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AN APPLICATION OF THE CASORATIAN TECHNIQUE TO THE 2D TODA LATTICE EQUATION

WEN-XIU MA

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100080, China
mawx@cas.usf.edu

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A general Casoratian formulation is proposed for the 2D Toda lattice equation, which involves two coupled eigenfunction systems. Various Casoratian type solutions are generated, through solving the resulting linear conditions and using a Bäcklund transformation.

Keywords: 2D Toda lattice equation; Casoratian formulation; soliton; complexiton.

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1. Introduction

It is well-known that Wronskian formulations show a common characteristic feature of continuous soliton equations, and provide a powerful tool to construct exact solutions to continuous soliton equations.[1–7] The resulting technique has been applied to many continuous soliton equations such as the KdV, MKdV, NLS, derivative NLS, Boussinesq, KP, sine-Gordon and sinh-Gordon equations. With Wronskian formulations, soliton solutions and rational solutions are usually expressed as some kind of logarithmic derivatives of Wronskian-type determinants with respect to space variables, and the involved determinants are generated by eigenfunctions satisfying linear systems of differential equations. A great help is that Wronskian formulations transform nonlinear problems into linear problems, and thus continuous soliton equations can be treated by means of linear theories.

There is a discrete version of Wronskian formulations, called Casoratian formulations, for discrete soliton equations such as the Volttera, nonlinear electrical network, and Toda lattice equations (see, for example, Refs. 8–11). With Casoratian formulations, soliton solutions and rational solutions are often expressed as some kind of rational functions of Casoratian type determinants, and the involved determinants are made of eigenfunctions satisfying linear systems of differential-difference equations. Therefore, the Casoratian technique offers a direct approach for constructing exact solutions to discrete soliton equations.

Besides soliton solutions and rational solutions, the Wronskian and Casoratian techniques can be used to construct positon solutions,[12–15] i.e., solutions involving one kind of transcendental functions: trigonometric functions. More generally, a novel kind of solutions called complexiton solutions has been introduced and generated using such techniques for continuous and discrete soliton equations[3,8] and soliton equations with sources.[16] Those solutions contain two kinds of transcendental waves: exponential waves and trigonometric waves, with different speeds, and they correspond to complex eigenvalues of associated characteristic linear problems and generate solitons and positons as limit cases of the complex eigenvalues.[5,17]

One of the intriguing discrete soliton equations is the 2D Toda lattice equation,[18]

$$\frac{\partial^2 Q_n}{\partial s \partial x} = V_{n+1} - 2V_n + V_{n-1},\ Q_n = \ln(1+V_n), \tag{1}$$

where $x, s \in \mathbb{R}$ and $n \in \mathbb{Z}$. Through the dependent variable transformation,

$$V_n = \frac{\partial^2}{\partial s \partial x} \ln \tau_n, \tag{2}$$

Eq. (1) may be integrated with respect to x and s to obtain

$$1 + \frac{\partial^2}{\partial s \partial x} \ln \tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \tag{3}$$

where the constants of integration are set to zero. This equation is equivalent to

$$\frac{\partial^2 \tau_n}{\partial s \partial x}\tau_n - \frac{\partial \tau_n}{\partial s}\frac{\partial \tau_n}{\partial x} = \tau_{n+1}\tau_{n-1} - \tau_n^2\,, \tag{4}$$

which can be written as

$$D_x D_s \tau_n \cdot \tau_n = 2(\tau_{n+1}\tau_{n-1} - \tau_n^2)\,, \tag{5}$$

in terms of Hirota's operator,[18]

$$(D_z f \cdot g) = (\partial_z - \partial_{z'}) f(z) g(z')|_{z'=z}\,. \tag{6}$$

If we set

$$y_n = \ln \frac{\tau_{n+1}}{\tau_n}\,, \tag{7}$$

then we obtain another form for the 2D Toda lattice equation:

$$\frac{\partial^2 y_n}{\partial s \partial x} = \mathrm{e}^{y_{n+1} - y_n} - \mathrm{e}^{y_n - y_{n-1}}\,. \tag{8}$$

Two forms (1) and (8) of the 2D Toda lattice equation are linked through

$$\frac{\partial^2 y_n}{\partial s \partial x} = V_{n+1} - V_n.$$

In this paper, we would like to establish a general Casoratian formulation for the 2D Toda lattice equation (4) and analyze its exact solutions based on the resulting Casoratian formulation and a Bäcklund transformation.

