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A six-component integrable hierarchy and its Hamiltonian formulation

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The aim of this paper is to construct a six-component integrable hierarchy associated with a matrix spatial spectral problem of arbitrary order. The adopted method is the zero curvature formulation. The corresponding Hamiltonian formulation is furnished by using the trace identity, which guarantees the Liouville integrability for the resulting hierarchy. Two illustrative examples of integrable equations of lower orders are six-component coupled nonlinear Schrödinger equations and modified Korteweg–de Vries equations.

Keywords: Matrix spectral problem; zero curvature equation; integrable hierarchy; Hamiltonian formulation; NLS equations; mKdV equations.

1. Introduction

The zero curvature formulation is a fundamental tool for generating integrable equations. It starts from a Lax pair of matrix spatial and temporal spectral problems[1] and the associated inverse scattering transform allows for the construction of solutions to Cauchy problems.[2,3]

Let us consider an n-dimensional potential: $u = (u_1, \dots, u_n)^T$ and assume that λ is the spectral parameter. A general procedure for constructing integrable equations in the zero curvature formulation is as follows (see Refs. 4 and 5). First, adopt a loop algebra $\tilde{g}$ to formulate a spectral matrix:

$$U = U(u, \lambda) = f_0(\lambda) + u_1 f_1(\lambda) + \cdots + u_n f_n(\lambda), \tag{1.1}$$

where $f_1, \ldots, f_n$ are linear independent elements in $\tilde{g}$ and f_0 is a pseudo-regular element in $\tilde{g}$:

$$\text{Ker ad}_{f_0} \oplus \text{Im ad}_{f_0} = \tilde{g}, \text{ and Ker ad}_{f_0} \text{ is commutative.}$$

This property guarantees that there exists a Laurent series solution $Y = \sum_{s=0}^{\infty} \lambda^{-s} Y^{[s]}$ to the stationary zero curvature equation:

$$Y_x = i[U, Y]. \tag{1.2}$$

Then, an integrable hierarchy can be presented through zero curvature equations:

$$U_{t_r} - V_x^{[r]} + i[U, V^{r]}] = 0, \quad r \geq 0, \tag{1.3}$$

which are the compatibility conditions between the spatial and temporal matrix spectral problems:

$$-i\phi_x = U\phi, \quad -i\phi_{t_r} = V^{[r]}\phi, \quad r \geq 0. \tag{1.4}$$

Hamiltonian structures of the resulting integrable equations could be furnished[4,5] by using the trace identity:

$$\frac{\delta}{\delta u} \int \text{tr}\left(Y \frac{\partial U}{\partial \lambda}\right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \text{tr}\left(Y \frac{\partial U}{\partial u}\right), \tag{1.5}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to u and γ is the constant determined by

$$\gamma = -\frac{\lambda}{2} \frac{\partial}{\partial \lambda} \ln |\, \text{tr}(Y^2)|. \tag{1.6}$$

Various integrable hierarchies are generated in such a way, on the basis of the special linear algebras (see Refs. 3 and 6–13), and the special orthogonal algebras (see Refs. 14–18). Bi-Hamiltonian formulations can often be furnished, which exhibit the Liouville integrability of the associated zero curvature equations.[19] Integrable hierarchies with two components, let us say p and q, are of great importance. The four well-known such integrable hierarchies are associated with the following spectral matrices:

$$U = \begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix}, \quad U = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix}, \quad U = \begin{bmatrix} \lambda & \lambda p \\ \lambda q & -\lambda \end{bmatrix}, \quad U = \begin{bmatrix} \lambda v & \lambda p \\ \lambda q & -\lambda v \end{bmatrix}, \tag{1.7}$$

where $pq + v^2 = 1$. The corresponding integrable hierarchies are the Ablowitz–Kaup–Newell–Segur hierarchy,[3] the Kaup–Newell hierarchy,[20] the Wadati–Konno–Ichikawa hierarchy[21] and the Heisenberg hierarchy,[22] respectively. The four counterparts of spectral matrices associated with so(3,$\mathbb{R}$) are

