Modern Physics Letters B
Vol. 33, No. 36 (2019) 1950457 (10 pages)

DOI: 10.1142/S0217984919504578

Lump and interaction solutions to linear PDEs in 2 + 1 dimensions via symbolic computation

Wen-Xiu Ma

Department of Mathematics, Zhejiang Normal University,
Jinhua 321004, Zhejiang, China
Department of Mathematics, King Abdulaziz University,
Jeddah, Saudi Arabia
Department of Mathematics and Statistics,
University of South Florida, Tampa, FL 33620, USA
College of Mathematics and Physics, Shanghai University of Electric Power,
Shanghai 200090, China
College of Mathematics and Systems Science,
Shandong University of Science and Technology,
Qingdao 266590, Shandong, China
International Institute for Symmetry Analysis and Mathematical Modelling,
Department of Mathematical Sciences, North-West University,
Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa
mawx@cas.usf.edu

Received 27 July 2019
Revised 17 August 2019
Accepted 20 August 2019
Published 27 November 2019

The aim of this paper is to show that there exist lump solutions and interaction solutions to linear partial differential equations in $2+1$ dimensions. Through symbolic computations with Maple, we exhibit a great variety of exact solutions to a class of $(2+1)$-dimensional linear partial differential equations, and present a specific example which possesses lump, lump-kink and lump-soliton solutions. This supplements the study on lump, rogue wave and breather solutions and their interaction solutions to nonlinear integrable equations.

Keywords: Lump solution; interaction solution; symbolic computation.

1. Introduction

Soliton and lump solutions describe various important nonlinear phenomena in nature.[1,2] Positons and complexitons are counterparts of solitons, which enrich the diversity of exact solutions to integrable equations.[3,4] Interaction solutions between two classes of solutions describe more diverse nonlinear phenomena.[5] Upon taking long wave limits, lump solutions can be generated from solitons.[1,6] The Hirota bilinear method in soliton theory provides a powerful approach to all those exact solutions.[7,8]

Mathematically, solitons are solutions usually exponentially localized in all directions in space and time, and lumps are solutions rationally localized in all directions in space. Based on a Hirota bilinear form in $2+1$ dimensions:

$$P(D_x, D_y, D_t)f\cdot f = 0\,, \tag{1.1}$$

where P is a polynomial in the indicated variables, and D_x, D_y and D_t are Hirota's bilinear derivatives, an N-soliton solution is determined by

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i\xi_i + \sum_{i<j} \mu_i\mu_j a_{ij}\right)\,, \tag{1.2}$$

with

$$\begin{cases} \xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, & 1\le i\le N\,,\\ \mathrm{e}^{a_{ij}} = -\dfrac{P(k_i-k_j, l_i-l_j, \omega_j-\omega_i)}{P(k_i+k_j, l_i+l_j, \omega_j+\omega_i)}, & 1\le i<j\le N\,, \end{cases} \tag{1.3}$$

where the dispersion relations hold: $P(k_i, l_i, -\omega_i) = 0$, $1\le i\le N$, and the $\xi_{i,0}$'s are arbitrary constants. The KPI equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{1.4}$$

possesses a lump solution[9]:

$$u = 2(\ln f)_{xx}, \quad f = \left(a_1x + a_2y + \frac{a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6}{a_1^2+a_5^2}t + a_4\right)^2 + \left(a_5x + a_6y + \frac{2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2}{a_1^2+a_5^2}t + a_8\right)^2 + \frac{3(a_1^2+a_5^2)^3}{(a_1a_6-a_2a_5)^2}\,, \tag{1.5}$$

where the parameters a_i's are arbitrary but need to satisfy $a_1a_6 - a_2a_5 \neq 0$, which guarantees rational localization in all directions in the (x, y)-plane. Other integrable equations, which possess lump solutions, contain the three-dimensional three-wave resonant interaction,[10] the BKP equation,[11,12] the Davey–Stewartson equation II,[6] the Ishimori-I equation,[13] the KP equation with a self-consistent source,[14] and many others.[15–17] Symbolic computations also show that various non-integrable equations possess lump solutions as well, including $(2+1)$-dimensional generalized KP, BKP and Bogoyavlensky–Konopelchenko equations (see, e.g. Refs. 18–23). Moreover, recent studies exhibit the existence of interaction solutions of lumps with another kind of dispersive waves to nonlinear integrable equations in $2+1$ dimensions (see, e.g. Refs. 24 and 25 for lump–soliton interaction solutions; and see, e.g. Refs. 26–28 for lump–kink interaction solutions).