The paper is organized as follows. In Sec. 2, a general Casoratian formulation is presented for the bilinear 2D Toda lattice equation (4). In Sec. 3, some specific cases of linear conditions are discussed and a Bäcklund transformation is furnished to construct exact solutions, and various examples of Casoratian type solutions are presented. Concluding remarks are given finally in Sec. 4.

2. A General Casoratian Formulation

The N-soliton solution to the bilinear 2D Toda lattice equation (4) is expressed as a Casorati determinant,[19]

$$\tau_n = \mathrm{Cas}(\phi_1, \phi_2, \ldots, \phi_N) = \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \phi_2(n) & \phi_2(n+1) & \cdots & \phi_2(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix}, \tag{9}$$

where each $\phi_i(n) = \phi_i(n, x, s)$ satisfies the linear differential-difference equations

$$\frac{\partial \phi_i(n)}{\partial x} = \phi_i(n+1), \ \frac{\partial \phi_i(n)}{\partial s} = -\phi_i(n-1), \ 1 \le i \le N\,. \tag{10}$$

The above Casorati determinant has been used in the theory of the 1D lattice equations.[8–11] We will adopt the notation[8]

$$k..l = k, k+1, \ldots, l\,, \tag{11}$$

where $k < l$, and it denotes the generalized Casorati determinant by

$$|i_1, \ldots, i_N| = \det([i_1, \ldots, i_N])\,, \tag{12}$$

where $i_j \in \mathbb{Z},\ 1 \le j \le N$, and the matrix $[i_1, \ldots, i_N]$ is defined by

$$[i_1, \ldots, i_N] = \begin{bmatrix} \phi_1(n+i_1) & \phi_1(n+i_2) & \cdots & \phi_1(n+i_N) \\ \phi_2(n+i_1) & \phi_2(n+i_2) & \cdots & \phi_2(n+i_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n+i_1) & \phi_N(n+i_2) & \cdots & \phi_N(n+i_N) \end{bmatrix}. \tag{13}$$

Obviously, the standard Casorati determinant is given by

$$\mathrm{Cas}(\phi_1, \phi_2, \ldots, \phi_N) = |0..N-1|\,.$$

Theorem 1. *Let* $\varepsilon = \pm 1$ *and* $\delta = \pm 1$, *i.e.,* $(\varepsilon, \delta) = (1,1),\ (1,-1),\ (-1,1)$ *or* $(-1,-1)$. *If a set of functions* $\phi_i(n) = \phi_i(n, x, s),\ 1 \le i \le N$, *satisfies the following* *coupled linear differential-difference equations:*

$$\frac{\partial \phi_i(n)}{\partial x} = \varepsilon \phi_i(n+\delta) + \sum_{j=1}^{N} \lambda_{ij}(x)\phi_j(n),\ 1 \le i \le N\,, \tag{14}$$

$$\frac{\partial \phi_i(n)}{\partial s} = -\varepsilon \phi_i(n-\delta) + \sum_{j=1}^{N} \mu_{ij}(s)\phi_j(n),\ 1 \le i \le N, \tag{15}$$

where $\lambda_{ij}(x)$ *and* $\mu_{ij}(s)$, $1 \le i,j \le N$, *are arbitrary real functions, then* $\tau_n = |0..N-1|$ *defined by Eq.* (9) *solves the bilinear 2D Toda lattice equation* (4).

Proof. Under an exchange of the variables x and s, the cases of linear conditions (14) and (15) with different values $\delta = \pm 1$ are transformed into each other, but the bilinear 2D Toda lattice equation (4) is invariant. Therefore, we only need to check the case under $\delta = 1$. In what follows, we set $\delta = 1$.

