Modern Physics Letters B
Vol. 36, No. 20 (2022) 2250094 (13 pages)

DOI: 10.1142/S0217984922500944

A novel kind of reduced integrable matrix mKdV equations and their binary Darboux transformations

Wen-Xiu Ma

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China

Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA

School of Mathematical and Statistical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa
mawx@cas.usf.edu

Received 26 February 2022
Revised 29 March 2022
Accepted 29 March 2022
Published 14 June 2022

A novel kind of reduced integrable matrix mKdV equations is generated from a group reduction replacing the spectral parameter λ with $-\lambda$ in the matrix AKNS spectral problems. The traditional replacement in making integrable reductions is to replace the spectral parameter λ with its complex conjugate λ^*. Binary Darboux transformations are constructed by use of Lax pairs and adjoint Lax pairs, and thus, soliton solutions are presented for the resulting reduced integrable matrix mKdV equations from the zero seed solution.

Keywords: Matrix spectral problem; group reduction; zero curvature equation; binary Darboux transformation; soliton solution.

PACS Number(s): 05.45.Yv, 02.30.Ik

1. Introduction

There exist a few efficient and effective methods to compute analytical solutions to integrable equations in soliton theory. Those include the inverse scattering transform, the Darboux transformation (DT), the Hirota bilinear method, the Painlevé test and the Riemann–Hilbert technique.[1–3] The DT is one of the most direct approaches for generating soliton solutions to integrable equations.[4,5] A pair of matrix spectral problems, called a Lax pair, plays a crucial role in formulating DTs (see, for example, Ref. 6), which is also the basis for establishing Riemann–Hilbert problems and the inverse scattering transform.[1–3] A binary DT is constructed from both a pair of matrix spectral problems being equivalent to a given integrable equation, and another pair of adjoint matrix spectral problems being equivalent to the same given equation, called an adjoint Lax pair.

Let x and t be two independent variables, and $u = u(x,t)$, a column vector of dependent variables. A Lax pair of spatial and temporal matrix spectral problems reads

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \quad -i\phi_t = V\phi = V(u,\lambda)\phi, \tag{1.1}$$

where i stands for the unit imaginary number, λ is the spectral parameter, U and V are square matrices, and ϕ is a column eigenfunction. We assume that such a Lax pair leads equivalently to an integrable equation

$$u_t = K(u), \tag{1.2}$$

via the zero curvature equation

$$U_t - V_x + i[U,V] = 0, \tag{1.3}$$

where $[\cdot,\cdot]$ denotes the matrix commutator, i.e. $[U,V] = UV - VU$. This is the compatibility condition of the involved two spectral problems. Such Lax pairs also possess beautiful algebraic structures, which guarantee the integrability of the associated nonlinear equations.[7,8] The adjoint Lax pair of the matrix spectral problems in (1.1) reads as follows:

$$i\tilde{\phi}_x = \tilde{\phi}U = \tilde{\phi}U(u,\lambda), \quad i\tilde{\phi}_t = \tilde{\phi}V = \tilde{\phi}V(u,\lambda). \tag{1.4}$$

The corresponding compatibility condition produces the same zero curvature equation as (1.3), i.e. it does not generate any new equations, and thus, we can use both the Lax pair and the adjoint Lax pair simultaneously.

Based on a Lax pair of matrix spectral problems, a reduced integrable equation can be presented, if the corresponding reduced zero curvature equation still holds while a group reduction is taken for the spectral matrix U. One example of such reductions takes the form

$$U^\dagger(x,t,\lambda^*) = (U(x,t,\lambda^*))^\dagger = CU(x,t,\lambda)C^{-1}, \tag{1.5}$$

where C is a constant Hermitian matrix and λ^* denotes the complex conjugate of λ (see, for example, Ref. 9). In the above condition, the key is to replace the spectral parameter λ with λ^* in the spectral matrix U, which works for the nonlinear Schrödinger (NLS) equations and the modified Korteweg–de Vries (mKdV) equations. In this paper, we would like to present another example of possible reductions which work for the mKdV equations. We will replace the spectral parameter λ with $-\lambda$ and so introduce

$$U^T(x,t,-\lambda) = (U(x,t,-\lambda))^T = -CU(x,t,\lambda)C^{-1}, \tag{1.6}$$

in the matrix Ablowitz–Kaup–Newell–Segur (AKNS) spectral problems, to reduce the mKdV equations. Unfortunately, this reduction idea does not work for the NLS equations. It is known that the other two replacements $\lambda \to -\lambda^*$ and $\lambda \to \lambda$ (i.e. no change) could only produce nonlocal integrable reductions, together with the reflection transformations of the independent variables: $x \to -x$, $t \to -t$ and $(x,t) \to (-x,-t)$ (see, for example, Ref. 10).