In this paper, we would like to show that like nonlinear integrable equations, linear partial differential equations can possess lump solutions and their interaction solutions with kink and soliton waves. A class of linear partial differential equations in $2+1$ dimensions will be analyzed and a specific equation in the class will be considered to verify such solution phenomena. More concretely, we will search for lump solutions and mixed lump-kink and lump-soliton solutions to a class of $(2+1)$-dimensional linear partial differential equations. Through making symbolic computations with Maple, sufficient conditions to guarantee the existence of the mentioned solutions will be given and a few examples of lump and interaction solutions will be explicitly presented and plotted. Concluding remarks will be given finally in the last section.

2. Lump and Interaction Solutions

Let $u = u(x, y, t)$ be a real function of $x, y, t \in \mathbb{R}$. We consider a class of linear partial differential equations (PDEs) in $2+1$ dimensions:

$$\alpha u_{tx} + \beta u_{xy} + \gamma u_{ty} = 0\,, \tag{2.1}$$

where α, β and γ are three given constants, and u_{tx}, u_{xy} and u_{ty} are all mixed second-order partial derivatives.

2.1. *A general criterion*

We search for a kind of exact solutions

$$u = v(\xi, \eta, \zeta)\,, \tag{2.2}$$

where v is an arbitrary function, and ξ, η and ζ are three linear variables:

$$\begin{cases} \xi = a_1 x + b_1 y + c_1 t + d_1\,, \\ \eta = a_2 x + b_2 y + c_2 t + d_2\,, \\ \zeta = a_3 x + b_3 y + c_3 t + d_3\,, \end{cases} \tag{2.3}$$

in which a_i, b_i, c_i and d_i, $1 \le i \le 3$, are constants to be determined. Then, the linear PDE (2.1) becomes

$$w_1 v_{\xi\xi} + w_2 v_{\eta\eta} + w_3 v_{\zeta\zeta} + w_4 v_{\xi\eta} + w_5 v_{\xi\zeta} + w_6 v_{\eta\zeta} = 0\,, \tag{2.4}$$

where w_i, $1 \le i \le 6$, are quadratic functions of the parameters a_i, b_i and c_i, $1 \le i \le 3$. Requiring all coefficients of the six second-order partial derivatives of v

to be zero, we obtain a system of conditions on the parameters:

$$\begin{cases} \alpha a_1 c_1 + \beta a_1 b_1 + \gamma b_1 c_1 = 0, \\ \alpha a_2 c_2 + \beta a_2 b_2 + \gamma b_2 c_2 = 0\,, \\ \alpha a_3 c_3 + \beta a_3 b_3 + \gamma b_3 c_3 = 0\,, \\ \alpha(a_1 c_2 + a_2 c_1) + \beta(a_1 b_2 + a_2 b_1) + \gamma(b_1 c_2 + b_2 c_1) = 0\,, \\ \alpha(a_1 c_3 + a_3 c_1) + \beta(a_1 b_3 + a_3 b_1) + \gamma(b_1 c_3 + b_3 c_1) = 0\,, \\ \alpha(a_2 c_3 + a_3 c_2) + \beta(a_2 b_3 + a_3 b_2) + \gamma(b_2 c_3 + b_3 c_2) = 0\,. \end{cases} \tag{2.5}$$

2.2. *Specific solutions*

By direct symbolic computations with Maple, we can get a few solutions to this system of quadratic equations. We list the following three of them which are fascinating.

