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Four-component combined integrable equations possessing bi-Hamiltonian formulations

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This paper aims to generate a Liouville integrable Hamiltonian hierarchy by introducing a specific matrix eigenvalue problem with four components. The adopted approach is the zero curvature formulation. A bi-Hamiltonian formulation is furnished through applying the trace identity, which shows the Liouville integrability of the resulting hierarchy. Two examples of generalized combined nonlinear Schrödinger equations and modified Korteweg–de Vries equations are presented.

Keywords: Matrix eigenvalue problem; zero curvature equation; integrable hierarchy; NLS equations; mKdV equations.

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1. Introduction

Integrable models are associated with Lax pairs of matrix eigenvalue problems,[1] based on which one can explore remarkable integrable properties, for example, infinitely many symmetries and conserved quantities.[2,3] They have various applications in physical and engineering sciences, including nonlinear optics, water waves and quantum mechanics.

Typical examples of integrable models include the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy[4] and its various hierarchies of integrable couplings.[5] Matrix algebras play a crucial role in formulating meaningful Lax pairs.[6,7] It is always intriguing to see what kind of Lax pairs could lead to integrable models. In this paper, we would like to propose a new matrix eigenvalue problem and compute an associated integrable hierarchy.

Let us discuss about the zero curvature formulation for constructing integrable hierarchies briefly (see Refs. 6 and 7 for details). First, let us denote a q-dimensional column potential vector by $u = (u_1, \dots, u_q)^T$ and the spectral parameter by λ. Using a given loop matrix algebra $\tilde{g}$ with the loop parameter λ, we formulate a spatial spectral matrix:

$$\mathcal{M} = \mathcal{M}(u, \lambda) = u_1 E_1(\lambda) + \cdots + u_q E_q(\lambda) + E_0(\lambda), \tag{1.1}$$

where the elements $E_1, \dots, E_q$ are linear independent in $\tilde{g}$. The above element E_0 is assumed to be pseudo-regular:

$$\mathrm{Im}\,\mathrm{ad}_{E_0} \oplus \mathrm{Ker}\,\mathrm{ad}_{E_0} = \tilde{g}, \quad [\mathrm{Ker}\ \mathrm{ad}_{E_0}, \mathrm{Ker}\,\mathrm{ad}_{E_0}] = 0,$$

where ad_{E_0} stands for the adjoint action of E_0 on $\tilde{g}$. This generally ensures that the stationary zero curvature equation

$$Y_x = [\mathcal{M}, Y] \tag{1.2}$$

will have a Laurent series solution $Y = \sum_{n\geq 0} \lambda^{-n} Y^{[n]}$ in the underlying loop algebra $\tilde{g}$.

Second, we introduce an infinite sequence of temporal spectral matrices

$$\mathcal{N}^{[m]} = (\lambda^m Y)_+ + \Delta_r = \sum_{n=0}^{m} \lambda^{m-n} Y^{[n]} + \Delta_m, \quad m \geq 0, \tag{1.3}$$

where $\Delta_m \in \tilde{g}, m \geq 0$, as the other parts of a sequence of Lax pairs, to form a hierarchy of integrable models:

$$u_{t_m} = X^{[m]} = X^{[m]}(u), \quad m \geq 0, \tag{1.4}$$

through the zero curvature equations:

$$\mathcal{M}_{t_m} - \mathcal{N}_x^{[m]} + [\mathcal{M}, \mathcal{N}^{[m]}] = 0, \quad m \geq 0. \tag{1.5}$$

They represent the solvability conditions of the spatial and temporal matrix eigenvalue problems:

$$\varphi_x = \mathcal{M}\varphi, \quad \varphi_{t_m} = \mathcal{N}^{[m]}\varphi, \quad m \geq 0. \tag{1.6}$$

Finally, applying the so-called trace identity:

$$\frac{\delta}{\delta u} \int \mathrm{tr}\Big(Y \frac{\partial \mathcal{M}}{\partial \lambda}\Big) dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa} \mathrm{tr}\Big(Y \frac{\partial \mathcal{M}}{\partial u}\Big), \tag{1.7}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to u, and κ is a constant, independent of the spectral parameter λ, we furnish the hierarchy (1.4) with a Hamiltonian formulation and thus explore its Liouville integrability.

Various hierarchies of Liouville integrable models have been presented.[4–17] The well-known integrable hierarchies with two components include the AKNS hierarchy,[4] the Heisenberg hierarchy,[18] the Kaup–Newell hierarchy[19] and the Wadati–Konno–Ichikawa hierarchy.[20] Their spectral matrices read

$$\begin{aligned} &\mathcal{M} = \begin{bmatrix} \lambda & u_1 \\ u_2 & -\lambda \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} \lambda u_3 & \lambda u_1 \\ \lambda u_2 & -\lambda u_3 \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} \lambda^2 & \lambda u_1 \\ \lambda u_2 & -\lambda^2 \end{bmatrix}, \\ &\mathcal{M} = \begin{bmatrix} \lambda & \lambda u_1 \\ \lambda u_2 & -\lambda \end{bmatrix}, \end{aligned} \tag{1.8}$$

where $u_1 u_2 + u_3^2 = 1$, respectively.