Let us use $(Ef)(n) = f(n+1)$ and define

$$(L_x\phi_i)(n) = \sum_{j=1}^{N} \lambda_{ij}\phi_j(n), \qquad (L_s\phi_i)(n) = \sum_{j=1}^{N} \mu_{ij}\phi_j(n),\ 1 \le i \le N\,. \tag{16}$$

Then, using Eq. (14), we can compute that

$$\frac{\partial \tau_n}{\partial x} = \sum_{i=1}^{N} \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x\phi_i(n) & \partial_x\phi_i(n+1) & \cdots & \partial_x\phi_i(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix}$$

$$= \varepsilon \sum_{i=1}^{N} \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ (E\phi_i)(n) & (E\phi_i)(n+1) & \cdots & (E\phi_i)(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix}$$

$$+ \sum_{i=1}^{N} \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ (L_x\phi_i)(n) & (L_x\phi_i)(n+1) & \cdots & (L_x\phi_i)(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix}$$

$$= \varepsilon \sum_{j=1}^{N} \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & (E\phi_1)(n+j-1) & \cdots & \phi_1(n+N-1) \\ \phi_2(n) & \phi_2(n+1) & \cdots & (E\phi_2)(n+j-1) & \cdots & \phi_2(n+N-1) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & (E\phi_N)(n+j-1) & \cdots & \phi_N(n+N-1) \end{vmatrix}$$

$$+ \sum_{i=1}^{N} \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{ii}\phi_i(n) & \lambda_{ii}\phi_i(n+1) & \cdots & \lambda_{ii}\phi_i(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix}$$

$$= \varepsilon|0..N-2, N| + \left(\sum_{i=1}^{N} \lambda_{ii}\right)\tau_n \,.$$

Using almost the same argument, we can obtain

$$\frac{\partial \tau_n}{\partial s} = -\varepsilon|-1, 1..N-1| + \left(\sum_{i=1}^{N} \mu_{ii}\right)\tau_n \,.$$

Further, we can similarly compute that

$$\begin{aligned} \frac{\partial^2 \tau_n}{\partial s \partial x} &= -|-1, 1..N-2, N| - \tau_n + \varepsilon\left(\sum_{i=1}^{N} \mu_{ii}\right)|0..N-2, N| \\ &\quad + \left(\sum_{i=1}^{N} \lambda_{ii}\right)\left[-\varepsilon|-1, 1..N-1| + \left(\sum_{i=1}^{N} \mu_{ii}\right)\tau_n\right] \\ &= -|-1, 1..N-2, N| - \tau_n + \varepsilon\left(\sum_{i=1}^{N} \mu_{ii}\right)|0..N-2, N| \\ &\quad - \varepsilon\left(\sum_{i=1}^{N} \lambda_{ii}\right)|-1, 1..N-1| + \left(\sum_{i=1}^{N} \lambda_{ii}\right)\left(\sum_{i=1}^{N} \mu_{ii}\right)\tau_n \,. \end{aligned}$$

Plugging these results into the bilinear equation (4) gives

$$\begin{aligned} &\frac{\partial^2 \tau_n}{\partial s \partial x}\tau_n - \frac{\partial \tau_n}{\partial s}\frac{\partial \tau_n}{\partial x} - \tau_{n+1}\tau_{n-1} + \tau_n^2 \\ &\quad = -|-1, 1..N-2, N||0..N-1| + |0..N-2, N||-1, 1..N-1| \\ &\qquad -|1..N||-1..N-2| \,. \end{aligned}$$

This sum is the Laplace expansion by $N \times N$ minors of the following $2N \times 2N$ determinant:

$$-\frac{1}{2}\begin{vmatrix} [\,-1,0, & 1..N-2\,] & [\quad \emptyset, & N-1,N] \\ [\,-1,0, & \emptyset\quad] & [\,1..N-2, & N-1,N] \end{vmatrix},$$

where $\emptyset$ indicates the $N \times (N-2)$ zero matrix, and $[\emptyset, N-1, N] = [\emptyset, \Phi(n+N-1), \Phi(n+N)]$ and $[-1, 0, \emptyset] = [\Phi(n-1), \Phi(n), \emptyset]$ with $\Phi(m) = (\phi_1(m), \ldots, \phi_N(m))^T$. Obviously, this determinant is zero. Therefore, the solution is verified. $\square$

The linear conditions (14) and (15) in the case of $(\varepsilon, \delta) = (1, 1)$ is a generalization of the conditions (10). Theorem 1 tells us that if a set of functions $\phi_i(n)$, $1 \le i \le N$, satisfies all linear conditions in (14) and (15), then we can get a Casoratian solution $\tau_n = |0..N-1|$ to the bilinear 2D Toda lattice equation (4). If we exchange x and s in τ_n, we can get another Casoratian solution, based on Theorem 1.