$$U = \begin{bmatrix} 0 & q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 \\ \lambda q & 0 & -\lambda p \\ \lambda^2 & \lambda p & 0 \end{bmatrix} \tag{1.8}$$

and

$$U = \begin{bmatrix} 0 & -\lambda q & -\lambda \\ \lambda q & 0 & -\lambda p \\ \lambda & \lambda p & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -\lambda q & -\lambda v \\ \lambda q & 0 & -\lambda p \\ \lambda v & \lambda p & 0 \end{bmatrix}, \tag{1.9}$$

where $p^2 + q^2 + v^2 = 1$ (see Refs. 23–26, respectively).

This paper aims to construct an integrable hierarchy of six-component equations associated with a matrix spectral problem of arbitrary order within the zero curvature formulation. By using the trace identity, we furnish a Hamiltonian formulation for the resulting hierarchy. Two illustrative examples of lower orders are six-component integrable coupled nonlinear Schrödinger equations and six-component integrable coupled modified Korteweg–de Vries equations, whose coefficients depend on numbers of copies of six potentials appeared in the spectral matrix. The final section provides a conclusion and some concluding remarks.

2. An Integrable Hierarchy with Six Potentials

Let us fix three natural numbers n_1, n_2 and n_3. To construct integrable equations within the zero curvature formulation, we need to pick up a matrix spectral problem, and in this paper, we begin with a matrix spectral problem of the following form:

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \ U = \begin{bmatrix} \lambda & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & 0 \\ \mathbf{w}_1 & 0 & 0 & 0 & \mathbf{v}_1^T \\ \mathbf{w}_2 & 0 & 0 & 0 & \mathbf{v}_2^T \\ \mathbf{w}_3 & 0 & 0 & 0 & \mathbf{v}_3^T \\ 0 & \mathbf{w}_1^T & \mathbf{w}_2^T & \mathbf{w}_3^T & -\lambda \end{bmatrix}, \tag{2.1}$$

where ϕ is the eigenfunction, λ is again the spectral parameter, u is the potential with six components:

$$u = u(x,t) = (v_1, v_2, v_3, w_1, w_2, w_3)^T, \tag{2.2}$$

and

$$\mathbf{v}_j = \underbrace{(v_j, \ldots, v_j)}, \quad \mathbf{w}_j = \underbrace{(w_j, \ldots, w_j)}^T, \quad 1 \le j \le 3. \tag{2.3}$$

The above spectral problem cannot be reduced from the matrix Ablowitz–Kaup–Newell–Segur spectral problem, though there are various meaningful reductions (see Refs. 27 and 28).

In order to construct an associated integrable hierarchy within the zero curvature formulation, we first solve the stationary zero curvature equation (1.2) among Laurent series matrices of the following form:

$$Y = \begin{bmatrix} a & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & 0 \\ \mathbf{c}_1 & 0 & d_1 E_{n_1,n_2} & d_2 E_{n_1,n_3} & \mathbf{b}_1^T \\ \mathbf{c}_2 & -d_1 E_{n_2,n_1} & 0 & d_3 E_{n_2,n_3} & \mathbf{b}_2^T \\ \mathbf{c}_3 & -d_2 E_{n_3,n_1} & -d_3 E_{n_3,n_2} & 0 & \mathbf{b}_3^T \\ 0 & \mathbf{c}_1^T & \mathbf{c}_2^T & \mathbf{c}_3^T & -a \end{bmatrix} = \sum_{s=0}^{\infty} \lambda^{-s} Y^{[s]}, \tag{2.4}$$

where

$$\mathbf{b}_j = (\underbrace{b_j, \ldots, b_j}_{n_j}), \quad \mathbf{c}_j = (\underbrace{c_j, \ldots, c_j}_{n_j})^T, \quad 1 \le j \le 3, \tag{2.5}$$