This paper aims to present a hierarchy of four-component Liouville integrable models through the zero curvature formulation. The corresponding bi-Hamiltonian formulation is furnished via the trace identity. Two illustrative examples of generalized combined integrable nonlinear Schrödinger and modified Korteweg–de Vries models are presented. The final section gives a conclusion and a few concluding remarks.

2. A Four-Component Integrable Hierarchy

Let δ be an arbitrary number, and T be a square matrix of order r such that

$$T^2 = -I_r, \tag{2.1}$$

where I_r denotes the identity matrix of order r. Let us form a set of block matrices

$$g = \left\{ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}_{2r\times 2r} \middle| A_4 = TA_1T^{-1}, A_3 = \delta T A_2 T^{-1} \right\}. \tag{2.2}$$

It is easy to see this forms a matrix Lie algebra under the matrix commutator $[A, B] = AB - BA$. We will use the Lie algebra with $r = 2$ and

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.3}$$

to formulate a specific spectral matrix below.

Let α_1 and α_2 be two arbitrary numbers, and $u = u(x,t) = (u_1, u_2, u_3, u_4)^T$ be a four-component potential vector. Assume that

$$\alpha = \alpha_1 - \alpha_2 \neq 0. \tag{2.4}$$

Motivated by recent studies on matrix eigenvalue problems with four potentials (see, e.g. Refs. 21–25), we consider a matrix eigenvalue problem of the following form:

$$\varphi_x = \mathcal{M}\varphi = \mathcal{M}(u,\lambda)\varphi, \quad \mathcal{M} = \begin{bmatrix} \alpha_1\lambda & u_1 & u_2 & 0 \\ u_3 & \alpha_2\lambda & 0 & u_4 \\ \delta u_4 & 0 & \alpha_2\lambda & -u_3 \\ 0 & \delta u_2 & -u_1 & \alpha_1\lambda \end{bmatrix}, \tag{2.5}$$

where λ is again the spectral parameter. This spectral matrix $\mathcal{M}$ belongs to the previous matrix algebra g, with $r = 2$ and T being defined by (2.3). The eigenvalue problem cannot be any reduction of the matrix AKNS eigenvalue problem (see, e.g. Ref. 26). Interestingly, from this eigenvalue problem, we can generate an integrable hierarchy of Hamiltonian equations, which shows particular combined structures of integrable models. Obviously, the case of $\delta = 0$ yields integrable couplings, which are not of perturbation type.

To construct an associated Liouville integrable hierarchy, we first solve the corresponding stationary zero curvature equation (1.2) by assuming Y to be of the following Laurent series type:

$$Y = \begin{bmatrix} a & b & e & f \\ c & -a & f & g \\ \delta g & -\delta f & -a & -c \\ -\delta f & \delta e & -b & a \end{bmatrix} = \sum_{n\geq 0} \lambda^{-n} Y^{[n]}, \tag{2.6}$$

where the basic objects can be stated as follows:

$$\begin{cases} a = \sum_{n\geq 0} \lambda^{-n} a^{[n]}, \quad b = \sum_{n\geq 0} \lambda^{-n} b^{[n]}, \quad c = \sum_{n\geq 0} \lambda^{-n} c^{[n]}, \\ e = \sum_{n\geq 0} \lambda^{-n} e^{[n]}, \quad f = \sum_{n\geq 0} \lambda^{-n} f^{[n]}, \quad g = \sum_{n\geq 0} \lambda^{-n} g^{[n]}. \end{cases} \tag{2.7}$$

It is sufficient to take a solution of the form in (2.6), since the commutator of a matrix $A \in g$ and U must be of that form. It is readily seen that the corresponding stationary zero curvature equation (1.2) engenders the initial conditions:

$$a_x^{[0]} = 0, \quad b^{[0]} = c^{[0]} = e^{[0]} = g^{[0]} = 0, \quad f_x^{[0]} = 0, \tag{2.8}$$

and the recursion relations to determine the Laurent series solution:

$$\begin{cases} b^{[n+1]} = \dfrac{1}{\alpha}[b_x^{[n]} + 2a^{[n]}u_1 + 2\delta f^{[n]}u_2], \\ c^{[n+1]} = -\dfrac{1}{\alpha}[c_x^{[n]} - 2a^{[n]}u_3 + 2\delta f^{[n]}u_4], \end{cases} \tag{2.9}$$