Let us observe how the Casoratian formulation generates solutions a little bit more carefully. From the compatibility conditions $\phi_{i,xs} = \phi_{i,sx}$, $1 \le i \le N$, of the conditions (14) and (15), we have the equalities

$$\sum_{j,k=1}^{N} (\lambda_{ij}\mu_{jk} - \mu_{ij}\lambda_{jk})\phi_k = 0,\ 1 \le i \le N, \tag{17}$$

and thus we see that the Casorati determinant $\mathrm{Cas}(\phi_1, \phi_2, \ldots, \phi_N)$ becomes zero at a point (x, s) where the coefficient matrices $A = A(x) = (\lambda_{ij}(x))_{N\times N}$ and $B = B(s) = (\mu_{ij}(s))_{N\times N}$ do not commute. Therefore, if A and B are constant and do not commute, then $\tau_n = |0..N-1|$ is zero. This shows that the reduced case of Eqs. (14) and (15) under

$$A(x)B(s) - B(s)A(x) = 0 \tag{18}$$

is important in generating non-trivial Casoratian solutions to the bilinear 2D Toda lattice equation (4).

3. Casoratian Type Solutions

We would like to construct exact solutions of the bilinear 2D Toda lattice equation (4) by using the resulting Casoratian formulation and introducing a Bäcklund transformation.

Theorem 2. *If $A(x) = (\lambda_{ij}(x))_{N\times N}$ and $B(s) = (\mu_{ij}(s))_{N\times N}$ are continuous and satisfy* (18) *and*

$$A(x)\int_0^x A(x')dx' = \int_0^x A(x')dx' A(x), \tag{19}$$

$$B(s)\int_0^s B(s')ds' = \int_0^s B(s')ds' B(s)\,, \tag{20}$$

then the linear differential-difference equations (14) *and* (15) *have the following solution*:

$$\Phi = \Phi(n) = \exp\left(\int_0^x A(x')dx' + \int_0^s B(s')ds'\right)$$
$$\times\, (p_1^n \mathrm{e}^{\varepsilon(p_1^\delta x - p_1^{-\delta}s)+q_1}, \ldots, p_N^n \mathrm{e}^{\varepsilon(p_N^\delta x - p_N^{-\delta}s)+q_N})^T\,, \tag{21}$$

where $\Phi = (\phi_1, \ldots, \phi_N)^T$ *and* $p_i \neq 0,\ q_i,\ 1 \le i \le N$, *are arbitrary real constants.*

Proof. The condition (18) implies that

$$\exp\left(\int_0^x A(x')dx' + \int_0^s B(s')ds'\right) = \exp\left(\int_0^x A(x')dx'\right)\exp\left(\int_0^s B(s')ds'\right)$$
$$= \exp\left(\int_0^s B(s')ds'\right)\exp\left(\int_0^x A(x')dx'\right),$$
$$A(x)\exp\left(\int_0^s B(s')ds'\right) = \exp\left(\int_0^s B(s')ds'\right)A(x),$$
$$B(s)\exp\left(\int_0^x A(x')dx'\right) = \exp\left(\int_0^x A(x')dx'\right)B(s)\,.$$

The other two conditions (19) and (20) guarantee that

$$\partial_x \exp\left(\int_0^x A(x')dx'\right) = A(x)\exp\left(\int_0^x A(x')dx'\right), \tag{22}$$
$$\partial_x \exp\left(\int_0^s B(s')ds'\right) = B(s)\exp\left(\int_0^s B(s')ds'\right), \tag{23}$$

respectively. Further, a direct computation shows that

$$\frac{\partial\Phi(n)}{\partial x} = \varepsilon\Phi(n+\delta) + A(x)\Phi(n), \qquad \frac{\partial\Phi(n)}{\partial s} = -\varepsilon\Phi(n-\delta) + B(s)\Phi(n)\,.$$