E_{n_j,n_k} is an $n_j \times n_k$ matrix with all entries being one, and we take Lauent expansions

$$\begin{gathered} a = \sum_{s=0}^{\infty} \lambda^{-s} a^{[s]}, \quad b_j = \sum_{s=0}^{\infty} \lambda^{-s} b_j^{[s]}, \quad c_j = \sum_{s=0}^{\infty} \lambda^{-s} c_j^{[s]}, \\ d_j = \sum_{s=0}^{\infty} \lambda^{-s} d_j^{[s]}, \quad 1 \le j \le 3. \end{gathered} \tag{2.6}$$

It is easy to check that the corresponding stationary zero curvature equation yields the initial conditions:

$$a_x^{[0]} = 0, \quad b_1^{[0]} = b_2^{[0]} = b_3^{[0]} = c_1^{[0]} = c_2^{[0]} = c_3^{[0]} = 0, \quad d_{1,x}^{[0]} = d_{2,x}^{[0]} = d_{3,x}^{[0]} = 0, \tag{2.7}$$

and the recursion relations:

$$\begin{cases} b_1^{[s+1]} = -ib_{1,x}^{[s]} + v_1 a^{[s]} + n_2 v_2 d_1^{[s]} + n_3 v_3 d_2^{[s]}, \\ b_2^{[s+1]} = -ib_{2,x}^{[s]} + v_2 a^{[s]} - n_1 v_1 d_1^{[s]} + n_3 v_3 d_3^{[s]}, \\ b_3^{[s+1]} = -ib_{3,x}^{[s]} + v_3 a^{[s]} - n_1 v_1 d_2^{[s]} - n_2 v_2 d_3^{[s]}, \end{cases} \tag{2.8}$$

$$\begin{cases} c_1^{[s+1]} = ic_{1,x}^{[s]} + w_1 a^{[s]} - n_2 w_2 d_1^{[s]} - n_3 w_3 d_2^{[s]}, \\ c_2^{[s+1]} = ic_{2,x}^{[s]} + w_2 a^{[s]} + n_1 w_1 d_1^{[s]} - n_3 w_3 d_3^{[s]}, \\ c_3^{[s+1]} = ic_{3,x}^{[s]} + w_3 a^{[s]} + n_1 w_1 d_2^{[s]} + n_2 w_2 d_3^{[s]}, \end{cases} \tag{2.9}$$

$$\begin{cases} d_{1,x}^{[s+1]} = i(w_1 b_2^{[s+1]} - w_2 b_1^{[s+1]} + v_1 c_2^{[s+1]} - v_2 c_1^{[s+1]}), \\ d_{2,x}^{[s+1]} = i(w_1 b_3^{[s+1]} - w_3 b_1^{[s+1]} + v_1 c_3^{[s+1]} - v_3 c_1^{[s+1]}), \\ d_{3,x}^{[s+1]} = i(w_2 b_3^{[s+1]} - w_3 b_2^{[s+1]} + v_2 c_3^{[s+1]} - v_3 c_2^{[s+1]}), \end{cases} \tag{2.10}$$

and

$$\begin{aligned} a_x^{[s+1]} &= i(-n_1w_1b_1^{[s+1]} - n_2w_2b_2^{[s+1]} - n_3w_3b_3^{[s+1]} + n_1v_1c_1^{[s+1]} \\ &\quad + n_2v_2c_2^{[s+1]} + n_3v_3c_3^{[s+1]}) \\ &= -(n_1w_1b_{1,x}^{[s]} + n_2w_2b_{2,x}^{[s]} + n_3w_3b_{3,x}^{[s]} + n_1v_1c_{1,x}^{[s]} + n_2v_2c_{2,x}^{[s]} + n_3v_3c_{3,x}^{[s]}), \end{aligned} \tag{2.11}$$

where $s \geq 0$. To have a unique Laurent series solution, we take the initial values

$$a^{[0]} = 1,\ d_1^{[0]} = d_2^{[0]} = d_3^{[0]} = 0 \tag{2.12}$$

and choose the constants of integration as zero

$$a^{[s]}|_{u=0} = 0, \quad d_1^{[s]}|_{u=0} = d_2^{[s]}|_{u=0} = d_3^{[s]}|_{u=0} = 0, \quad s \geq 1. \tag{2.13}$$