$$\begin{cases} e^{[n+1]} = \dfrac{1}{\alpha}[e_x^{[n]} - 2f^{[n]}u_1 + 2a^{[n]}u_2], \\ g^{[n+1]} = -\dfrac{1}{\alpha}[g_x^{[n]} - 2f^{[n]}u_3 - 2a^{[n]}u_4], \end{cases} \tag{2.10}$$

$$\begin{cases} a_x^{[n+1]} = c^{[n+1]}u_1 + \delta g^{[n+1]}u_2 - b^{[n+1]}u_3 - \delta e^{[n+1]}u_4, \\ f_x^{[n+1]} = g^{[n+1]}u_1 - c^{[n+1]}u_2 + e^{[n+1]}u_3 - b^{[n+1]}u_4, \end{cases} \tag{2.11}$$

where $n \geq 0$. To have a unique Laurent series solution, we go with the initial data

$$a^{[0]} = \frac{1}{2}\beta, \quad f^{[0]} = \frac{1}{2}\gamma, \tag{2.12}$$

where β and γ are two arbitrary constants, and take the constants of integration to be zero

$$a^{[n]}|_{u=0} = 0, \quad f^{[n]}|_{u=0} = 0, \quad n \geq 1. \tag{2.13}$$

Then, one can work out that

$$\begin{cases} b^{[1]} = \frac{1}{\alpha}(\beta u_1 + \delta\gamma u_2), \quad c^{[1]} = \frac{1}{\alpha}(\beta u_3 - \delta\gamma u_4), \\ e^{[1]} = -\frac{1}{\alpha}(\gamma u_1 - \beta u_2), \quad g^{[1]} = \frac{1}{\alpha}(\gamma u_3 + \beta u_4), \\ a^{[1]} = f^{[1]} = 0, \end{cases}$$

$$\begin{cases} b^{[2]} = \frac{1}{\alpha^2}(\beta u_{1,x} + \delta\gamma u_{2,x}), \quad c^{[2]} = -\frac{1}{\alpha^2}(\beta u_{3,x} - \delta\gamma u_{4,x}), \\ e^{[2]} = -\frac{1}{\alpha^2}(\gamma u_{1,x} - \beta u_{2,x}), \quad g^{[2]} = -\frac{1}{\alpha^2}(\gamma u_{3,x} + \beta u_{4,x}), \end{cases}$$

$$\begin{cases} a^{[2]} = -\frac{1}{\alpha^2}[(\gamma u_3 - \delta\gamma u_4)u_1 + \delta(\gamma u_3 + \beta u_4)u_2], \\ f^{[2]} = -\frac{1}{\alpha^2}[(\gamma u_3 + \beta u_4)u_1 - (\beta u_3 - \delta\gamma u_4)u_2], \end{cases}$$

$$\begin{cases} b^{[3]} = \frac{1}{\alpha^3}[\beta u_{1,xx} + \delta\gamma u_{2,xx} - 2(\beta u_3 - \delta\gamma u_4)u_1^2 - 4\delta(\gamma u_3 + \beta u_4)u_1u_2 \\ \qquad +2\delta(\beta u_3 - \delta\gamma u_4)u_2^2], \\ c^{[3]} = \frac{1}{\alpha^3}[\beta u_{3,xx} - \delta\gamma u_{4,xx} - 2(\beta u_1 + \delta\gamma u_2)u_3^2 + 4\delta(\gamma u_1 - \beta u_2)u_3u_4 \\ \qquad + 2\delta(\beta u_1 + \delta\gamma u_2)u_4^2], \end{cases}$$

$$\begin{cases} e^{[3]} = \frac{1}{\alpha^3}[-\gamma u_{1,xx} + \beta u_{2,xx} + 2(\gamma u_3 + \beta u_4)u_1^2 - 4(\beta u_3 - \delta\gamma u_4)u_1u_2 \\ \qquad - 2\delta(\gamma u_3 + \beta u_4)u_2^2], \\ g^{[3]} = \frac{1}{\alpha^3}[\gamma u_{3,xx} + \beta u_{4,xx} - 2(\gamma u_1 - \beta u_2)u_3^2 - 4(\beta u_1 + \delta\gamma u_2)u_3u_4 \\ \qquad + 2\delta(\gamma u_1 - \beta u_2)u_4^2], \end{cases}$$

$$\begin{cases} a^{[3]} = \dfrac{1}{\alpha^3}[-(\beta u_3 - \delta\gamma u_4)u_{1,x} - \delta(\gamma u_3 + \beta u_4)u_{2,x} + (\beta u_1 + \delta\gamma u_2)u_{3,x} \\ \qquad - \delta(\gamma u_1 - \beta u_2)u_{4,x}], \\ f^{[3]} = \dfrac{1}{\alpha^3}[-(\gamma u_3 + \beta u_4)u_{1,x} + (\beta u_3 - \delta\gamma u_4)u_{2,x} + (\gamma u_1 - \beta u_2)u_{3,x} \\ \qquad + (\beta u_1 + \delta\gamma u_2)u_{4,x}], \end{cases}$$