By taking advantage of a Lax pair, a binary DT can be determined for an associated integrable equation by

$$\phi' = T^+\phi = T^+(u,\lambda)\phi, \quad \tilde{\phi}' = \tilde{\phi}T^- = \tilde{\phi}T^-(u,\lambda), \tag{1.7}$$

where $T^- = (T^+)^{-1}$, provided that ϕ' and $\tilde{\phi}'$ satisfy the new matrix spectral problems:

$$-i\phi'_x = U'\phi', \quad -i\phi'_t = V'\phi', \tag{1.8}$$

and the new adjoint matrix spectral problems:

$$i\tilde{\phi}'_x = \tilde{\phi}'U', \quad i\tilde{\phi}'_t = \tilde{\phi}'V', \tag{1.9}$$

where the new Lax of spectral matrices is defined by

$$U' = U(u',\lambda), \quad V' = V(u',\lambda), \quad u' = f(u). \tag{1.10}$$

The above condition for producing a binary DT just requires that the Darboux matrices T^+ and T^- should satisfy

$$-iT^+_xT^- + T^+UT^- = U', \quad -iT^+_tT^- + T^+VT^- = V'. \tag{1.11}$$

Obviously, either (1.8) or (1.9) guarantees that the new Lax pairs, U' and V', generate the same zero curvature Eq. (1.3) with u replaced with u'. Thus, u' produces a new solution to the same integrable equation when so does u. That is to say, $u' = f(u)$ presents a Bäcklund transformation of the associated integrable equation. There exist abundant examples of binary DTs for scalar or multi-component integrable equations (see, for example, Refs. 4 and 11–17), though very few examples for non-commutative cases, such as cases producing matrix integrable equations, in the existing literature (see, for example, Refs. 18–20).

In this paper, we would like to construct a novel kind of integrable reductions of the general integrable matrix mKdV equations, and binary DTs for the resulting reduced integrable matrix mKdV equations, beginning with a Lax pair of arbitrary-order matrix AKNS spectral problems. The resulting binary DTs also possess an N-fold Darboux characteristics, in the regular case where eigenvalues are different from adjoint eigenvalues. Upon taking the zero seed solution, the obtained binary DTs generate soliton solutions to the reduced integrable matrix mKdV equations. A conclusion and several concluding remarks will be given in Sec. 5.

2. Reduced Integrable Matrix mKdV Equations

Let us recall the general integrable matrix mKdV equations (see, for example, Ref. 21). Assume that $m, n \geq 0$ are two arbitrary integers, and I_s denotes the identity matrix of size s ($s \in \mathbb{N}$). We consider a Lax pair of matrix AKNS spectral problems:

$$-i\phi_x = U\phi = U(p,q;\lambda)\phi, \quad -i\phi_t = V\phi = V(p,q;\lambda)\phi, \tag{2.1}$$

where

$$p = (p_{jl})_{m\times n}, \quad q = (q_{lj})_{n\times m}, \tag{2.2}$$

are two matrices of dependent variables, and

$$U = \lambda\Lambda + P, \quad V = \lambda^3\Omega + Q, \tag{2.3}$$

are two spectral matrices. The involved square matrices, Λ, Ω, P and Q, read

$$\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n), \tag{2.4}$$

$$\Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \tag{2.5}$$

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.6}$$

$$\begin{aligned} Q = Q(u,\lambda) &= \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P,P_x] + P_{xx} + 2P^3) \\ &= \frac{\beta}{\alpha}\lambda^2 \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2}\lambda \begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix} \\ &\quad - \frac{\beta}{\alpha^3}\begin{bmatrix} i(pq_x - p_x q) & p_{xx} + 2pqp \\ q_{xx} + 2qpq & i(qp_x - q_x p) \end{bmatrix}, \end{aligned} \tag{2.7}$$

where α_1, α_2 and β_1, β_2 are two pairs of different constants, $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ and $I_{m,n} = \mathrm{diag}(I_m, -I_n)$.

With only one nonzero pair (p_{jl}, q_{lj}) ($1 \leq j, l \leq n$), the spatial spectral problem in (2.1) becomes the standard AKNS spectral problem.[22] Because of the existence of a multiple eigenvalue of Λ, the spatial matrix spectral problem in (2.1) with matrix potentials, p and q, is degenerate.

