \tag{3.17}$$

or

$$-i\phi_t = V^{[2s+1]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}\phi, \quad s \geq 0, \tag{3.18}$$

and together with the spatial spectral problems, these form the Lax pairs for the reduced p-based hierarchy and the q-based hierarchy.

4. An Application

In this section, we introduce a class of dual group reductions aimed at constructing novel integrable mKdV models. We concentrate on the cases where $m = 2$ and $n = 3$, for which the spectral matrix takes the following form:

$$U(u, \lambda) = \begin{bmatrix} \alpha_1\lambda I_2 & p \\ q & \alpha_2\lambda I_3 \end{bmatrix}, \tag{4.1}$$

where p is a 2×3 matrix-valued potential, q is a 3×2 matrix-valued potential, and λ is the spectral parameter.

To begin, we choose two pairs of matrix blocks:

$$\Delta_1 = \begin{bmatrix} 0 & \delta_1 \\ \delta_1 & 0 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0 & 0 & \delta_1 \\ 0 & \delta_1 & 0 \\ \delta_1 & 0 & 0 \end{bmatrix}; \quad \Sigma_1 = \begin{bmatrix} \sigma_1 & \sigma_5 \\ \sigma_5 & \sigma_1 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} \sigma_3 & 0 & \sigma_4 \\ 0 & \sigma_2 & 0 \\ \sigma_4 & 0 & \sigma_3 \end{bmatrix}, \tag{4.2}$$

where δ_1, σ_2 are arbitrary but nonzero constants, and $\sigma_1, \sigma_3, \sigma_4$ and σ_5 are arbitrary, subject to the conditions $\sigma_1^2 \neq \sigma_5^2$ and $\sigma_3^2 \neq \sigma_4^2$ to ensure the invertibility of Δ_1, Δ_2, Σ_1 and Σ_2. Based on the preceding general procedure, we obtain the following forms for the matrix potentials:

$$p = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_3 & p_2 & p_1 \end{bmatrix}, \tag{4.3}$$

and

$$q = \begin{bmatrix} \dfrac{(\sigma_4\sigma_5 - \sigma_1\sigma_3)p_1 + (\sigma_1\sigma_4 - \sigma_3\sigma_5)p_3}{\sigma_3^2 - \sigma_4^2} & \dfrac{(\sigma_3\sigma_5 - \sigma_1\sigma_4)p_1 + (\sigma_4\sigma_5 - \sigma_1\sigma_3)p_3}{\sigma_3^2 - \sigma_4^2} \\ -\dfrac{(\sigma_1 + \sigma_5)p_2}{\sigma_2} & -\dfrac{(\sigma_1 + \sigma_5)p_2}{\sigma_2} \\ \dfrac{(\sigma_3\sigma_5 - \sigma_1\sigma_4)p_1 + (\sigma_4\sigma_5 - \sigma_1\sigma_3)p_3}{\sigma_3^2 - \sigma_4^2} & \dfrac{(\sigma_4\sigma_5 - \sigma_1\sigma_3)p_1 + (\sigma_1\sigma_4 - \sigma_3\sigma_5)p_3}{\sigma_3^2 - \sigma_4^2} \end{bmatrix}. \tag{4.4}$$

Thus, the reduced matrix spectral problem takes the form:

$$-i\phi_x = \begin{bmatrix} \alpha_1\lambda & 0 & p_1 & p_2 & p_3 \\ 0 & \alpha_1\lambda & p_3 & p_2 & p_1 \\ \dfrac{(\sigma_4\sigma_5 - \sigma_1\sigma_3)p_1 + (\sigma_1\sigma_4 - \sigma_3\sigma_5)p_3}{\sigma_3^2 - \sigma_4^2} & \dfrac{(\sigma_3\sigma_5 - \sigma_1\sigma_4)p_1 + (\sigma_4\sigma_5 - \sigma_1\sigma_3)p_3}{\sigma_3^2 - \sigma_4^2} & \alpha_2\lambda & 0 & 0 \\ -\dfrac{(\sigma_1 + \sigma_5)p_2}{\sigma_2} & -\dfrac{(\sigma_1 + \sigma_5)p_2}{\sigma_2} & 0 & \alpha_2\lambda & 0 \\ \dfrac{(\sigma_3\sigma_5 - \sigma_1\sigma_4)p_1 + (\sigma_4\sigma_5 - \sigma_1\sigma_3)p_3}{\sigma_3^2 - \sigma_4^2} & \dfrac{(\sigma_4\sigma_5 - \sigma_1\sigma_3)p_1 + (\sigma_1\sigma_4 - \sigma_3\sigma_5)p_3}{\sigma_3^2 - \sigma_4^2} & 0 & 0 & \alpha_2\lambda \end{bmatrix} \phi, \tag{4.5}$$

where p_1, p_2, p_3 are the potentials, and λ is the spectral parameter. Consequently, the corresponding class of reduced coupled integrable mKdV models is given by