and

$$\begin{cases} b^{[4]} = \dfrac{1}{\alpha^4}\{\beta u_{1,xxx} + \delta\gamma u_{2,xxx} - 6[u_1(\beta u_3 - \delta\gamma u_4) + \delta u_2(\gamma u_3 + \beta u_4)]u_{1,x} \\ \qquad - 6\delta[u_1(\gamma u_3 + \beta u_4) - u_2(\beta u_3 - \delta\gamma u_4)]u_{2,x}\}, \\ c^{[4]} = -\dfrac{1}{\alpha^4}\{\beta u_{3,xxx} - \delta\gamma u_{4,xxx} - 6[u_1(\beta u_3 - \delta\gamma u_4) + \delta u_2(\gamma u_3 + \beta u_4)]u_{3,x} \\ \qquad + 6\delta[u_1(\gamma u_3 + \beta u_4) - u_2(\beta u_3 - \delta\gamma u_4)]u_{4,x}\}, \end{cases}$$

$$\begin{cases} e^{[4]} = \dfrac{1}{\alpha^4}\{-\gamma u_{1,xxx} + \beta u_{2,xxx} + 6[u_1(\gamma u_3 + \beta u_4) - u_2(\beta u_3 - \delta\gamma u_4)]u_{1,x} \\ \qquad - 6[u_1(\beta u_3 - \delta\gamma u_4) + \delta u_2(\gamma u_3 + \beta u_4)]u_{2,x}\}, \\ g^{[4]} = -\dfrac{1}{\alpha^4}\{\gamma u_{3,xxx} + \beta u_{4,xxx} - 6[u_1(\gamma u_3 + \beta u_4) - u_2(\beta u_3 - \delta\gamma u_4)]u_{3,x} \\ \qquad - 6[u_1(\beta u_3 - \delta\gamma u_4) + \delta u_2(\gamma u_3 + \beta u_4)]u_{4,x}\}, \end{cases}$$

$$\begin{aligned} a^{[4]} = \frac{1}{\alpha^4}&[-(\beta u_3 - \delta\gamma u_4)u_{1,xx} - \delta(\gamma u_3 + \beta u_4)u_{2,xx} - (\beta u_1 + \delta\gamma u_2)u_{3,xx} \\ &+ \delta(\gamma u_1 - \beta u_2)u_{4,xx} + (\beta u_{3,x} - \delta\gamma u_{4,x})u_{1,x} + \delta(\gamma u_{3,x} + \beta u_{4,x})u_{2,x} \\ &+3(\beta u_3^2 - 2\delta\gamma u_3u_4 - \delta\beta u_4^2)u_1^2 + 6\delta(\gamma u_3^2 + 2\beta u_3u_4 - \delta\gamma u_4^2)u_1u_2 \\ &-3\delta(\beta u_3^2 - 2\delta\gamma u_3u_4 - \delta\beta u_4^2)u_2^2], \end{aligned}$$

$$\begin{aligned} f^{[4]} = \frac{1}{\alpha^4}&[-(\gamma u_3 + \beta u_4)u_{1,xx} + (\beta u_3 - \delta\gamma u_4)u_{2,xx} - (\gamma u_1 - \beta u_2)u_{3,xx} \\ &- (\beta u_1 + \delta\gamma u_2)u_{4,xx} + (\gamma u_{3,x} + \beta u_{4,x})u_{1,x} - (\beta u_{3,x} - \delta\gamma u_{4,x})u_{2,x} \\ &+ 3(\gamma u_3^2 + 2\beta u_3u_4 - \delta\gamma u_4^2)u_1^2 - 6(\beta u_3^2 - 2\delta\gamma u_3u_4 - \delta\beta u_4^2)u_1u_2 \\ &- 3\delta(\gamma u_3^2 + 2\beta u_3u_4 - \delta\gamma u_4^2)u_2^2]. \end{aligned}$$

Upon considering these results, we can impose $\Delta_r = 0,\ m \geq 0$, to formulate

$$\varphi_{t_m} = \mathcal{N}^{[m]}\varphi = \mathcal{N}^{[m]}(u,\lambda)\varphi, \quad \mathcal{N}^{[m]} = (\lambda^m Y)_+ = \sum_{n=0}^{m} \lambda^n Y^{[m-n]}, \quad m \geq 0, \tag{2.14}$$

which are the temporal matrix eigenvalue problems within the zero curvature formulation. The conditions that guarantee the solvability of the spatial and temporal matrix eigenvalue problems in (2.5) and (2.14) are given by the zero curvature equations in (1.5). They generate a hierarchy of evolution models with four potentials:

$$u_{t_m} = X^{[m]} = X^{[m]}(u) = (\alpha b^{[m+1]}, \alpha e^{[m+1]}, -\alpha c^{[m+1]}, -\alpha g^{[m+1]})^T, \quad m \geq 0, \tag{2.15}$$

or more concretely,

$$\begin{aligned} &u_{1,t_m} = \alpha b^{[m+1]}, \quad u_{2,t_m} = \alpha e^{[m+1]}, \quad u_{3,t_m} = -\alpha c^{[m+1]}, \\ &u_{4,t_m} = -\alpha g^{[m+1]}, \quad m \geq 0. \end{aligned} \tag{2.16}$$