Now, it is direct to see that the compatibility condition of the matrix spectral problems in (2.1) yields the following integrable matrix mKdV equations:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_x qp), \quad q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_x pq + 3qpq_x). \tag{2.8}$$

When $m = 1$ and $n = 1$, we can have

$$\begin{cases} p_{11,t} = p_{11,xxx} + 6p_{11}q_{11}p_{11,x}, \\ q_{11,t} = q_{11,xxx} + 6p_{11}q_{11}q_{11,x}. \end{cases} \tag{2.9}$$

When $m = 2$ and $n = 1$, we can obtain

$$\begin{cases} p_{j1,t} = p_{j1,xxx} + 3(p_{11}q_{11} + p_{21}q_{12})p_{j1,x} + 3(p_{11,x}q_{11} + p_{21,x}q_{12})p_{j1}, \\ q_{1j,t} = q_{1j,xxx} + 3(p_{11}q_{11} + p_{21}q_{12})q_{1j,x} + 3(p_{11}q_{11,x} + p_{21}q_{12,x})q_{1j}, \end{cases} \tag{2.10}$$

where $1 \leq j \leq 2$. When $m = 2$ and $n = 2$, we can get

$$\begin{cases} p_{jl,t} = p_{jl,xxx} + 3\sum_{r,s=1}^{2} p_{jr}q_{rs}p_{sl,x} + 3\sum_{r,s=1}^{2} p_{jr,x}q_{rs}p_{sl}, \\ q_{lj,t} = q_{lj,xxx} + 3\sum_{r,s=1}^{2} q_{lr,x}p_{rs}q_{sj} + 3\sum_{r,s=1}^{2} q_{lr}p_{rs}q_{sj,x}, \end{cases} \tag{2.11}$$

where $1 \leq j, l \leq 2$. In these three examples, we have taken $\alpha = -\beta = 1$.

Let us then make integrable reductions of the general integrable matrix mKdV equations (2.8). We take two constant invertible symmetric matrices Σ_1, Σ_2 and introduce a particular reduction for the spectral matrix U defined in (2.3):

$$U^T(x,t,-\lambda) = (U(x,t,-\lambda))^T = -CU(x,t,\lambda)C^{-1}, \tag{2.12}$$

where C is a constant square matrix by

$$C = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Sigma_j^T = \Sigma_j, \quad j = 1, 2. \tag{2.13}$$

This group reduction precisely requires

$$P^T(x,t) = -CP(x,t)C^{-1}, \tag{2.14}$$

which leads equivalently to the reduction for the potential matrices:

$$q(x,t) = -\Sigma_2^{-1}p^T(x,t)\Sigma_1. \tag{2.15}$$

Moreover, it is direct to see that such a reduction in (2.12) guarantees that

$$V^T(x,t,-\lambda) = (V(x,t,-\lambda))^T = -CV(x,t,\lambda)C^{-1} \tag{2.16}$$

and

$$Q^T(x,t,-\lambda) = (Q(x,t,-\lambda))^T = -CQ(x,t,\lambda)C^{-1}, \tag{2.17}$$

where V and Q are given by (2.3) and (2.7). Therefore, under the reduction (2.15), the matrix integrable mKdV equations (2.8) reduce to the following reduced matrix mKdV equations:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} - 3p\Sigma_2^{-1}p^T\Sigma_1p_x - 3p_x\Sigma_2^{-1}p^T\Sigma_1p), \tag{2.18}$$

where $p = (p_{jl})_{m\times n}$, and Σ_1, Σ_2 are two arbitrary invertible symmetric matrices of sizes m and n, respectively. This system possesses a Lax pair of the reduced spatial and temporal matrix spectral problems of (2.1).

When $n = 1$, taking $\alpha = -\beta = 1$ and $\Sigma_1 = 1, \Sigma_2 = \sigma^{-1}, \sigma = \pm 1$, we obtain the two integrable scalar mKdV equations:

$$p_{11,t} = p_{11,xxx} + 6\sigma p_{11}^2 p_{11,x}, \quad \sigma = \pm 1. \tag{2.19}$$