Then, we can uniquely determine that

$$\begin{cases} b_1^{[1]} = v_1, \quad b_2^{[1]} = v_2, \quad b_3^{[1]} = v_3, \\ c_1^{[1]} = w_1, \quad c_2^{[1]} = w_2, \quad c_3^{[1]} = w_3, \\ d_1^{[1]} = d_2^{[1]} = d_3^{[1]} = 0, \quad a^{[1]} = 0; \end{cases}$$

$$\begin{cases} b_1^{[2]} = -iv_{1,x}, \quad b_2^{[2]} = -iv_{2,x}, \quad b_3^{[2]} = -iv_{3,x}, \\ c_1^{[2]} = iw_{1,x}, \quad c_2^{[2]} = iw_{2,x}, \quad c_3^{[2]} = iw_{3,x}, \\ d_1^{[2]} = -v_1w_2 + v_2w_1, \quad d_2^{[2]} = -v_1w_3 + v_3w_1, \quad d_3^{[2]} = -v_2w_3 + v_3w_2, \\ a^{[2]} = -n_1v_1w_1 - n_2v_2w_2 - n_3v_3w_3; \end{cases}$$

$$\begin{cases} b_1^{[3]} = -v_{1,xx} + (-n_1v_1^2 + n_2v_2^2 + n_3v_3^2)w_1 - 2(n_2v_2w_2 + n_3v_3w_3)v_1, \\ b_2^{[3]} = -v_{2,xx} + (n_1v_1^2 - n_2v_2^2 + n_3v_3^2)w_2 - 2(n_1v_1w_1 + n_3v_3w_3)v_2, \\ b_3^{[3]} = -v_{3,xx} + (n_1v_1^2 + n_2v_2^2 - n_3v_3^2)w_3 - 2(n_1v_1w_1 + n_2v_2w_2)v_3, \end{cases}$$

$$\begin{cases} c_1^{[3]} = -w_{1,xx} + (-n_1w_1^2 + n_2w_2^2 + n_3w_3^2)v_1 - 2(n_2v_2w_2 + n_3v_3w_3)w_1, \\ c_2^{[3]} = -w_{2,xx} + (n_1w_1^2 - n_2w_2^2 + n_3w_3^2)v_2 - 2(n_1v_1w_1 + n_3v_3w_3)w_2, \\ c_3^{[3]} = -w_{3,xx} + (n_1w_1^2 + n_2w_2^2 - n_3w_3^2)v_3 - 2(n_1v_1w_1 + n_2v_2w_2)w_3, \end{cases}$$

$$\begin{cases} d_1^{[3]} = -i(v_1w_{2,x} - v_2w_{1,x} - v_{1,x}w_2 + v_{2,x}w_1), \\ d_2^{[3]} = -i(v_1w_{3,x} - v_3w_{1,x} - v_{1,x}w_3 + v_{3,x}w_1), \\ d_3^{[3]} = -i(v_2w_{3,x} - v_3w_{2,x} - v_{2,x}w_3 + v_{3,x}w_2), \end{cases}$$

$$a^{[3]} = -i(n_1v_1w_{1,x} - n_1v_{1,x}w_1 + n_2v_2w_{2,x} - n_2v_{2,x}w_2 + n_3v_3w_{3,x} - n_3v_{3,x}w_3);$$

and

$$\begin{cases} b_1^{[4]} = i(v_{1,xxx} + 3n_1v_1v_{1,x}w_1 + 3n_2v_1v_{2,x}w_2 + 3n_3v_1v_{3,x}w_3 \\ \qquad\quad - 3n_2v_2v_{2,x}w_1 - 3n_3v_3v_{3,x}w_1 + 3n_2v_{1,x}v_2w_2 + 3n_3v_{1,x}v_3w_3), \\ b_2^{[4]} = i(v_{2,xxx} + 3n_1v_1v_{2,x}w_1 - 3n_1v_1v_{1,x}w_2 + 3n_1v_{1,x}v_2w_1 \\ \qquad\quad + 3n_2v_2v_{2,x}w_2 + 3n_3v_2v_{3,x}w_3 + 3n_3v_{2,x}v_3w_3 - 3n_3v_3v_{3,x}w_2), \\ b_3^{[4]} = i(v_{3,xxx} + 3n_1v_1v_{3,x}w_1 - 3n_1v_1v_{1,x}w_3 + 3n_1v_{1,x}v_3w_1 \\ \qquad\quad - 3n_2v_2v_{2,x}w_3 + 3n_2v_2v_{3,x}w_2 + 3n_2v_{2,x}v_3w_2 + 3n_3v_3v_{3,x}w_3), \end{cases}$$