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3} p_{1,xxx} + \dfrac{3\beta}{\alpha^3\sigma_2(\sigma_3^2 - \sigma_4^2)} \{[2\sigma_2(\sigma_1\sigma_3 - \sigma_4\sigma_5)p_1^2 - 4\sigma_2(\sigma_1\sigma_4 - \sigma_3\sigma_5)p_1p_3 \\ \qquad + (\sigma_1 + \sigma_5)(\sigma_3^2 - \sigma_4^2)p_2^2 + 2\sigma_2(\sigma_1\sigma_3 - \sigma_4\sigma_5)p_3^2]p_{1,x} \\ \qquad + (\sigma_1 + \sigma_5)(\sigma_3^2 - \sigma_4^2)p_2(p_1 + p_3)p_{2,x} + [-2\sigma_2(\sigma_1\sigma_4 - \sigma_3\sigma_5)p_1^2 \\ \qquad + 4\sigma_2(\sigma_1\sigma_3 - \sigma_4\sigma_5)p_1p_3 + (\sigma_1 + \sigma_5)(\sigma_3^2 - \sigma_4^2)p_2^2 - 2\sigma_2(\sigma_1\sigma_4 - \sigma_3\sigma_5)p_3^2]p_{3,x}\}, \\ p_{2,t} = -\dfrac{\beta}{\alpha^3} p_{2,xxx} + \dfrac{3\beta}{\alpha^3\sigma_2(\sigma_3 + \sigma_4)} \{\sigma_2(\sigma_1 + \sigma_5)p_2(p_1 + p_3)(p_{1,x} + p_{3,x}) \\ \qquad + (\sigma_1 + \sigma_5)[\sigma_2(p_1 + p_3)^2 + 4(\sigma_3 + \sigma_4)p_2^2]p_{2,x}\}, \\ p_{3,t} = -\dfrac{\beta}{\alpha^3} p_{3,xxx} + \dfrac{3\beta}{\alpha^3\sigma_2(\sigma_3^2 - \sigma_4^2)} \{[-2\sigma_2(\sigma_1\sigma_4 - \sigma_3\sigma_5)p_1^2 + 4\sigma_2(\sigma_1\sigma_3 - \sigma_4\sigma_5)p_1p_3 \\ \qquad + (\sigma_1 + \sigma_5)(\sigma_3^2 - \sigma_4^2)p_2^2 - 2\sigma_2(\sigma_1\sigma_4 - \sigma_3\sigma_5)p_3^2]p_{1,x} \\ \qquad + (\sigma_1 + \sigma_5)(\sigma_3^2 - \sigma_4^2)p_2(p_1 + p_3)p_{2,x} + [2\sigma_2(\sigma_1\sigma_3 - \sigma_4\sigma_5)p_1^2 \\ \qquad - 4\sigma_2(\sigma_1\sigma_4 - \sigma_3\sigma_5)p_1p_3 + (\sigma_1 + \sigma_5)(\sigma_3^2 - \sigma_4^2)p_2^2 + 2\sigma_2(\sigma_1\sigma_3 - \sigma_4\sigma_5)p_3^2]p_{3,x}\}. \end{cases} \tag{4.6}$$

Here, α, β, σ_2 are arbitrary nonzero constants, while $\sigma_1, \sigma_3, \sigma_4$ and σ_5 are arbitrary, with the constraints $\sigma_1^2 \neq \sigma_5^2$ and $\sigma_3^2 \neq \sigma_4^2$. Interestingly, δ_1 does not appear in the resulting integrable mKdV models. The higher-order integrable models in the corresponding reduced integrable hierarchy can also be computed directly.

Let us now work out a few specific examples. First, consider the case where

$$\alpha = -\beta = \sigma_1 = 1, \quad \sigma_2 = \pm 1, \quad \sigma_3 = \pm 2, \quad \sigma_4 = \mp 1, \quad \sigma_5 = -2. \tag{4.7}$$

Then the corresponding spectral matrix takes the form

$$U = U(u, \lambda) = \begin{bmatrix} \alpha_1 \lambda & 0 & p_1 & p_2 & p_3 \\ 0 & \alpha_1 \lambda & p_3 & p_2 & p_1 \\ \pm p_3 & \pm p_1 & (\alpha_1 - 1)\lambda & 0 & 0 \\ \pm p_2 & \pm p_2 & 0 & (\alpha_1 - 1)\lambda & 0 \\ \pm p_1 & \pm p_3 & 0 & 0 & (\alpha_1 - 1)\lambda \end{bmatrix}, \tag{4.8}$$

where $u = (p_1, p_2, p_3)^T$ and α_1 is an arbitrary constant. The resulting coupled integrable mKdV model is