When $m = 2$ and $n = 1$, we can get the following four new systems of integrable two-component mKdV equations:

$$\begin{cases} p_{11,t} = p_{11,xxx} + d_1(c_1 p_{11}^2 + c_2 p_{21}^2)p_{11,x} + d_1(c_1 p_{11} p_{11,x} + c_2 p_{21} p_{21,x})p_{11}, \\ p_{21,t} = p_{21,xxx} + d_2(c_1 p_{11}^2 + c_2 p_{21}^2)p_{21,x} + d_2(c_1 p_{11} p_{11,x} + c_2 p_{21} p_{21,x})p_{21}, \end{cases} \tag{2.20}$$

$$\begin{cases} p_{11,t} = p_{11,xxx} + (c_1 p_{11}^2 + c_2 p_{21}^2)p_{21,x} + (c_1 p_{11} p_{11,x} + c_2 p_{21} p_{21,x})p_{21}, \\ p_{21,t} = p_{21,xxx} + (c_1 p_{11}^2 + c_2 p_{21}^2)p_{11,x} + (c_1 p_{11} p_{11,x} + c_2 p_{21} p_{21,x})p_{11}, \end{cases} \tag{2.21}$$

$$\begin{cases} p_{11,t} = p_{11,xxx} + d_1(3p_{21} p_{11,x} + p_{11} p_{21,x})p_{11}, \\ p_{21,t} = p_{21,xxx} + d_2(p_{21} p_{11,x} + 3p_{11} p_{21,x})p_{21}, \end{cases} \tag{2.22}$$

and

$$\begin{cases} p_{11,t} = p_{11,xxx} + d_1(3p_{11} p_{21,x} + p_{21} p_{11,x})p_{21}, \\ p_{21,t} = p_{21,xxx} + d_1(p_{21} p_{11,x} + 3p_{21,x} p_{11})p_{11}, \end{cases} \tag{2.23}$$

where $c_j, d_j, 1 \le j \le 2$, are arbitrary nonzero real constants. The system (2.20) and the system in (2.22) contain the two systems of mKdV equations presented in Ref. 23. One system of such mixed type mKdV equations of (2.20) has been solved by the inverse scattering transform.[24] When $m = 2$ and $n = 2$, we can generate a more general system of integrable mKdV equations

$$p_{jl,t} = p_{jl,xxx} + \sum_{r,s=1}^{2} c_r d_s p_{jr} p_{sr} p_{sl,x} + \sum_{r,s=1}^{2} c_r d_s p_{jr,x} p_{sr} p_{sl}, \tag{2.24}$$

where $1 \le j, l \le 2$, and $c_j, d_j, 1 \le j \le 2$, are arbitrary nonzero real constants.

3. Binary Darboux Transformations

Let us start to formulate Darboux matrices in a general case, where eigenvalues could be equal to adjoint eigenvalues.

Let $N \ge 1$ be another arbitrary integer. First, we take two sets of eigenvalues and adjoint eigenvalues:

$$\lambda_k \in \mathbb{C}, \quad 1 \le k \le N, \quad \text{and} \quad \hat{\lambda}_k = -\lambda_k, \quad 1 \le k \le N, \tag{3.1}$$

and two sets of the associated eigenfunctions and adjoint eigenfunctions:

$$-iv_{k,x} = U(p, q; \lambda_k)v_k, \quad -iv_{k,t} = V(p, q; \lambda_k)v_k, \quad 1 \le k \le N \tag{3.2}$$

and

$$i\hat{v}_{k,x} = \hat{v}_k U(p, q; \hat{\lambda}_k), \quad i\hat{v}_{k,t} = \hat{v}_k V(p, q; \hat{\lambda}_k), \quad 1 \le k \le N. \tag{3.3}$$

For the sake of convenience, we introduce

$$v = (v_1, \dots, v_N), \quad \hat{v} = (\hat{v}_1^T, \dots, \hat{v}_N^T)^T \tag{3.4}$$

and

$$A = \mathrm{diag}(\lambda_1, \dots, \lambda_N), \quad \hat{A} = \mathrm{diag}(\hat{\lambda}_1, \dots, \hat{\lambda}_N). \tag{3.5}$$

Then the equations for the eigenfunctions read

$$\begin{cases} -iv_x = \Lambda v A + P v, \\ i\hat{v}_x = \hat{A}\hat{v}\Lambda + \hat{v}P \end{cases} \tag{3.6}$$

and

$$\begin{cases} -iv_t = \Omega v A^3 + (Q(\lambda_1)v_1, \dots, Q(\lambda_N)v_N), \\ i\hat{v}_t = \hat{A}^3\hat{v}\Omega + (\hat{v}_1 Q(\hat{\lambda}_1), \dots, \hat{v}_N Q(\hat{\lambda}_N)), \end{cases} \tag{3.7}$$

where the four matrices Λ, Ω, P and Q are defined by (2.4)–(2.7).