$$\begin{cases} c_1^{[4]} = -i(w_{1,xxx} + 3n_1v_1w_1w_{1,x} - 3n_2v_1w_2w_{2,x} - 3n_3v_1w_3w_{3,x} \\ \qquad\quad + 3n_2v_2w_{1,x}w_2 + 3n_2v_2w_1w_{2,x} + 3n_3v_3w_{1,x}w_3 + 3n_3v_3w_1w_{3,x}), \\ c_2^{[4]} = -i(w_{2,xxx} - 3n_1v_2w_1w_{1,x} + 3n_1v_1w_1w_{2,x} + 3n_1v_1w_{1,x}w_2 \\ \qquad\quad + 3n_2v_2w_2w_{2,x} - 3n_3v_2w_3w_{3,x} + 3n_3v_3w_2w_{3,x} + 3n_3v_3w_{2,x}w_3), \\ c_3^{[4]} = -i(w_{3,xxx} + 3n_1v_1w_1w_{3,x} + 3n_1v_1w_{1,x}w_3 - 3n_1v_3w_1w_{1,x} \\ \qquad\quad + 3n_2v_2w_2w_{3,x} + 3n_2v_2w_{2,x}w_3 - 3n_2v_3w_2w_{2,x} + 3n_3v_3w_3w_{3,x}), \end{cases}$$

$$\begin{cases} d_1^{[4]} = 3(n_1v_1w_1 + n_2v_2w_2 + n_3v_3w_3)(v_1w_2 - v_2w_1) + v_{1,xx}w_2 - v_{2,xx}w_1 \\ \qquad\quad - v_2w_{1,xx} + v_1w_{2,xx} - v_{1,x}w_{2,x} + v_{2,x}w_{1,x}, \\ d_2^{[4]} = 3(n_1v_1w_1 + n_2v_2w_2 + n_3v_3w_3)(v_1w_3 - v_3w_1) + v_{1,xx}w_3 - v_{3,xx}w_1 \\ \qquad\quad - v_3w_{1,xx} + v_1w_{3,xx} - v_{1,x}w_{3,x} + v_{3,x}w_{1,x}, \\ d_3^{[4]} = 3(n_1v_1w_1 + n_2v_2w_2 + n_3v_3w_3)(v_2w_3 - v_3w_2) + v_{2,xx}w_3 - v_{3,xx}w_2 \\ \qquad\quad - v_3w_{2,xx} + v_2w_{3,xx} - v_{2,x}w_{3,x} + v_{3,x}w_{2,x}, \end{cases}$$

$$\begin{aligned} a^{[4]} = {} & \frac{3}{2}n_1(n_1w_1^2 - n_2w_2^2 - n_3w_3^2)v_1^2 - \frac{3}{2}n_2(n_1w_1^2 - n_2w_2^2 + n_3w_3^2)v_2^2 \\ & - \frac{3}{2}n_3(n_1w_1^2 + n_2w_2^2 - n_3w_3^2)v_3^2 + 6n_1v_1(n_2v_2w_2 + n_3v_3w_3)w_1 \\ & + 6n_2n_3v_2v_3w_2w_3 + n_1v_1w_{1,xx} + n_1v_{1,x,x}w_1 + n_2v_2w_{2,xx} + n_2v_{2,xx}w_2 \\ & + n_3v_3w_{3,xx} + n_3v_{3,xx}w_3 - n_1v_{1,x}w_{1,x} - n_2v_{2,x}w_{2,x} - n_3v_{3,x}w_{3,x}. \end{aligned}$$

At this moment, we notice that we can assume the temporal matrix spectral problems to be of the following form:

$$-i\phi_{t_r} = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi, \quad V^{[r]} = (\lambda^r Y)_+ = \sum_{s=0}^{r} \lambda^s Y^{[r-s]}, \quad r \geq 0, \tag{2.14}$$

which are the other parts of Lax pairs of matrix spectral problems in the zero curvature formulation. The compatibility conditions of the spatial and temporal matrix spectral problems in (2.1) and (2.14) are exactly the zero curvature equations in (1.3).