As particular examples, this hierarchy contains the model of combined integrable nonlinear Schrödinger equations:

$$\begin{cases} u_{1,t_2} = \dfrac{1}{\alpha^2}[\beta u_{1,xx} + \delta\gamma u_{2,xx} - 2(\beta u_3 - \delta\gamma u_4)u_1^2 - 4\delta u_1 u_2(\gamma u_3 + \beta u_4) \\ \qquad + 2\delta(\beta u_3 - \delta\gamma u_4)u_2^2], \\ u_{2,t_2} = \dfrac{1}{\alpha^2}[-\gamma u_{1,xx} + \beta u_{2,xx} + 2(\gamma u_3 + \beta u_4)u_1^2 - 4\delta u_1 u_2(\beta u_3 - \delta\gamma u_4) \\ \qquad - 2\delta(\gamma u_3 + \beta u_4)u_2^2], \\ u_{3,t_2} = \dfrac{1}{\alpha^2}[-\beta u_{3,xx} + \delta\gamma u_{4,xx} + 2(\beta u_1 + \delta\gamma u_2)u_3^2 - 4\delta(\gamma u_1 - \beta u_2)u_3 u_4 \\ \qquad - 2\delta(\beta u_1 + \delta\gamma u_2)u_4^2], \\ u_{4,t_2} = \dfrac{1}{\alpha^2}[-\gamma u_{3,xx} - \beta u_{4,xx} + 2(\gamma u_1 - \beta u_2)u_3^2 + 4\delta(\beta u_1 + \delta\gamma u_2)u_3 u_4 \\ \qquad - 2\delta(\gamma u_1 - \beta u_2)u_4^2], \end{cases} \tag{2.17}$$

and the model of combined integrable modified Korteweg–de Vries equations:

$$\begin{cases} u_{1,t_3} = \dfrac{1}{\alpha^3}\{\beta u_{1,xxx} + \delta\gamma u_{2,xxx} - 6[u_1(\beta u_3 - \delta\gamma u_4) + \delta u_2(\gamma u_3 + \beta u_4)]u_{1,x} \\ \qquad - 6\delta[u_1(\gamma u_3 + \beta u_4) - u_2(\beta u_3 - \delta\gamma u_4)]u_{2,x}\}, \\ u_{2,t_3} = \dfrac{1}{\alpha^3}\{-\gamma u_{1,xxx} + \beta u_{2,xxx} + 6[u_1(\gamma u_3 + \beta u_4) - u_2(\beta u_3 - \delta\gamma u_4)]u_{1,x} \\ \qquad - 6[u_1(\beta u_3 - \delta\gamma u_4) + \delta u_2(\gamma u_3 + \beta u_4)]u_{2,x}\}, \\ u_{3,t_3} = \dfrac{1}{\alpha^3}\{\beta u_{3,xxx} - \delta\gamma u_{4,xxx} - 6[u_1(\beta u_3 - \delta\gamma u_4) + \delta u_2(\gamma u_3 + \beta u_4)]u_{3,x} \\ \qquad + 6\delta[u_1(\gamma u_3 + \beta u_4) - u_2(\beta u_3 - \delta\gamma u_4)]u_{4,x}\}, \\ u_{4,t_3} = \dfrac{1}{\alpha^3}\{\gamma u_{3,xxx} + \beta u_{4,xxx} - 6[u_1(\gamma u_3 + \beta u_4) - u_2(\beta u_3 - \delta\gamma u_4)]u_{3,x} \\ \qquad - 6[u_1(\beta u_3 - \delta\gamma u_4) + \delta u_2(\gamma u_3 + \beta u_4)]u_{4,x}\}. \end{cases} \tag{2.18}$$

These systems provide two typical coupled integrable models, which extend the category of coupled integrable models of nonlinear Schrödinger equations and modified Korteweg–de Vries equations (see, e.g. Refs. 23, 27 and 28). One interesting character is that every equation contains two derivative terms of the highest order, and so, we call them combined models.

Three special subcases of $\delta = 0$, $\beta = 0$ and $\gamma = 0$ in the above hierarchy are interesting. The first subcase presents novel integrable couplings of the AKNS hierarchy, which are not of perturbation type. The other two subcases produce reduced hierarchies of uncombined integrable models.