To present Darboux matrices in a general situation, where eigenvalues could equal adjoint eigenvalues, we introduce a square matrix $M = (m_{kl})_{N\times N}$, whose entries are determined by

$$m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k} & \text{if } \lambda_l \neq \hat{\lambda}_k, \\ 0 & \text{if } \lambda_l = \hat{\lambda}_k, \end{cases} \quad \text{where } 1 \leq k, \ \ l \leq N. \tag{3.8}$$

Such an M-matrix involves zero entries, when $\lambda_l = \hat{\lambda}_k$ for a pair $1 \leq k, l \leq N$. Therefore, it generalizes the traditional case without zero entries (see, for example, Refs. 3, 25 and 26). This new kind of M-matrices is required, indeed, while generating soliton solutions to nonlocal integrable equations (see, for example, Refs. 27 and 28).

When M is invertible, we can formulate two Darboux matrices as follows:

$$\begin{cases} T^+ = T^+(\lambda) = I_{m+n} - \displaystyle\sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l}, \\ T^- = T^-(\lambda) = I_{m+n} + \displaystyle\sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}. \end{cases} \tag{3.9}$$

Those two Darboux matrices can be stated in a compact form, by means of partial fractional decompositions:

$$T^+ = I_{m+n} - \sum_{l=1}^{N} \frac{v_l^M \hat{v}_l}{\lambda - \hat{\lambda}_l}, \quad T^- = I_{m+n} + \sum_{k=1}^{N} \frac{v_k \hat{v}_k^M}{\lambda - \lambda_k}, \tag{3.10}$$

where we set

$$\begin{cases} (v_1^M, \dots, v_N^M) = (v_1, \dots, v_N) M^{-1}, \\ ((\hat{v}_1^M)^T, \dots, (\hat{v}_N^M)^T)^T = M^{-1}(\hat{v}_1^T, \dots, \hat{v}_N^T)^T. \end{cases} \tag{3.11}$$

At this moment, we can have the compact forms for the Darboux matrices:

$$T^+ = I_{m+n} - vM^{-1}\hat{R}\hat{v}, \quad T^- = I_{m+n} + vRM^{-1}\hat{v}, \tag{3.12}$$

upon defining

$$R = \mathrm{diag}\left(\frac{1}{\lambda - \lambda_1}, \dots, \frac{1}{\lambda - \lambda_N}\right), \quad \hat{R} = \mathrm{diag}\left(\frac{1}{\lambda - \hat{\lambda}_1}, \dots, \frac{1}{\lambda - \hat{\lambda}_N}\right). \tag{3.13}$$

Let us then take

$$T_1^{\pm} = \lim_{\lambda\to\infty} [\lambda(T^{\pm}(\lambda) - I_{m+n})]. \tag{3.14}$$

Obviously, we have

$$T_1^+ = -vM^{-1}\hat{v}, \quad T_1^- = vM^{-1}\hat{v}, \tag{3.15}$$

which also yields

$$T_1^+ = -T_1^-.$$

Moreover, we can have the following basic properties for the two Darboux matrices T^+ and T^-.

(i) A spectral property holds:

$$\begin{aligned} &\left(\prod_{l=1}^{N}(\lambda - \hat{\lambda}_l)T^+\right)(\lambda_k)v_k = 0, \\ &\hat{v}_k\left(\prod_{l=1}^{N}(\lambda - \lambda_l)T^-\right)(\hat{\lambda}_k) = 0, \quad 1 \le k \le N. \end{aligned} \tag{3.16}$$

(ii) If an orthogonal condition is satisfied:

$$\hat{v}_k v_l = 0 \quad \text{when } \lambda_l = \hat{\lambda}_k, \tag{3.17}$$

where $1 \le k, l \le N$, then we have $\hat{R}\hat{v}vR = MR - \hat{R}M$, and thus, T^+ and T^- are inverse to each other:

$$T^+(\lambda)T^-(\lambda) = I_{m+n}. \tag{3.18}$$

Now, in order to present a binary DT, we need to compute the derivatives of the M-matrix with respect to the two independent variables, x and t. It is direct to show that

$$\hat{v}_k\Lambda v_l = 0 \quad \text{when } \lambda_l = \hat{\lambda}_k, \tag{3.19}$$

where $1 \le k, l \le N$, guarantees that

$$M_x = i\hat{v}\Lambda v; \tag{3.20}$$

and it is also straightforward to check that

$$\hat{v}_k\Omega_{[k,l]}v_l = 0 \quad \text{when } \lambda_l = \hat{\lambda}_k, \tag{3.21}$$

where $1 \le k, l \le N$, and

$$\Omega_{[k,l]} = (\hat{\lambda}_k^2 + \hat{\lambda}_k \lambda_l + \lambda_l^2)\Omega + \frac{\beta}{\alpha}(\hat{\lambda}_k + \lambda_l)P - \frac{\beta}{\alpha^2} I_{m,n}(P^2 + iP_x), \quad 1 \le k, \ l \le N, \tag{3.22}$$

guarantees that

$$M_t = i\left[\hat{v}\hat{A}^2\Omega v + \hat{v}\hat{A}\Omega A v + \hat{v}\Omega A^2 v + \frac{\beta}{\alpha}(\hat{v}\hat{A}Pv + \hat{v}PAv) - \frac{\beta}{\alpha^2}\hat{v} I_{m,n}(P^2 + iP_x)v\right]. \tag{3.23}$$