$$\begin{cases} p_{1,t} = p_{1,xxx} \pm 3[(4p_1p_3 + p_2^2)p_{1,x} + p_2(p_1 + p_3)p_{2,x} + (2p_1^2 + p_2^2 + 2p_3^2)p_{3,x}], \\ p_{2,t} = p_{2,xxx} \pm 3\{p_2(p_1 + p_3)(p_{1,x} + p_{3,x}) + [(p_1 + p_3)^2 + 4p_2^2]p_{2,x}\}, \\ p_{3,t} = p_{3,xxx} \pm 3[(2p_1^2 + p_2^2 + 2p_3^2)p_{1,x} + p_2(p_1 + p_3)p_{2,x} + (4p_1p_3 + p_2^2)p_{3,x}]. \end{cases} \tag{4.9}$$

Note that the coefficients are structured in a specific and symmetric pattern, reflecting the chosen parameter symmetries.

Next, consider the parameter choice:

$$\alpha = -\beta = 1, \quad \sigma_1 = -3, \quad \sigma_2 = 1, \quad \sigma_3 = 2, \quad \sigma_4 = -1, \quad \sigma_5 = -6. \tag{4.10}$$

The spectral matrix becomes

$$U = U(u, \lambda)$$
$$= \begin{bmatrix} \alpha_1 \lambda & 0 & p_1 & p_2 & p_3 \\ 0 & \alpha_1 \lambda & p_3 & p_2 & p_1 \\ 4p_1 + 5p_3 & 5p_1 + 4p_3 & (\alpha_1 - 1)\lambda & 0 & 0 \\ 9p_2 & 9p_2 & 0 & (\alpha_1 - 1)\lambda & 0 \\ 5p_1 + 4p_3 & 4p_1 + 5p_3 & 0 & 0 & (\alpha_1 - 1)\lambda \end{bmatrix}, \tag{4.11}$$

along with $u = (p_1, p_2, p_3)^T$ and α_1 is arbitrary. The corresponding coupled integrable mKdV system is

$$\begin{cases} p_{1,t} = p_{1,xxx} + 3(8p_1^2 + 20p_1p_3 + 9p_2^2 + 8p_3^2)p_{1,x} + 27p_2(p_1 + p_3)p_{2,x} \\ \qquad + 3(10p_1^2 + 16p_1p_3 + 9p_2^2 + 10p_3^2)p_{3,x}, \\ p_{2,t} = p_{2,xxx} + 27p_2(p_1 + p_3)(p_{1,x} + p_{3,x}) + 27[(p_1 + p_3)^2 + 4p_2^2]p_{2,x}, \\ p_{3,t} = 3(10p_1^2 + 16p_1p_3 + 9p_2^2 + 10p_3^2)p_{1,x} + 27p_2(p_1 + p_3)p_{2,x} \\ \qquad + 3(8p_1^2 + 20p_1p_3 + 9p_2^2 + 8p_3^2)p_{3,x}. \end{cases} \tag{4.12}$$

Here, the distribution of coefficients differs significantly from the previous example, illustrating the structural diversity induced by different group reductions.

Extending this approach to larger values of m and n increases the complexity of the analysis but significantly expands the potential for discovering new matrix spectral problems and associated integrable models. Symbolic computation, for instance using Maple, can be effectively employed to identify new configurations of matrix blocks suitable for formulating dual group reductions. This, in turn, facilitates the construction and analysis of additional integrable models characterized by reduced Lax pairs.

5. Conclusion and Remarks

A set of dual group reductions has been introduced and analyzed to reduce the AKNS matrix spectral problems, leading to associated reduced matrix mKdV integrable hierarchies. The first nonlinear system in each hierarchy corresponds to a new integrable coupled mKdV model. In this reduction framework, one group reduction imposes a constraint between the two matrix potentials in the original AKNS spectral problem, while the other applies a constraint to each potential individually. This type of dual group reduction differs from those considered in earlier studies, which typically focus on nonlocal group reductions.

Various methods, such as the Darboux transformation, Hirota's bilinear method, Bäcklund transformations, and the Wronskian determinant technique, may be employed to generate soliton-type solutions (see, e.g. Refs. 31–33). These include rational solutions,[34] lump solutions,[35–37] breather and rogue waves,[38–40] and interaction solutions.[41] Additionally, the Riemann–Hilbert approach provides a powerful tool for constructing soliton solutions, especially in the presence of multiple poles in the scattering data.[42]

It is worth noting that the reduced integrable models correspond to specific constraints on the potentials in the original system. While these constraints can make the reduced models more challenging to derive and solve, they also produce special classes of solutions to the unreduced system that are of significant interest in their own right. We hope that this work contributes to the classification of integrable models and enriches the theory of integrable models through the lens of dual group reductions and reduced Lax pairs.