3. Recursion Operator and bi-Hamiltonian Formulation

Let $\delta \neq 0$ now. To furnish a bi-Hamiltonian formulation and show the Liouville integrability[29,30] for the soliton hierarchy (2.16), one can apply the so-called trace identity (1.7) to the spatial matrix eigenvalue problem (2.5). The trace identity uses the Laurent series solution Y determined by (2.6). One can then easily work out

$$\operatorname{tr}\left(Y\frac{\partial \mathcal{M}}{\partial \lambda}\right) = 2\alpha a, \quad \operatorname{tr}\left(Y\frac{\partial \mathcal{M}}{\partial u}\right) = (2c, 2\delta g, 2b, 2\delta e)^T, \tag{3.1}$$

and accordingly, the trace identity presents

$$\frac{\delta}{\delta u}\int \lambda^{-(n+1)}\alpha a^{[n+1]}\,dx = \lambda^{-\kappa}\frac{\partial}{\partial \lambda}\lambda^{\kappa-n}(c^{[n]}, \delta g^{[n]}, b^{[n]}, \delta e^{[n]})^T, \quad n \geq 0. \tag{3.2}$$

Checking with $n = 2$ yields $\kappa = 0$, and consequently, one arrives at

$$\frac{\delta}{\delta u}\mathcal{H}^{[n]} = (c^{[n+1]}, \delta g^{[n+1]}, b^{[n+1]}, \delta e^{[n+1]})^T, \quad n \geq 0, \tag{3.3}$$

where the Hamiltonian functionals are taken as

$$\mathcal{H}^{[n]} = -\int \frac{\alpha a^{[n+2]}}{n+1}\,dx, \quad n \geq 0. \tag{3.4}$$

This allows us to propose a Hamiltonian formulation for the hierarchy (2.16):

$$u_{t_m} = X^{[m]} = J_1\frac{\delta \mathcal{H}^{[m]}}{\delta u}, \quad m \geq 0, \tag{3.5}$$

where J_1 is the Hamiltonian operator:

$$J_1 = \left[\begin{array}{cc|cc} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \frac{\alpha}{\delta} \\ \hline -\alpha & 0 & 0 & 0 \\ 0 & -\frac{\alpha}{\delta} & 0 & 0 \end{array}\right], \tag{3.6}$$

and $\mathcal{H}^{[m]}$ are the functionals defined by (3.4). It follows directly from the Hamiltonian theory that there exists an interrelation $S = J_1\frac{\delta \mathcal{H}}{\delta u}$ between a symmetry S and a conserved functional $\mathcal{H}$ of the same model.

It is known that these vector fields $X^{[n]}$ satisfy a characteristic commutative property:

$$[\![X^{[n_1]}, X^{[n_2]}]\!] = X^{[n_1]\prime}(u)[X^{[n_2]}] - X^{[n_2]\prime}(u)[X^{[n_1]}] = 0, \quad n_1, n_2 \geq 0. \tag{3.7}$$

This property can been derived from an algebra of Lax operators:

$$[\![\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}]\!] = \mathcal{N}^{[n_1]\prime}(u)[X^{[n_2]}] - \mathcal{N}^{[n_2]\prime}(u)[X^{[n_1]}] + [\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}] = 0, \quad n_1, n_2 \geq 0, \tag{3.8}$$

which can directly be checked by analyzing the relation between the isospectral zero curvature equations (see Ref. 31 for the detailed proof).

Furthermore, from the recursion relation $X^{[m+1]} = \Phi X^{[m]}$, we can work out a hereditary recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$[29] for the hierarchy (2.16) as follows:

$$\begin{cases} \Phi_{11} = \dfrac{1}{\alpha}(\partial_x - 2u_1\partial^{-1}u_3 - 2\delta u_2\partial^{-1}u_4), \quad \Phi_{12} = \dfrac{1}{\alpha}(-2\delta u_1\partial^{-1}u_4 + 2\delta u_2\partial^{-1}u_3), \\ \Phi_{13} = \dfrac{1}{\alpha}(-2u_1\partial^{-1}u_1 + 2\delta u_2\partial^{-1}u_2), \quad \Phi_{14} = \dfrac{1}{\alpha}(-2\delta u_1\partial^{-1}u_2 - 2\delta u_2\partial^{-1}u_1), \end{cases} \tag{3.9}$$

$$\begin{cases} \Phi_{21} = \dfrac{1}{\alpha}(2u_1\partial^{-1}u_4 - 2u_2\partial^{-1}u_3), \quad \Phi_{22} = \dfrac{1}{\alpha}(\partial_x - 2u_1\partial^{-1}u_3 - 2\delta u_2\partial^{-1}u_4), \\ \Phi_{23} = \dfrac{1}{\alpha}(-2u_1\partial^{-1}u_2 - 2u_2\partial^{-1}u_1), \quad \Phi_{24} = \dfrac{1}{\alpha}(2u_1\partial^{-1}u_1 - 2\delta u_2\partial^{-1}u_2), \end{cases} \tag{3.10}$$