Let us also take the adjoint eigenfunctions:

$$\hat{v}_k(x,t,\hat{\lambda}_k) = v_k^T(x,t,\lambda_k)C, \quad 1 \le k \le N. \tag{3.24}$$

Then, we can find that T_1^+ satisfies a required involution property:

$$(T_1^+(x,t))^T = -CT_1^+(x,t)C^{-1}, \tag{3.25}$$

where C is defined by (2.13), and the three conditions in (3.17), (3.19) and (3.21) become

$$v_k^T C v_l = v_k^T C \Lambda v_l = v_k^T C \Omega_{[k,l]} v_l = 0 \quad \text{when } \lambda_l = \hat{\lambda}_k, \tag{3.26}$$

where $1 \le k, l \le N$.

All those properties above lead us to establish a general binary DT formulation as follows.

Theorem 3.1. *Let* Λ *and* $\Omega_{[k,l]}$ *be defined by* (2.4) *and* (3.22), *and let the adjoint eigenvalues* $\{\hat{\lambda}_k | 1 \le k \le N\}$ *be taken as in* (3.1) *and the associated adjoint eigenfunctions* $\{\hat{v}_k | 1 \le k \le N\}$ *be determined by* (3.24). *Assume that* $T^{\pm}$ *and* $T_1^{\pm}$ *are formulated by* (3.9) *and* (3.14). *Then if the conditions in* (3.26) *are satisfied, we have a binary DT*:

$$\phi' = T^+\phi, \quad \tilde{\phi}' = \tilde{\phi}T^-, \tag{3.27}$$

yielding a Bäcklund transformation

$$P' = P + [T_1^+, \Lambda], \tag{3.28}$$

for the redeuced integrable matrix mKdV equations (2.18).

Finally, we would like to exhibit an N-fold decomposition feature for the above binary DT in the standard case $\{\lambda_k | 1 \le k \le N\} \cap \{\hat{\lambda}_k | 1 \le k \le N\} = \emptyset$.

To the end, we introduce two new sets of single binary Darboux matrices $T^+[\![k]\!]$ and $T^-[\![k]\!], 1 \le k \le N$, recursively as follows:

$$\begin{cases} T^+[\![k]\!] = T^+[\![k]\!](\lambda) = I_{m+n} - \dfrac{\lambda_k - \hat{\lambda}_k}{\lambda - \hat{\lambda}_k} \dfrac{v_k' \hat{v}_k'}{\hat{v}_k' v_k'}, \quad 1 \le k \le N, \\ T^-[\![k]\!] = T^-[\![k]\!](\lambda) = I_{m+n} + \dfrac{\lambda_k - \hat{\lambda}_k}{\lambda - \lambda_k} \dfrac{v_k' \hat{v}_k'}{\hat{v}_k' v_k'}, \quad 1 \le k \le N, \end{cases} \tag{3.29}$$

with new pairs of eigenfunctions and adjoint eigenfunctions:

$$v_k' = T^+\{k-1\}(\lambda_k)v_k, \quad \hat{v}_k' = \hat{v}_k T^-\{k-1\}(\hat{\lambda}_k), \quad 1 \le k \le N, \tag{3.30}$$

where

$$\begin{cases} T^+\{0\} = T^-\{0\} = I_{m+n}, \ T^+\{k\} = T^+[\![k]\!] \dots T^+[\![2]\!] T^+[\![1]\!], \quad 1 \le k \le N, \\ T^-\{k\} = T^-[\![1]\!] T^-[\![2]\!] \dots T^-[\![k]\!], \quad 1 \le k \le N. \end{cases} \tag{3.31}$$

Then we can have the following N-fold decompositions for the two Darboux matrices T^+ and T^-:

$$T^+ = T^+[\![N]\!] T^+[\![N-1]\!] \dots T^+[\![1]\!], \quad T^- = T^-[\![1]\!] \dots T^-[\![N-1]\!] T^-[\![N]\!], \tag{3.32}$$

where $T^+[\![k]\!]$ and $T^-[\![k]\!]$, $1 \le k \le N$, are defined by (3.29).