This verifies the solution in Eq. (21). □

Noting that Eqs. (14) and (15) are linear, any linear combination of Φ defined by Eq. (21) with different sets of p_i and q_i, $1 \le i \le N$, is again a solution to Eqs. (14) and (15). One example is the set of functions

$$\phi_i = \sum_{j=1}^M p_{ij}^n \mathrm{e}^{(\varepsilon p_{ij}^\delta + \lambda_i)x - (\varepsilon p_{ij}^{-\delta} - \mu_i)s + q_{ij}},\ 1 \le i \le N, \tag{24}$$

where p_{ij} are arbitrary non-zero real constants and q_{ij}, λ_i and μ_i are arbitrary real constants. Actually, $\Phi = (\phi_1, \ldots, \phi_N)^T$ satisfies the linear conditions (14) and (15) with $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and $B = \mathrm{diag}(\mu_1, \ldots, \mu_N)$. Thus, we have a Casoratian solution $\tau_n = |0..N-1| = \mathrm{Cas}(\phi_1, \ldots, \phi_N)$. The N-soliton solutions correspond to $M = 2$.[20] The situation with a general integer M yields new Casoratian solutions involving many free parameters.

If for each $l \le N$, we further take $\lambda_i = \mu_i,\ 1 \le i \le l$, then

$$\Phi = \left(\phi_1, \partial_{\lambda_1}\phi_1, \ldots, \frac{1}{k_1!}\partial_{\lambda_1}^{k_1}\phi_1; \cdots; \phi_l, \partial_{\lambda_l}\phi_l, \ldots, \frac{1}{k_l!}\partial_{\lambda_l}^{k_l}\phi_l\right), \tag{25}$$

where $k_1 + \cdots + k_l = N$, satisfies the linear conditions (14) and (15) with

$$A = B = \mathrm{diag}(C_1, \cdots, C_l),\ C_i = \begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i \times k_i},\ 1 \le i \le l\,.$$

Thus, this gives us the following Casoratian solution:

$$\tau_n = \mathrm{Cas}\left(\phi_1, \partial_{\lambda_1}\phi_1, \ldots, \frac{1}{k_1!}\partial_{\lambda_1}^{k_1}\phi_1; \cdots; \phi_l, \partial_{\lambda_l}\phi_l, \ldots, \frac{1}{k_l!}\partial_{\lambda_l}^{k_l}\phi_l\right). \tag{26}$$

Theorem 3. *If $\tau_n = \tau_n(x, s)$ solves the bilinear 2D Toda lattice equation* (4), *and $\sigma_n = \sigma_n(x, s)$ satisfies*

$$\frac{\partial^2 \sigma_n}{\partial s \partial x}\sigma_n = \frac{\partial \sigma_n}{\partial x}\frac{\partial \sigma_n}{\partial s}\,, \qquad \sigma_{n+1}\sigma_{n-1} = \sigma_n^2\,, \tag{27}$$

then the function $\tilde{\tau}_n$ defined by

$$\tilde{\tau}_n = \tilde{\tau}_n(x, s) = \sigma_n(\alpha x, \alpha^{-1}s)\tau_n(\alpha x, \alpha^{-1}s), \tag{28}$$

where α is a non-zero real constant, presents another solution to the bilinear 2D Toda lattice equation (4).

Proof. Under the first condition in Eq. (27), a direct computation tells that

$$\left(\frac{\partial^2 \tilde{\tau}_n}{\partial s \partial x}\tilde{\tau}_n - \frac{\partial \tilde{\tau}_n}{\partial x}\frac{\partial \tilde{\tau}_n}{\partial s}\right)(x, s) = \left[\sigma_n^2\left(\frac{\partial^2 \tau_n}{\partial s \partial x}\tau_n - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial s}\right)\right](\alpha x, \alpha^{-1}s)\,.$$