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References

1. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
2. A. Das, *Integrable Models* (World Scientific, Teaneck, 1989).
3. W. X. Ma, *Nonlinear Anal.: Real World Appl.* **47** (2019) 1.
4. W. X. Ma, *Partial Differ. Equ. Appl. Math.* **4** (2021) 100190.
5. M. J. Ablowitz and Z. H. Musslimani, *Nonlinearity* **29** (2016) 915.
6. W. X. Ma, Y. H. Hung and F. D. Wang, *Stud. Appl. Math.* **145** (2020) 563.
7. L. M. Ling and W. X. Ma, *Symmetry* **13** (2021) 512.
8. J. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems* (SIAM, Philadelphia, 2010).
9. C. Q. Song, D. M. Xiao and Z. N. Zhu, *Commun. Nonlinear Sci. Numer. Simul.* **45** (2017) 13.
10. M. Gürses, A. Pekcan, M. Gürses and A. Pekcan, *J. Math. Phys.* **59** (2018) 051501.
11. M. Gürses and A. Pekcan, *Commun. Nonlinear Sci. Numer. Simul.* **67** (2019) 427.
12. C. Athorne and A. Fordy, *J. Phys. A: Math. Gen.* **20** (1987) 1377.
13. V. S. Gerdjikov, *Turk. J. Math.* **46** (2022) 1821.
14. M. J. Ablowitz and Z. H. Musslimani, *Stud. Appl. Math.* **139** (2017) 7.
15. W. X. Ma, Nonlocal integrable equations in soliton theory, in *Nonlinear and Modern Mathematical Physics*, eds. S. Manukure and W. X. Ma (Springer, Cham, Switzerland, 2024), pp. 251–66.
16. G. Z. Tu, *J. Phys. A: Math. Gen.* **22** (1989) 2375.
17. W. X. Ma, *East Asian J. Appl. Math.* **14** (2024) 281.
18. W. X. Ma, *Appl. Math. Lett.* **168** (2025) 109574.
19. W. X. Ma, *Mathematics* **13** (2025) 1438.
20. W. X. Ma, *Phys. D* **430** (2022) 133078.
21. J. Zhang, C. P. Zhang and Y. N. Cui, *Adv. Math. Phys.* **2017** (2017) 9743475.
22. S. Manukure, *Commun. Nonlinear Sci. Numer. Simul.* **57** (2018) 125.
23. H. F. Wang and Y. F. Zhang, *Commun. Nonlinear Sci. Numer. Simul.* **99** (2021) 105822.
24. Q. L. Zhao, Y. D. Zhong and X. Y. Li, *J. Appl. Anal. Comput.* **12** (2022) 1353.
25. M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, *Stud. Appl. Math.* **53** (1974) 249.
26. H. F. Wang and Y. F. Zhang, *Int. J. Theoret. Phys.* **60** (2021) 1797.
27. T. S. Liu and T. C. Xia, *Nonlinear Anal. Real World Appl.* **68** (2022) 103667.
28. X. M. Zhu and J. B. Zhang, *Adv. Math. Phys.* **2022** (2022) 2200092.
29. W. X. Ma, *Chin. Phys. B* **32** (2023) 020201.
30. W. X. Ma, *Mod. Phys. Lett. B* **36** (2022) 2250094.
31. D. Gao, W. X. Ma and X. Lü, *Z. Naturforsch. A* **79** (2024) 887.
32. J. Y. Chu, Y. Q. Liu and W. X. Ma, *Mod. Phys. Lett. B* **39** (2025) 2550060.
33. L. Cheng, Y. Zhang and W. X. Ma, *Commun. Theor. Phys.* **77** (2025) 035002.
34. B. D. V. Campos, *Math. Comput. Simul.* **201** (2022) 396.
35. W. X. Ma and Y. Zhou, *J. Differ. Equ.* **264** (2018) 2633.
36. T. A. Sulaiman, A. Yusuf, A. Abdeljabbar and M. Alquran, *J. Geom. Phys.* **169** (2021) 104347.
37. H. C. Ma, Y. X. Bai and A. P. Deng, *Math. Methods Appl. Sci.* **43** (2020) 7135.

38. S. Manukure, A. Chowdhury and Y. Zhou, *Int. J. Mod. Phys. B* **33** (2019) 950098.
39. A. Yusuf, T. A. Sulaiman, A. Abdeljabbar and M. Alquran, *J. Ocean Eng. Sci.* **8** (2022) 145.
40. Y. H. Yin, X. Lü and W. X. Ma, *Nonlinear Dyn.* **108** (2022) 4181.
41. B. Liu, X. E. Zhang, B. Wang and X. Lü, *Mod. Phys. Lett. B* **36** (2022) 2250057.
42. J. J. Yang, S. F. Tian and Z. Q. Li, *J. Math. Anal. Appl.* **511** (2022) 126103.