4. Soliton Solutions

Let us take the adjoint eigenvalues $\{\hat{\lambda}_k | \, 1 \le k \le N\}$ as in (3.1). Starting from the zero seed solution $P = 0$, we can readily obtain the associated eigenfunctions and adjoint eigenfunctions

$$v_k(x,t) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^3 \Omega t} w_k, \quad 1 \le k \le N, \tag{4.1}$$

$$\hat{v}_k(x,t) = w_k^T \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^3 \Omega t} C, \quad 1 \le k \le N, \tag{4.2}$$

where w_k, $1 \le k \le N$, are arbitrary constant column vectors, and the three orthogonal conditions in (3.26) reduce to

$$w_k^T C w_l = w_k^T C \Lambda w_l = (\hat{\lambda}_k^2 + \hat{\lambda}_k \lambda_l + \lambda_l^2) w_k^T C \Omega w_l = 0 \quad \text{when } \lambda_l = \hat{\lambda}_k, \tag{4.3}$$

where $1 \le k, l \le N$, and Λ, Ω and C are given by (2.4), (2.5) and (2.13), respectively.

Now following the binary DT theory in Theorem 3.1, a new potential matrix can be generated by

$$P' = [T_1^+, \Lambda], \quad T_1^+ = -vM^{-1}\hat{v} = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l. \tag{4.4}$$

As a consequence, this generates a kind of soliton solutions to the reduced integrable matrix mKdV equations (2.18):

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2, \tag{4.5}$$

where we split $v_k = ((v_k^1)^T, (v_k^2)^T)^T$ and $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2)$, of which v_k^1 and v_k^2 are m- and n-dimensional column vectors, respectively, and $\hat{v}_k^1$ and $\hat{v}_k^2$ are m- and n-dimensional row vectors, respectively.

We remark that the situation of $\lambda_k = \hat{\lambda}_k$ occurs only when taking $\lambda_k = 0$. Because of $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$, we can easily observe that the three conditions in (4.3) are equivalent to

$$(w_k^1)^T \Sigma_1 w_l^1 = 0, \quad (w_k^2)^T \Sigma_2 w_l^2 = 0, \quad \text{if } \lambda_l = \hat{\lambda}_k, \quad \text{where } 1 \leq k,\ l \leq N, \quad (4.6)$$

where we split $w_k = ((w_k^1)^T, (w_k^2)^T)^T$, $1 \leq k \leq N$, as we did for v_k before.

Finally, we see that once the conditions in (4.6) are satisfied, the formula (4.5), together with (3.8), (4.1) and (4.2), presents soliton solutions to the reduced integrable matrix mKdV equations (2.18).

5. Concluding Remarks

The paper aims to present a novel kind of integrable reductions of the mKdV equations, based on the corresponding matrix AKNS spectral problems of arbitrary order, and to compute binary DTs for the resulting reduced integrable matrix mKdV equations, starting from the Lax pair and the adjoint Lax pair of matrix spectral problems. The resulting binary DTs have been used to construct soliton solutions to the reduced integrable matrix mKdV equations, which provides an amendment to the binary DT theory for the other kind of reduced integrable matrix mKdV equations.

The basic idea in our formulation of producing reduced integrable equations is to replace the spectral parameter λ with $-\lambda$. Such a simple idea generates a novel kind of reduced integrable matrix mKdV equations, based on the matrix AKNS spectral problems. Also, in our construction of binary DTs, we have used a generalized M-matrix, where eigenvalues could be equal to adjoint eigenvalues. The thought of doing that comes from recent studies on Riemann–Hilbert problems for nonlocal integrable equations (see, for example, Refs. 27 and 28). The resulting general formulation of binary DTs can be applied to either local or nonlocal integrable equations (see, for example, Refs. 27–32 for nonlocal theories). We remark that taking repeated eigenvalues or repeated adjoint eigenvalues engenders Darboux matrices with higher-order poles, and taking derivatives with respect to eigenvalues or adjoint eigenvalues yields generalized DTs.