$$\begin{cases} \Phi_{31} = \dfrac{1}{\alpha}(2u_3\partial^{-1}u_3 - 2\delta u_4\partial^{-1}u_4), \quad \Phi_{32} = \dfrac{1}{\alpha}(2\delta u_3\partial^{-1}u_4 + 2\delta u_4\partial^{-1}u_3), \\ \Phi_{33} = \dfrac{1}{\alpha}(-\partial_x + 2u_3\partial^{-1}u_1 + 2\delta u_4\partial^{-1}u_2), \quad \Phi_{34} = \dfrac{1}{\alpha}(2\delta u_3\partial^{-1}u_2 - 2\delta u_4\partial^{-1}u_1), \end{cases} \tag{3.11}$$

$$\begin{cases} \Phi_{41} = \dfrac{1}{\alpha}(2u_3\partial^{-1}u_4 + 2u_4\partial^{-1}u_3), \quad \Phi_{42} = \dfrac{1}{\alpha}(-2u_3\partial^{-1}u_3 + 2\delta u_4\partial^{-1}u_4), \\ \Phi_{43} = \dfrac{1}{\alpha}(-2u_3\partial^{-1}u_2 + 2u_4\partial^{-1}u_1), \quad \Phi_{44} = \dfrac{1}{\alpha}(-\partial_x + 2u_3\partial^{-1}u_1 + 2\delta u_4\partial^{-1}u_2). \end{cases} \tag{3.12}$$

With some analysis, we can observe that J_1 and $J_2 = \Phi J_1$ constitute a Hamiltonian pair, that is, an arbitrary linear combination of J_1 and J_2 is again Hamiltonian, and thus the hierarchy (2.16) possesses a bi-Hamiltonian formulation[30]:

$$u_{t_m} = X^{[m]} = J_1\frac{\delta\mathcal{H}^{[m]}}{\delta u} = J_2\frac{\delta\mathcal{H}^{[m-1]}}{\delta u}, \quad m \geq 1. \tag{3.13}$$

Further, we can see that the associated Hamiltonian functionals commute with each other under the corresponding two Poisson brackets[6]:

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_1} = \int \left(\frac{\delta \mathcal{H}^{[n_1]}}{\delta u}\right)^T J_1 \frac{\delta \mathcal{H}^{[n_2]}}{\delta u}\, dx = 0, \quad n_1, n_2 \geq 0 \tag{3.14}$$

and

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_2} = \int \left(\frac{\delta \mathcal{H}^{[n_1]}}{\delta p}\right)^T J_2 \frac{\delta \mathcal{H}^{[n_2]}}{\delta u}\, dx = 0, \quad n_1, n_2 \geq 0. \tag{3.15}$$

To conclude, each model in the hierarchy (2.16) is Liouville integrable and possesses infinitely many commuting symmetries $\{X^{[n]}\}_{n=0}^{\infty}$ and conserved functionals $\{\mathcal{H}^{[n]}\}_{n=0}^{\infty}$. In particular, the systems in (2.17) and (2.18) provide two specific examples of nonlinear combined Liouville integrable Hamiltonian models.

4. Concluding Remarks

A Liouville four-component integrable hierarchy with Hamiltonian formulations has been generated from a specific special matrix eigenvalue problem. The success comes from the existence of a particular Laurent series solution of the corresponding stationary zero curvature equation. The resulting integrable models have been shown to be bi-Hamiltonian via the trace identity to the underlying matrix eigenvalue problem.

We point out that the case of $\delta = 0$ corresponds to integrable couplings and the variational identity is required to furnish a Hamiltonian formulation (see, e.g. Ref. 5 for details). It should be particularly interesting to explore mathematical structures of solitons to the presented integrable models. Powerful and effective approaches to solitons include the Riemann–Hilbert technique,[32] the determinant approach,[33] the Zakharov–Shabat dressing method,[34] and the Darboux transformation.[35–37] Besides solitons, lump, kink, breather and rogue wave solutions, including their interaction solutions (see, e.g. Refs. 38–43), are also interesting, and one can compute them from solitons by wave number reductions. On the other hand, considering nonlocal group reductions of matrix eigenvalue problems under similarity transformations, one can generate nonlocal reduced integrable models and study their solitons (see, e.g. Refs. 44–46).

Integrable models are of great interest. They significantly advance our understanding of complex nonlinear mathematical and physical problems (see, e.g. Ref. 47). They offer insights into the dynamic behavior of physical systems and have close connections to various areas of mathematics, including algebraic geometry, representation theory, and the theory of special functions.