When $\gamma = 0$, we can have

$$\alpha c_1 + \beta b_1 = 0, \quad \alpha c_2 + \beta b_2 = 0, \quad \alpha c_3 + \beta b_3 = 0\,. \tag{2.6}$$

When $a_3 = b_3 = c_3 = 0$, we can have

$$\begin{cases} b_1 = -\dfrac{a_1(\alpha c_2 + \beta b_2)}{\gamma c_2}, \quad c_1 = -\dfrac{a_1(\alpha c_2 + \beta b_2)}{\gamma b_2}\,, \\ a_2 = -\dfrac{\gamma b_2 c_2}{\alpha c_2 + \beta b_2}\,. \end{cases} \tag{2.7}$$

Otherwise, we can generally have

$$\begin{cases} a_2 = -\dfrac{b_2(\beta a_1 + \gamma c_1)}{\alpha c_1}, \quad a_3 = -\dfrac{b_3(\beta a_1 + \gamma c_1)}{\alpha c_1}\,, \\ b_1 = -\dfrac{\alpha a_1 c_1}{\beta a_1 + \gamma c_1}, \quad c_2 = -\dfrac{b_2(\beta a_1 + \gamma c_1)}{\alpha a_1}\,, \\ c_3 = -\dfrac{b_3(\beta a_1 + \gamma c_1)}{\alpha a_1}\,. \end{cases} \tag{2.8}$$

In each set of the above solutions, the parameters not determined in the set are arbitrary provided that all expressions in the set will be meaningful. Some straightforward computations can show that all those solutions satisfy a determinant equation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0\,, \tag{2.9}$$

which implies that none of the above three sets of parameters can generate rogue wave solutions to the PDEs (2.1).

When three mixed second-order partial derivatives of v with respect to ξ, η and ζ are all zero, i.e.

$$v_{\xi\eta} = v_{\xi\zeta} = v_{\eta\zeta} = 0\,,$$

then the general system (2.5) is reduced to a half set of conditions on the parameters:

$$\begin{cases} \alpha a_1 c_1 + \beta a_1 b_1 + \gamma b_1 c_1 = 0\,, \\ \alpha a_2 c_2 + \beta a_2 b_2 + \gamma b_2 c_2 = 0\,, \\ \alpha a_3 c_3 + \beta a_3 b_3 + \gamma b_3 c_3 = 0\,. \end{cases} \tag{2.10}$$

This generates a different kind of solutions: $u = v_1(\xi) + v_2(\eta) + v_3(\zeta)$, with separable variables ξ, η and ζ in three arbitrary functions v_1, v_2 and v_3. When $\gamma = 0$, the above system is simplified into

$$\alpha a_1 c_1 + \beta a_1 b_1 = 0, \quad \alpha a_2 c_2 + \beta a_2 b_2 = 0, \quad \alpha a_3 c_3 + \beta a_3 b_3 = 0\,, \tag{2.11}$$

which is a little bit more general than (2.6). Obviously, this set of equations can have solutions, which do not need to satisfy the determinant equation (2.9).

2.3. *An illustrative example*

Let us now consider a specific equation in the above class of $(2+1)$-dimensional linear PDEs:

$$u_{tx} + u_{xy} = 0\,. \tag{2.12}$$

Due to the linearity, combining two solutions can yield a class of solutions to (2.12):

$$u = v_1(y,t) + v_2(x, t-y)\,, \tag{2.13}$$

where v_1 and v_2 are arbitrary. This kind of solutions does not need to satisfy (2.9), either. The two solutions $v_1(y,t)$ and $v_2(x,t-y)$ are generated by (2.5) and (2.6), respectively. By transforming the PDE (2.12) into a canonical one, we can directly show that this class of solutions in (2.13) is general. That is, all solutions of (2.12) must be of the type in (2.13).

Particularly, when the conditions in (2.6) hold, we can have the following subclass of solutions:

$$u = v_2 = (\ln f)_{xx}, \quad f = \xi^{2m} + \eta^{2n} + g(\zeta) + 1\,, \tag{2.14}$$

where m and n are arbitrary natural numbers, and the function g is arbitrary. Therefore, upon taking

$$g(\zeta) = 0, \quad \mathrm{e}^{\zeta} \;\text{ or }\; \cosh\zeta\,, \tag{2.15}$$

from (2.14), we can present lump solutions, and interaction solutions: lump-kink and lump-soliton solutions, for the linear PDE (2.12). The resulting solution with $m = n = 1$ reads

$$u = \frac{f_{xx}f - f_x^2}{f^2} = \frac{[2a_1^2 + 2a_2^2 + a_3^2 g''(\zeta)]f - [2a_1\xi + 2a_2\eta + a_3 g'(\zeta)]^2}{f^2}\,. \tag{2.16}$$

This supplements various theories of soliton solutions and dromion-type solutions, through basic techniques including the Painlevé analysis (see, e.g. Ref. 29), the Riemann–Hilbert approach (see, e.g. Refs. 30 and 31), symmetry constraints (see, e.g. Refs. 32–34) and binary nonlinearization (see, e.g. Refs. 35–37).