These equations generate a six-component integrable hierarchy:

$$u_{t_r} = X^{[r]} = (ib_1^{[r+1]}, ib_2^{[r+1]}, ib_3^{[r+1]}, -ic_1^{[r+1]}, -ic_2^{[r+1]}, -ic_3^{[r+1]})^T, \quad r \geq 0 \tag{2.15}$$

or more concretely,

$$\begin{cases} v_{1,t_r} = ib_1^{[r+1]}, \quad v_{2,t_r} = ib_2^{[r+1]}, \quad v_{3,t_r} = ib_3^{[r+1]}, \\ w_{1,t_r} = -ic_1^{[r+1]}, \quad w_{2,t_r} = -ic_2^{[r+1]}, \quad w_{3,t_r} = -ic_3^{[r+1]}, \end{cases} r \geq 0. \tag{2.16}$$

The first two nonlinear examples in this integrable hierarchy are the integrable coupled nonlinear Schrödinger equations:

$$\begin{cases} iv_{1,t_2} = v_{1,xx} + (n_1 v_1^2 - n_2 v_2^2 - n_3 v_3^2) w_1 + 2(n_2 v_2 w_2 + n_3 v_3 w_3) v_1, \\ iv_{2,t_2} = v_{2,xx} - (n_1 v_1^2 - n_2 v_2^2 + n_3 v_3^2) w_2 + 2(n_1 v_1 w_1 + n_3 v_3 w_3) v_2, \\ iv_{3,t_2} = v_{3,xx} - (n_1 v_1^2 + n_2 v_2^2 - n_3 v_3^2) w_3 + 2(n_1 v_1 w_1 + n_2 v_2 w_2) v_3 \end{cases} \tag{2.17}$$

and

$$\begin{cases} iw_{1,t_2} = -w_{1,xx} + (-n_1 w_1^2 + n_2 w_2^2 + n_3 w_3^2) v_1 - 2(n_2 v_2 w_2 + n_3 v_3 w_3) w_1, \\ iw_{2,t_2} = -w_{2,xx} + (n_1 w_1^2 - n_2 w_2^2 + n_3 w_3^2) v_2 - 2(n_1 v_1 w_1 + n_3 v_3 w_3) w_2, \\ iw_{3,t_2} = -w_{3,xx} + (n_1 w_1^2 + n_2 w_2^2 - n_3 w_3^2) v_3 - 2(n_1 v_1 w_1 + n_2 v_2 w_2) w_3 \end{cases} \tag{2.18}$$

and the integrable coupled modified Korteweg–de Vries equations:

$$\begin{cases} v_{1,t_3} = -v_{1,xxx} - 3n_1 v_1 v_{1,x} w_1 - 3n_2 v_1 v_{2,x} w_2 - 3n_3 v_1 v_{3,x} w_3 \\ \qquad + 3n_2 v_2 v_{2,x} w_1 + 3n_3 v_3 v_{3,x} w_1 - 3n_2 v_{1,x} v_2 w_2 - 3n_3 v_{1,x} v_3 w_3, \\ v_{2,t_3} = -v_{2,xxx} - 3n_1 v_1 v_{2,x} w_1 + 3n_1 v_1 v_{1,x} w_2 - 3n_1 v_{1,x} v_2 w_1 \\ \qquad - 3n_2 v_2 v_{2,x} w_2 - 3n_3 v_2 v_{3,x} w_3 - 3n_3 v_{2,x} v_3 w_3 + 3n_3 v_3 v_{3,x} w_2, \\ v_{3,t_3} = -v_{3,xxx} - 3n_1 v_1 v_{3,x} w_1 + 3n_1 v_1 v_{1,x} w_3 - 3n_1 v_{1,x} v_3 w_1 \\ \qquad + 3n_2 v_2 v_{2,x} w_3 - 3n_2 v_2 v_{3,x} w_2 - 3n_2 v_{2,x} v_3 w_2 - 3n_3 v_3 v_{3,x} w_3 \end{cases} \tag{2.19}$$