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References

1. P. D. Lax, *Commun. Pure Appl. Math.* **21** (1968) 467.
2. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
3. A. Das, *Integrable Models* (World Scientific, Teaneck, NJ, 1989).
4. M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, *Stud. Appl. Math.* **53** (1974) 249.
5. W. X. Ma, Integrable couplings and matrix loop algebras, in *Nonlinear and Modern Mathematical Physics*, eds. W. X. Ma and D. Kaup, AIP Conference Proceedings, Vol. 1562 (American Institute of Physics, Melville, NY, 2013), pp. 105–122.
6. G. Z. Tu, *J. Phys. A: Math. Gen.* **22** (1989) 2375.
7. W. X. Ma, *Chin. Ann. Math. Ser. A* **13** (1992) 115.
8. M. Antonowicz and A. P. Fordy, *Physica D* **28** (1987) 345.
9. T. C. Xia, F. J. Yu and Y. Zhang, *Physica A* **343** (2004) 238.
10. S. Manukure, *Commun. Nonlinear Sci. Numer. Simul.* **57** (2018) 125.
11. T. S. Liu and T. C. Xia, *Nonlinear Anal. Real World Appl.* **68** (2022) 103667.
12. H. F. Wang and Y. F. Zhang, *J. Comput. Appl. Math.* **420** (2023) 114812.
13. W. X. Ma, *Phys. Lett. A* **367** (2007) 473.
14. W. X. Ma, *Appl. Math. Comput.* **220** (2013) 117.
15. V. S. Gerdjikov, *Turkish J. Math.* **46** (2022) 1828.
16. W. X. Ma, *Appl. Math. Lett.* **145** (2023) 108775.
17. W. X. Ma, *Rom. J. Phys.* **68** (2023) 116.
18. L. A. Takhtajan, *Phys. Lett. A* **64** (1977) 235.
19. D. J. Kaup and A. C. Newell, *J. Math. Phys.* **19** (1978) 798.
20. M. Wadati, K. Konno and Y. H. Ichikawa, *J. Phys. Soc. Jpn.* **47** (1979) 1698.
21. W. X. Ma, *Rom. Rep. Phys.* **75** (2023) 115.
22. W. X. Ma, *Commun. Nonlinear Sci. Numer. Simul.* **126** (2023) 107460.
23. W. X. Ma, *Theor. Math. Phys.* **216** (2023) 1180.
24. Y. F. Zhang, *Commun. Theor. Phys.* **55** (2011) 273.
25. Zhaqilao, *Commun. Nonlinear Sci. Numer. Simul.* **17** (2012) 2319.
26. W. X. Ma, *Acta Appl. Math.* **187** (2023) 17.
27. W. X. Ma, *Chin. J. Phys.* **86** (2023) 292.
28. W. X. Ma, *Mod. Phys. Lett. B* **37** (2023) 2350143.
29. B. Fuchssteiner and A. S. Fokas, *Physica D* **4** (1981) 47.
30. F. Magri, *J. Math. Phys.* **19** (1978) 1156.
31. W. X. Ma, *J. Phys. A: Math. Gen.* **26** (1993) 2573.
32. S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method* (Consultants Bureau, New York, NY, 1984).
33. T. Aktosun, T. Busse, F. Demontis and C. van der Mee, *J. Phys. A: Math. Theor.* **43** (2010) 025202.
34. E. V. Doktorov and S. B. Leble, *A Dressing Method in Mathematical Physics* (Springer, Dordrecht, 2007).
35. V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer-Verlag, Berlin, 1991).
36. X. G. Geng, R. M. Li and B. Xue, *J. Nonlinear Sci.* **30** (2020) 991.

37. R. S. Ye and Y. Zhang, *Chaos Solitons Fractals* **169** (2023) 113233.
38. L. Cheng, Y. Zhang and M. J. Lin, *Anal. Math. Phys.* **9** (2019) 1741.
39. T. A. Sulaiman, A. Yusuf, A. Abdeljabbar and M. Alquran, *J. Geom. Phys.* **169** (2021) 104347.
40. S. Manukure, A. Chowdhury and Y. Zhou, *Int. J. Mod. Phys. B* **33** (2019) 1950098.
41. Y. Zhou, S. Manukure and M. McAnally, *J. Geom. Phys.* **167** (2021) 104275.
42. W. X. Ma, *Mathematics* **11** (2023) 4664.
43. S. X. Yang, Y. F. Wang and X. Zhang, *Chaos Solitons Fractals* **169** (2023) 113272.
44. W. X. Ma, *Commun. Theor. Phys.* **74** (2022) 065002.
45. W. X. Ma, *Rep. Math. Phys.* **92** (2023) 19.
46. W. X. Ma, *Int. J. Geom. Methods Mod. Phys.* **20** (2023) 2350098.
47. E. Kengne and W. M. Liu, *Nonlinear Waves: From Dissipative Solitons to Magnetic Solitons* (Springer, Singapore, 2022).