Further, taking

$$\begin{cases} a_1 = 1, \quad b_1 = -2, \quad c_1 = 2, \quad d_1 = 10\,, \\ a_2 = -2, \quad b_2 = -1, \quad c_2 = 1, \quad d_2 = -3\,, \\ a_3 = -6, \quad b_3 = -5, \quad c_3 = 5, \quad d_3 = -2\,, \end{cases} \tag{2.17}$$

we get, from (2.16), the three specific solutions to (2.12):

$$\begin{cases} u_1 = \dfrac{10 f_1 - (10x+32)^2}{f_1^2}\,, \\ f_1 = (x-2y+2t+10)^2 + (-2x-y+t-3)^2 + 1\,, \end{cases} \tag{2.18}$$

$$\begin{cases} u_2 = \dfrac{(10+36\mathrm{e}^{-6x-5y+5t-2}) f_2 - (10x+32-6\mathrm{e}^{-6x-5y+5t-2})^2}{f_2^2}\,, \\ f_2 = (x-2y+2t+10)^2 + (-2x-y+t-3)^2 + \mathrm{e}^{-6x-5y+5t-2} + 1\,, \end{cases} \tag{2.19}$$

and

$$\begin{cases} u_3 = \dfrac{[10+36\cosh(-6x-5y+5t-2)]}{f_3} \\ \qquad - \dfrac{[10x+32-6\sinh(-6x-5y+5t-2)]^2}{f_3^2}\,, \\ f_3 = (x-2y+2t+10)^2 + (-2x-y+t-3)^2 \\ \qquad + \cosh(-6x-5y+5t-2) + 1\,. \end{cases} \tag{2.20}$$

The solution u_2 is also called a lumpoff solution.[38] Three 3D plots and contour plots of those solutions are made in Figs. 1–3, which exhibit characteristics of lumps and interactions of lumps with kink and soliton waves in soliton theory.

2.4. *Two questions*

We have produced many lump solutions and their interaction solutions with kink and soliton type dispersive waves. Moreover, we can easily produce three-wave rational solutions by (2.13) which are analytical in the x, y and t space, i.e. the whole spatial and temporal space, and does not satisfy the determinant equation (2.9).

However, we still don't know how to guarantee the localization of such rational solutions in all directions in the whole spatial and temporal space, i.e.

$$\lim_{kx+ly-\omega t\to\pm\infty} u(x,y,t) = 0, \quad \forall k, l, \omega \in \mathbb{R}\,,$$

so that we can produce rogue wave solutions.