and

$$\begin{cases} w_{1,t_3} = -w_{1,xxx} - 3n_1 v_1 w_1 w_{1,x} + 3n_2 v_1 w_2 w_{2,x} + 3n_3 v_1 w_3 w_{3,x} \\ \qquad - 3n_2 v_2 w_{1,x} w_2 - 3n_2 v_2 w_1 w_{2,x} - 3n_3 v_3 w_{1,x} w_3 - 3n_3 v_3 w_1 w_{3,x}, \\ w_{2,t_3} = -w_{2,xxx} + 3n_1 v_2 w_1 w_{1,x} - 3n_1 v_1 w_1 w_{2,x} - 3n_1 v_1 w_{1,x} w_2 \\ \qquad - 3n_2 v_2 w_2 w_{2,x} + 3n_3 v_2 w_3 w_{3,x} - 3n_3 v_3 w_2 w_{3,x} - 3n_3 v_3 w_{2,x} w_3, \\ w_{3,t_3} = -w_{3,xxx} - 3n_1 v_1 w_1 w_{3,x} - 3n_1 v_1 w_{1,x} w_3 + 3n_1 v_3 w_1 w_{1,x} \\ \qquad - 3n_2 v_2 w_2 w_{3,x} - 3n_2 v_2 w_{2,x} w_3 + 3n_2 v_3 w_2 w_{2,x} - 3n_3 v_3 w_3 w_{3,x}. \end{cases} \tag{2.20}$$

The coefficients in these systems depend on three arbitrary numbers, n_1, n_2 and n_3, and so we can have abundant novel integrable coupled nonlinear Schrödinger equations and modified Korteweg–de Vries equations with six potentials.

3. Hamiltonian Formulation

To establish a Hamiltonian formulation for the integrable hierarchy (2.15), we apply the trace identity (1.5) to the matrix spatial spectral problem (2.1). Using the solution Y given by (2.4), we can directly compute that

$$\operatorname{tr}\left(Y\frac{\partial U}{\partial \lambda}\right)=2a,\quad \operatorname{tr}\left(Y\frac{\partial U}{\partial u}\right)=2(n_1c_1,n_2c_2,n_3c_3,n_1b_1,n_2b_2,n_3b_3)^T, \tag{3.1}$$

and consequently, we have

$$\frac{\delta}{\delta u}\int \lambda^{-s-1}a^{[s+1]}dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma-s}(n_1c_1^{[s]},n_2c_2^{[s]},n_3c_3^{[s]},n_1b_1^{[s]},n_2b_2^{[s]},n_3b_3^{[s]})^T,\quad s\geq 0. \tag{3.2}$$

Upon considering the case with $s=2$, we obtain $\gamma=0$, and further, we arrive at

$$\frac{\delta}{\delta u}\mathcal{H}^{[s]}=(n_1c_1^{[s+1]},n_2c_2^{[s+1]},n_3c_3^{[s+1]},n_1b_1^{[s+1]},n_2b_2^{[s+1]},n_3b_3^{[s+1]})^T,\quad s\geq 0, \tag{3.3}$$

where based on (3.2), the Hamiltonian functionals are given by

$$\mathcal{H}^{[s]}=-\int\frac{a^{[s+2]}}{s+1}dx,\quad s\geq 0. \tag{3.4}$$

This allows us to furnish the Hamiltonian formulation for the integrable hierarchy (2.15) as follows:

$$u_{t_r}=X^{[r]}=J\frac{\delta\mathcal{H}^{[r]}}{\delta u},\quad J=\left[\begin{array}{ccc|ccc} & & & \frac{1}{n_1}i & 0 & 0\\ & 0 & & 0 & \frac{1}{n_2}i & 0\\ & & & 0 & 0 & \frac{1}{n_3}i\\ \hline -\frac{1}{n_1}i & 0 & 0 & & & \\ 0 & -\frac{1}{n_2}i & 0 & & 0 & \\ 0 & 0 & -\frac{1}{n_3}i & & & \end{array}\right],\quad r\geq 0, \tag{3.5}$$

where J is a Hamiltonian operator, i.e., $\{\mathcal{F},\mathcal{G}\}_J=\int(\frac{\delta\mathcal{F}}{\delta u})^TJ\frac{\delta\mathcal{G}}{\delta u}dx$ defines a Poisson bracket, and the Hamiltonian functionals $\mathcal{H}^{[r]}$ are defined by (3.4). This Hamiltonian formulation exhibits a relation $S=J\frac{\delta\mathcal{H}}{\delta u}$ from a conserved functional $\mathcal{H}$ to a symmetry S. The commuting property of these symmetries:

$$[[X^{[s_1]},X^{[s_2]}]]=X^{[s_1]\prime}(u)[X^{[s_2]}]-X^{[s_2]\prime}(u)[X^{[s_1]}]=0,\quad s_1,s_2\geq 0, \tag{3.6}$$

comes from a Lax operator algebra:

$$[[V^{[s_1]},V^{[s_2]}]]=V^{[s_1]\prime}(u)[X^{[s_2]}]-V^{[s_2]\prime}(u)[X^{[s_1]}]+[V^{[s_1]},V^{[s_2]}]=0,\quad s_1,s_2\geq 0, \tag{3.7}$$

which is guaranteed by an algebraic structure associated with the isospectral zero curvature equations (see Ref. 29 for details). Further, a consequence of the Hamiltonian formulation is that the conserved functionals also commute under the corresponding Poisson bracket:

$$\{\mathcal{H}^{[s_1]}, \mathcal{H}^{[s_2]}\}_J = \int \left(\frac{\delta \mathcal{H}^{[s_1]}}{\delta u}\right)^T J \frac{\delta \mathcal{H}^{[s_2]}}{\delta u} dx = 0, \quad s_1, s_2 \geq 0. \tag{3.8}$$

By combining J with a recursion operator Φ,[30] generated from $K^{[s+1]} = \Phi K^{[s]}$, a bi-Hamiltonian formulation[19] can also be furnished for the integrable hierarchy (2.16). This ensures the Liouville integrability of every member in the hierarchy.

4. Concluding Remarks

An integrable hierarchy of Hamiltonian equations with six components had been generated from a matrix spectral problem of arbitrary order within the zero curvature formulation. It is crucial to determine a Laurent series solution to the corresponding stationary zero curvature equation. The resulting integrable hierarchy possesses a Hamiltonian formulation, furnished by the trace identity, which exhibits its Liouville integrability.

It would be of great importance to explore structures of soliton solutions to the resulting integrable equations. The Riemann–Hilbert technique,[31] the Zakharov–Shabat dressing method,[32] the Darboux transformation[33,34] and the determinant approach[35,36] could be helpful. Taking wave number reductions of soliton solutions can lead to other types of interesting solutions, including lump and breather wave solutions (see Refs. 37–40). Conducting nonlocal group reductions of matrix spectral problems will yield local and nonlocal reduced integrable equations (see Refs. 28 and 41–44, respectively). It needs further investigation to determine mathematical structures of soliton solutions to the resulting new integrable equations and their corresponding nonlocal integrable counterparts, even with numerical approaches that are efficient for solving algebraic-differential equations and fractional differential equations (see Refs. 45 and 46).

We also remark that our matrix spectral problem could be generalized further by including conjugate copies of v_j, $1 \leq j \leq 3$, or involving more potentials. Generalized matrix spectral problems will lead to more general integrable coupled equations (see Refs. 47 and 48), including integrable couplings (see Refs. 17, 49 and 50).

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