We also remark that on the one hand, there are only two kinds of group reductions for the matrix AKNS spectral problems which produce reduced integrable equations. One is to make $\lambda \to \lambda^*$ and the other is to make $\lambda \to -\lambda$. It should be interesting to apply such ideas to other kinds of matrix spectral problems to generate reduced integrable equations. On the other hand, there are many interesting problems in the theory of DTs, which include applications of DTs to other kinds of exact solutions, including rogue wave solutions, and more generally, lump solutions[33]; and formulations of binary DTs for integrable equations associated with non-semisimple Lie algebras (see Ref. 34 for DTs for continuous integrable couplings). It would also be desirable in exploring soliton dynamics, to establish connections between binary DT theories and Hirota bilinear forms.[35,36]

Acknowledgments

The work was supported in part by NSFC under the Grants No. 11975145 and 11972291, the Ministry of Science and Technology of China (G2021016032L), and the Natural Science Foundation for Colleges and Universities in Jiangsu Province (17 KJB 110020). The author would also like to thank Morgan McAnally, Solomon Manukure and Fudong Wang Yong Zhang for their valuable discussions.

References

1. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
2. F. Calogero and A. Degasperis, *Solitons and Spectral Transform I* (North-Holland, Amsterdam, 1982).
3. S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method* (Consultants Bureau, New York, 1984).
4. V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, 1991).
5. C. H. Gu, H. S. Hu and Z. X. Zhou, *Darboux Transformations in Integrable Systems* (Springer, Netherlands, 2005).
6. W. X. Ma, *Lett. Math. Phys.* **39** (1997) 33.
7. W. X. Ma, *J. Phys. A: Math. Gen.* **25** (1992) 5329.
8. W. X. Ma, *J. Phys. A: Math. Gen.* **26** (1993) 2573.
9. W. X. Ma, *Math. Methods Appl. Sci.* **42** (2019) 1099.
10. W. X. Ma, *Partial Differ. Equ. Appl. Math.* **4** (2021) 100190.
11. M. Mañas, *J. Phys. A: Math. Gen.* **29** (1996) 7721.
12. Y. B. Zeng, W. X. Ma and Y. J. Shao, *J. Math. Phys.* **42** (2001) 2113.
13. E. V. Doktorov and S. B. Leble, *A Dressing Method in Mathematical Physics* (Springer, Dordrecht, 2007).
14. H. Q. Zhang, Y. Wang and W. X. Ma, *Chaos* **27** (2007) 073102.
15. Y. Zhang, R. S. Ye and W. X. Ma, *Math. Methods Appl. Sci.* **43** (2020) 613.
16. L. Xu, D. S. Wang, X. Y. Wen and Y. L. Jiang, *J. Nonlinear Sci.* **30** (2020) 537.
17. X. G. Geng, Y. H. Li, J. Wei and Y. Y. Zhai, *Math. Methods Appl. Sci.* **44** (2021) 12727.
18. A. Degasperis and S. Lombardo, *J. Phys. A: Math. Theor.* **40** (2007) 961.
19. A. Degasperis and S. Lombardo, *J. Phys. A: Math. Theor.* **42** (2009) 385206.
20. R. M. Li and X. G. Geng, *Commun. Nonlinear Sci. Numer. Simul.* **90** (2020) 105408.
21. W. X. Ma, *Physica D* **430** (2022) 133078.
22. M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, *Stud. Appl. Math.* **53** (1974) 249.
23. U. Bandelow, A. Ankiewicz, Sh. Amiranashvili and N. Akhmediev, *Chaos* **28** (2018) 053108.
24. W. X. Ma, *J. Math. Anal. Appl.* **471** (2019) 796.
25. T. Kawata, Riemann spectral method for the nonlinear evolution equation, in *Advances in Nonlinear Waves*, ed. L. Debnath, Vol. I (Pitman, Boston, MA, 1984), pp. 210–225.
26. J. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems* (SIAM, Philadelphia, 2010).
27. W. X. Ma, Y. H. Huang and F. D. Wang, *Stud. Appl. Math.* **145** (2020) 563.
28. W. X. Ma, *Proc. Am. Math. Soc.* **149** (2021) 251.

29. M. J. Ablowitz and Z. H. Musslimani, *Phys. Rev. Lett.* **110** (2013) 064105.
30. M. Gürses and A. Pekcan, *J. Math. Phys.* **59** (2018) 051501.
31. A. S. Fokas, *Nonlinearity* **29** (2016) 319.
32. C. Q. Song, D. M. Xiao and Z. N. Zhu, *Commun. Nonlinear Sci. Numer. Simul.* **45** (2017) 13.
33. W. X. Ma and Y. Zhou, *J. Differ. Equ.* **264** (2018) 2633.
34. W. X. Ma and Y. J. Zhang, *Rev. Math. Phys.* **30** (2018) 1850003.
35. W. X. Ma, *Int. J. Nonlinear Sci. Numer. Simul.* **23** (2022) 123.
36. W. X. Ma, *Opt. Quantum Electron.* **52** (2020) 511.