Thus, the second condition in Eq. (27) ensures that

$$\begin{aligned} &\left(\frac{\partial^2 \tilde{\tau}_n}{\partial s \partial x}\tilde{\tau}_n - \frac{\partial \tilde{\tau}_n}{\partial x}\frac{\partial \tilde{\tau}_n}{\partial s} - \tilde{\tau}_{n+1}\tilde{\tau}_{n-1} + \tilde{\tau}_n^2\right)(x, s) \\ &\quad = \left[\sigma_n^2\left(\frac{\partial^2 \tau_n}{\partial s \partial x}\tau_n - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial s} - \tau_{n+1}\tau_{n-1} + \tau_n^2\right)\right](\alpha x, \alpha^{-1}s) = 0\,. \end{aligned}$$

The theorem is proved. □

This theorem provides us with an auto-Bäcklund transformation of the bilinear 2D Toda lattice equation (4). Generally, it also generates new solutions to the nonlinear 2D Toda lattice equations (1) and (8) from a given solution to the bilinear 2D Toda lattice equation (4), through the transformations given in the introduction. However, the case of $\alpha = 1$ does not lead to new solutions to the nonlinear 2D Toda lattice equation (1).

A particular selection of σ_n in Theorem 3 engenders the following corollary.

Corollary 1. *Let $\tau_n = \tau_n(x, s)$ be a solution to the bilinear 2D Toda lattice equation* (4) *and α be a non-zero real constant. If $a_n(x)$ and $b_n(s)$ satisfy*

$$a_{n+1}(x)a_{n-1}(x)b_{n+1}(s)b_{n-1}(s) = (a_n(x))^2(b_n(s))^2, \tag{29}$$

then $\tilde{\tau}_n$ with $\sigma_n(x, s) = a_n(x)b_n(s)$:

$$\tilde{\tau}_n = \tilde{\tau}_n(x, s) = a_n(\alpha x)b_n(\alpha^{-1}s)\tau_n(\alpha x, \alpha^{-1}s) \tag{30}$$

solves the bilinear 2D Toda lattice equation (4).

In particular, if $a(x)$, $b(s)$, $f(x)$ and $g(s)$ are real functions but $f(x)$ and $g(s)$ are positive or negative, then $\tilde{\tau}_n$ with $a_n(x) = a(x)(f(x))^n$ and $b_n(s) = b(s)(g(s))^n$:

$$\tilde{\tau}_n = \tilde{\tau}_n(x, s) = a(\alpha x)b(\alpha^{-1}s)(f(\alpha x))^n(g(\alpha^{-1}s))^n\tau_n(\alpha x, \alpha^{-1}s) \tag{31}$$

solves the bilinear 2D Toda lattice equation (4).

In the above corollary, the assumption that $f(x)$ and $g(s)$ are positive or negative is just to guarantee that $\tilde{\tau}_n$ is well-defined over the domain of $x, s \in \mathbb{R}$ and $n \in \mathbb{Z}$.

A combination of Theorems 1, 2 and 3 offers us an approach for constructing Casoratian type solutions to the bilinear 2D Toda lattice equation (4).

If we take $a(x) = b(s) = 1$, $f(x) = x^\beta$ and $g(s) = s^\alpha$, the resulting solution $\tilde{\tau}_n$ with $\alpha = 1$ gives the solution presented in Ref. 20.

Let us take

$$\bar{\phi}_i = p_i^n e^{\lambda_i x + \mu_i s + q_i} = e^{-\varepsilon p_i^\delta x + \varepsilon p_i^{-\delta} s}\phi_i, \ 1 \le i \le N, \tag{32}$$

where $p_i \neq 0$, q_i, λ_i and β_i, $1 \le i \le N$, are arbitrary real constants. The set of functions $\{\phi_i\}_{i=1}^N$ satisfies Eqs. (14) and (15) with $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and $B = \mathrm{diag}(\mu_1, \ldots, \mu_N)$ as showed before, and obviously, we have

$$\begin{aligned}\bar{\tau}_n &= \mathrm{Cas}(\bar{\phi}_1, \ldots, \bar{\phi}_N) \\ &= \exp\left(\sum_{i=1}^N (\lambda_i x + \mu_i s + q_i)\right) \prod_{i=1}^N p_i^n \prod_{i>j} (p_i - p_j) \\ &= \exp\left(-\varepsilon \sum_{i=1}^N p_i^\delta x\right) \exp\left(\varepsilon \sum_{i=1}^N p_i^{-\delta} s\right) \mathrm{Cas}(\phi_1, \ldots, \phi_N) \,. \end{aligned} \tag{33}$$