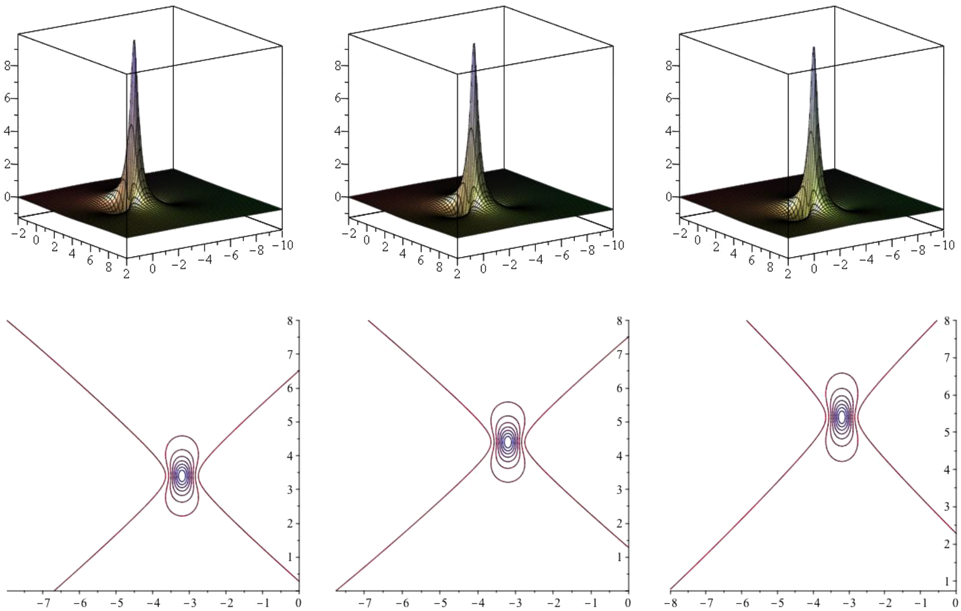

Fig. 1. (Color online) Profiles of u_1 when $t = 0, 1, 2$: 3D plots (top) and contour plots (bottom).

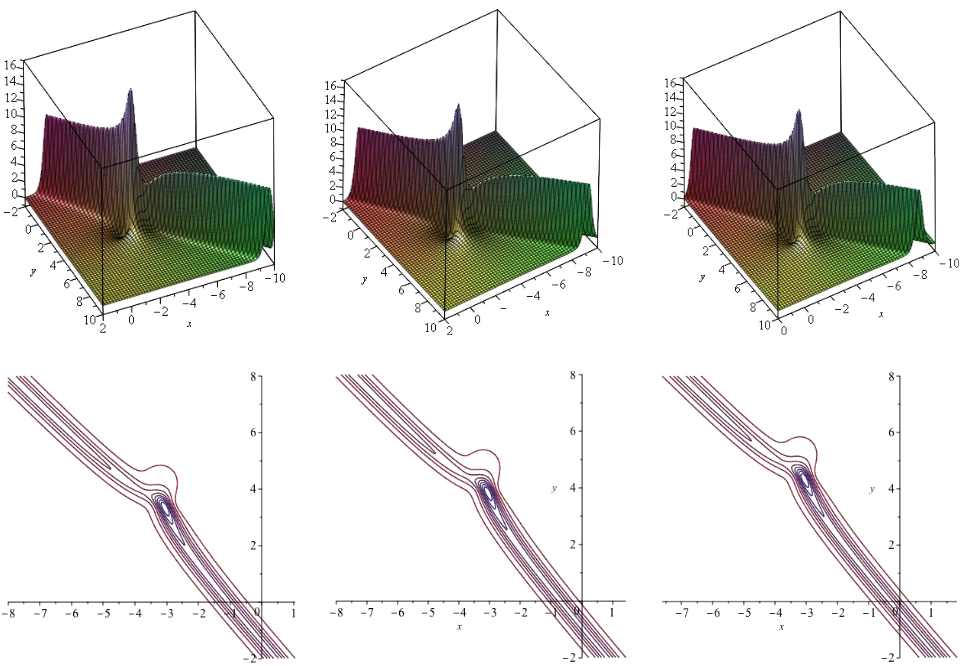

Fig. 2. (Color online) Profiles of u_2 when $t = 0, 0.5, 1$: 3D plots (top) and contour plots (bottom).