The last equality in Eq. (33) also tells us a formula for $\tau_n = \mathrm{Cas}(\phi_1, \ldots, \phi_N)$, where ϕ_i are defined by (24) with $M = 1$. It follows from the above corollary with $\alpha = 1$ that $\bar{\tau}_n$ is a Casoratian solution to the bilinear 2D Toda lattice equation (4). Again from the above corollary, we have a class of Casoratian type solutions to the bilinear 2D Toda lattice equation (4):

$$\begin{aligned}\tilde{\tau}_n &= a(\alpha x)b(\alpha^{-1}s)(f(\alpha x))^n(g(\alpha^{-1}s))^n \\ &\quad \times \exp\left(\sum_{i=1}^N (\lambda_i \alpha x + \mu_i \alpha^{-1} s + q_i)\right) \prod_{i=1}^N p_i^n \prod_{i>j} (p_i - p_j) \,. \end{aligned} \tag{34}$$

Obviously, these solutions $\tilde{\tau}_n$ are all just special cases of (31) with $\tau_n = 1$. They generate non-constant solutions to the nonlinear 2D Toda lattice equation (8), but only the zero solution to the nonlinear 2D Toda lattice equation (1).

If we now take the functions ϕ_i, $1 \le i \le N$, in Eq. (24) with $M = 2$, i.e.,

$$\phi_i = p_{i1}^n \mathrm{e}^{(\varepsilon p_{i1}^{\delta}+\lambda_i)x-(\varepsilon p_{i1}^{-\delta}-\mu_i)s+q_{i1}} + p_{i2}^n \mathrm{e}^{(\varepsilon p_{i2}^{\delta}+\lambda_i)x-(\varepsilon p_{i2}^{-\delta}-\mu_i)s+q_{i2}}, \ 1 \le i \le N, \quad (35)$$

then by the above corollary, we have a class of Casoratian type solutions to the bilinear 2D Toda lattice equation (4):

$$\tilde{\tau}_n = \tilde{\tau}_n(x,s) = a(\alpha x)b(\alpha^{-1}s)(f(\alpha x))^n(g(\alpha^{-1}s))^n \mathrm{Cas}(\phi_1,\ldots,\phi_N)(\alpha x, \alpha^{-1}s)\,. \quad (36)$$

A general case of M in Eq. (24) can produce more general Casoratian type solutions to the bilinear 2D Toda lattice equation (4). Such solutions $\tilde{\tau}_n$ can also generate new solutions to the nonlinear 2D Toda lattice equation (8), and if $\alpha \neq 1$, new solutions to the nonlinear 2D Toda lattice equation (1).

4. Concluding Remarks

A general Casoratian formulation of the bilinear 2D Toda lattice equation (4) has been presented by means of the bilinear form of (4). The resulting theory provides us with an effective approach for constructing exact solutions to the bilinear 2D Toda lattice equation (4). Special classes of functions satisfying (14) and (15), e.g., the functions defined by Eqs. (24) and (25), were used to generate Casoratian solutions, and further using the Bäcklund transformation in Theorem 3, various examples of Casoratian type solutions were presented.

We remark that the solutions $\tilde{\tau}_n$ presented in Corollary 1 may not be exactly Casoratian, even if τ_n is Casoratian. For example, $\tilde{\tau}_n$ is non-Casoratian when $f(x)$ and $g(s)$ are not constant functions. On the other hand, taking different types of functions for $a(x)$, $b(s)$, $f(x)$ and $g(s)$ can yield positon and complexiton type solutions.

There are also two other questions that we are interested in. The first question is how to solve the system of differential-difference equations in Eqs. (14) and (15) generally, in particular, in the case where the conditions (19) and (20) are not satisfied, or more generally, Eqs. (22) and (23) do not hold. This will bring us very different Casoratian solutions to the bilinear 2D Toda lattice equation (4). The second question is what kind of Casoratian formulations can exist for Pfaffianization of discrete soliton equations,[21,22] for example, for Pfaffianization of the 2D Toda lattice equation.[21] Any answers to these two questions will enhance our understanding of both diversity of Casoratian type solutions and university of Casoratian formulations.

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