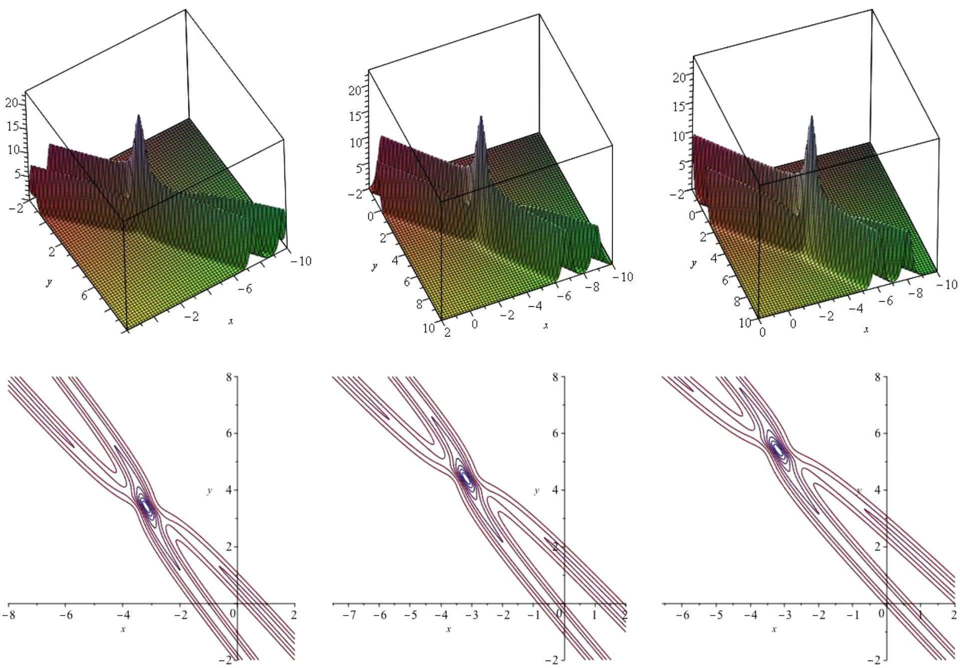

Fig. 3. (Color online) Profiles of u_3 when $t = 0, 1, 2$: 3D plots (top) and contour plots (bottom).

The other question is whether or not there exist lump solutions to the $(2+1)$-dimensional linear wave equation

$$u_{tt} = u_{xx} + u_{yy} .$$

This equation does not contain any mixed second-order partial derivative.

3. Concluding Remarks

We have focused on a class of linear partial differential equations to show the existence of lump solutions and their interaction solutions with kink and soliton waves via symbolic computation with Maple, as shown for nonlinear integrable equations in soliton theory. A few concrete lump and interaction solutions to a specific equation in the class were explicitly presented, together with three 3D plots and contour plots of the three specific solutions.

We remarked that we have never seen lumps and interactions of lumps with kink and soliton waves in a linear world. The obtained lump, lump-kink and lump-soliton solutions are valuable supplements to exact solutions generated from different kinds of combinations.[39–41] We know that integrable equations can be solved by the Wronskian technique.[42] Therefore, our study creates a new question: how can one generalize Wronskian solutions by introducing matrix entries of new type? It is also interesting to look for lump and interaction solutions to other generalized bilinear and tri-linear differential equations involving generalized bilinear derivatives.[43] The corresponding interaction solutions will normally not be resonant solutions presented through the linear superposition principle.[44,45] Integrable equations determined through generalized bilinear derivatives will have different interaction solutions, but lump solutions derived from quadratic functions remain the same as in the Hirota derivative case (see Ref. 46 for more discussions).

Diversity of interaction solutions should exhibit the existence of abundant Lie–Bäcklund symmetries which amends symmetry theories on differential equations. Absolutely, it is also important to explore examples of both linear and nonlinear discrete differential equations which exhibit lump and interaction solution phenomena. These are all interesting problems that deserve our further investigation.

Acknowledgments

This work was supported in part by NSFC under Grant Nos. 11371326, 11301331, 11371086 and 51771083, NSF under Grant No. DMS-1664561, Natural Science Fund for Colleges and Universities of Jiangsu Province under Grant No. 17KJB110020, Emphasis Foundation of Special Science Research on Subject Frontiers of CUMT under Grant No. 2017XKZD11, and the Distinguished Professorships by Zhejiang Normal University, China and North-West University, South Africa.

References

1. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
2. S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons — The Inverse Scattering Method* (Consultants Bureau, New York, 1984).
3. W. X. Ma, *Discrete Contin. Dyn. Syst.* (Suppl.) (2009) 506.
4. A.-M. Wazwaz and S. A. El-Tantawy, *Nonlinear Dyn.* **87** (2017) 2457.
5. W. X. Ma and Y. You, *Trans. Am. Math. Soc.* **357** (2005) 1753.
6. J. Satsuma and M. J. Ablowitz, *J. Math. Phys.* **20** (1979) 1496.
7. R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, New York, 2004).
8. P. J. Caudrey, *Philos. Trans. R. Soc. A* **369** (2011) 1215.
9. W. X. Ma, *Phys. Lett. A* **379** (2015) 1975.
10. D. J. Kaup, *J. Math. Phys.* **22** (1981) 1176.
11. C. R. Gilson and J. J. C. Nimmo, *Phys. Lett. A* **147** (1980) 472.
12. J. Y. Yang and W. X. Ma, *Int. J. Mod. Phys. B* **30** (2016) 1640028.
13. K. Imai, *Prog. Theor. Phys.* **98** (1997) 1013.
14. X. L. Yong, W. X. Ma, Y. H. Huang and Y. Liu, *Comput. Math. Appl.* **75** (2018) 3414.
15. Y. Zhang, H. H. Dong, X. E. Zhang and H. W. Yang, *Comput. Math. Appl.* **73** (2017) 246.
16. H. Q. Zhang and W. X. Ma, *Nonlinear Dyn.* **87** (2017) 2305.
17. W. X. Ma and Y. Zhou, *J. Differ. Equ.* **264** (2018) 2633.
18. W. X. Ma, Z. Y. Qin and X. Lü, *Nonlinear Dyn.* **84** (2016) 923.
19. X. Lü and W. X. Ma, *Nonlinear Dyn.* **85** (2016) 1217.

20. J. P. Yu and Y. L. Sun, *Nonlinear Dyn.* **87** (2017) 2755.
21. S. T. Chen and W. X. Ma, *Front. Math. China* **13** (2018) 525.
22. S. Manukure, Y. Zhou and W. X. Ma, *Comput. Math. Appl.* **75** (2018) 2414.
23. S. T. Chen and W. X. Ma, *Comput. Math. Appl.* **76** (2018) 1680.
24. J. Y. Yang, W. X. Ma and Z. Y. Qin, *Anal. Math. Phys.* **8** (2018) 427.
25. W. X. Ma, X. L. Yong and H. Q. Zhang, *Comput. Math. Appl.* **75** (2018) 289.
26. Y. N. Tang, S. Q. Tao and G. Qing, *Comput. Math. Appl.* **72** (2016) 2334.
27. H. Q. Zhao and W. X. Ma, *Comput. Math. Appl.* **74** (2017) 1399.
28. J. B. Zhang and W. X. Ma, *Comput. Math. Appl.* **74** (2017) 591.
29. B. Dorizzi, B. Grammaticos, A. Ramani and P. Winternitz, *J. Math. Phys.* **27** (1986) 2848.
30. W. X. Ma, *J. Geom. Phys.* **132** (2018) 45.
31. B. B. Hu, T. C. Xia and W. X. Ma, *Appl. Math. Comput.* **332** (2018) 148.
32. B. Konopelchenko and W. Strampp, *Inverse Probl.* **7** (1991) L17.
33. Q. L. Zhao and X. Y. Li, *Anal. Math. Phys.* **6** (2016) 237.
34. H. H. Dong, Y. Zhang and X. E. Zhang, *Commun. Nonlinear Sci. Numer. Simul.* **36** (2016) 354.
35. W. X. Ma and W. Strampp, *Phys. Lett. A* **185** (1994) 277.
36. X. Y. Li, Q. L. Zhao, Y. X. Li and H. H. Dong, *J. Nonlinear Sci. Appl.* **8** (2015) 496.
37. X. Y. Li and Q. L. Zhao, *J. Geom. Phys.* **121** (2017) 123.
38. M. Jia and S. Y. Lou, arXiv:1710.06604.
39. W. X. Ma and E. G. Fan, *Comput. Math. Appl.* **61** (2011) 950.
40. Z. H. Xu, H. L. Chen and Z. D. Dai, *Appl. Math. Lett.* **37** (2014) 34.
41. Ö. Ünsal and W. X. Ma, *Comput. Math. Appl.* **71** (2016) 1242.
42. N. C. Freeman and J. J. C. Nimmo, *Phys. Lett. A* **95** (1983) 1.
43. W. X. Ma, *Stud. Nonlinear Sci.* **2** (2011) 140.
44. W. X. Ma, *Rep. Math. Phys.* **72** (2013) 41.
45. W. X. Ma, *Front. Math. China* **8** (2013) 1139.
46. W. X. Ma, Y. Zhou and R. Dougherty, *Int. J. Mod. Phys. B* **30** (2016) 1640018.