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Riemann-Hilbert problems and soliton solutions of a multicomponent mKdV system and its reduction

Wen-Xiu Ma[1,2,3,4,5,6]

[1]College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai, China
[2]Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
[3]Department of Mathematics and Statistics, University of South Florida, Tampa, Florida
[4]Department of Mathematics, Zhejiang Normal University, Jinhua, China
[5]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, China
[6]International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho, South Africa

Correspondence
Wen-Xiu Ma, Department of Mathematics and Statistics, University of South Florida, Tampa FL 33620-5700.
Email: mawx@cas.usf.edu

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An arbitrary order matrix spectral problem is introduced and its associated multicomponent AKNS integrable hierarchy is constructed. Based on this matrix spectral problem, a kind of Riemann-Hilbert problems is formulated for a multicomponent mKdV system in the resulting AKNS integrable hierarchy. Through special corresponding Riemann-Hilbert problems with an identity jump matrix, soliton solutions to the presented multicomponent mKdV system are explicitly worked out. A specific reduction of the multicomponent mKdV system is made, together with its reduced Lax pair and soliton solutions.

KEYWORDS
matrix spectral problem, Riemann-Hilbert problem, soliton solution

1 | INTRODUCTION

The Riemann-Hilbert approach is one of the most powerful techniques to study integrable equations and particularly generate soliton solutions.[1] The approach is closely connected with the nonlinear Fourier method, called the inverse scattering method, in soliton theory.[2] It starts from a kind of matrix spectral problems possessing bounded eigenfunctions analytically extendable to the upper or lower half plane. The normalization conditions at infinity on the real axis in constructing the scattering coefficients is used in solving the corresponding Riemann-Hilbert problems.[1] Once taking the identity jump matrix, reduced Riemann-Hilbert problems yield soliton solutions, whose special limits can generate lump solutions, periodic solutions, and complexiton solutions. A few integrable equations, including the multiple wave interaction equations,[1] the general coupled nonlinear Schrödinger equations,[3] the Harry Dym equation,[4] and the generalized Sasa-Satsuma equation,[5] have been studied by solving the associated Riemann-Hilbert problems.

To state the standard procedure for establishing Riemann-Hilbert problems on the real axis, we start from a pair of matrix spectral problems of the following form:

$$-i\phi_x = U\phi, -i\phi_t = V\phi, \quad U = A(\lambda) + P(u, \lambda), \quad V = B(\lambda) + Q(u, \lambda),$$

where i is the unit imaginary number, λ is a spectral parameter, u is a potential, ϕ is an $m \times m$ matrix eigenfunction, A, B are constant commuting $m \times m$ matrices, and P, Q are trace-less $m \times m$ matrices. The compatibility condition of the two matrix spectral problems is the zero curvature equation

$$U_t - V_x + i[U, V] = 0,$$

where $[\cdot, \cdot]$ is the matrix commutator. To formulate a Riemann-Hilbert problem for this zero curvature equation, we adopt the following pair of equivalent matrix spectral problems:

$$\psi_x = i[A(\lambda), \psi] + \check{P}(u, \lambda)\psi, \psi_t = i[B(\lambda), \psi] + \check{Q}(u, \lambda)\psi,$$

where ψ is an $m \times m$ matrix eigenfunction, $\check{P} = iP$ and $\check{Q} = iQ$. The commutativity of A and B guarantees this equivalence. The relation between the two matrix eigenfunctions ϕ and ψ is given by

$$\phi = \psi E_g, \quad E_g = \mathrm{e}^{iA(\lambda)x + iB(\lambda)t}.$$

This way, we can have two analytical matrix eigenfunctions with the asymptotic conditions

$$\psi^{\pm} \to I_m, \quad \text{when} \quad x, t \to \pm\infty,$$

where I_m stands for the identity matrix of size m. Then, based on those two matrix eigenfunctions $\psi^{\pm}$, we try to determine two analytical matrix functions $P^{\pm}(x, t, \lambda)$, which are analytical in the upper and lower half-planes of λ, respectively, to formulate a Riemann-Hilbert problem on the real axis:

$$G^+(x, t, \lambda) = G^-(x, t, \lambda)G(x, t, \lambda), \quad \lambda \in \mathbb{R},$$

with

$$G^+(x, t, \lambda) = \lim_{\mu \in \mathbb{C}^+, \mu \to \lambda} P^+(x, t, \mu), \quad (G^-)^{-1}(x, t, \lambda) = \lim_{\mu \in \mathbb{C}^-, \mu \to \lambda} P^-(x, t, \mu), \quad \lambda \in \mathbb{R},$$

where $\mathbb{C}^+$ is the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ and $\mathbb{C}^-$ is the lower half-plane $\mathbb{C}^- = \{z \in \mathbb{C} | \operatorname{Im}(z) < 0\}$. Upon taking the jump matrix G to be the identity matrix I_m, the corresponding Riemann-Hilbert problem can be normally solved to generate soliton solutions, through observing asymptotic behaviors of the matrix functions $P^{\pm}$ at infinity of λ, which also provide the canonical normalization conditions of the Riemann-Hilbert problems.

In this paper, we shall present an application example by focusing on a multicomponent system of modified Korteweg-de Vries (mKdV) equations and generate its soliton solutions by special associated Riemann-Hilbert problems. The rest of the paper is structured as follows. In section 2, within the zero-curvature formulation, we rederive the Ablowitz-Kaup-Newell-Segur (AKNS) integrable hierarchy with multiple potentials and furnish its bi-Hamiltonian structure, based on a new arbitrary order matrix spectral problem suited for the Riemann-Hilbert theory. In section 3, taking a system of coupled mKdV equations as an example, we analyze analytical properties of matrix eigenfunctions for an equivalent matrix spectral problem and build a kind of Riemann-Hilbert problems associated with the newly introduced matrix spectral problem. In section 4, we compute soliton solutions to the considered multicomponent system of coupled mKdV equations from special associated Riemann-Hilbert problems on the real axis, in which the jump matrix is taken as the identity matrix. In section 5, we make a specific reduction and present soliton solutions to the reduced mKdV systems by the reduced special Riemann-Hilbert problems. In the last section, we give a conclusion, together with some remarks.

2 | MULTICOMPONENT AKNS INTEGRABLE HIERARCHY

2.1 | Zero curvature scheme

The zero curvature scheme to generate integrable hierarchies is stated as follows.[6-8] Let u be a vector potential and λ, a spectral parameter. Choose a square matrix spectral matrix $U = U(u, \lambda)$ from a given matrix loop algebra, whose underlying Lie algebra could be either semisimple[6,7] or non-semisimple.[8] Assume that there is a formal Laurent series solution

$$W = W(u,\lambda) = \sum_{m=0}^{\infty} W_m \lambda^{-m} = \sum_{m=0}^{\infty} W_m(u)\lambda^{-m} \tag{2.1}$$

to the corresponding stationary zero curvature equation

$$W_x = i[U, W]. \tag{2.2}$$

Based on this solution W, we introduce a series of Lax matrices

$$V^{[r]} = V^{[r]}(u,\lambda) = (\lambda^r W)_+ + \Delta_r, \quad r \geq 0, \tag{2.3}$$

where the subscript + denotes the operation of taking a polynomial part in λ, and Δ_r, $r \geq 0$, are appropriate modification terms. The appropriateness of selecting Δ_r is required to generate an integrable hierarchy

$$u_t = K_r(u) = K_r(x, t, u, u_x, \cdots), \quad r \geq 0, \tag{2.4}$$

from a series of zero curvature equations

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \quad r \geq 0. \tag{2.5}$$

The two matrices U and $V^{[r]}$ are called a Lax pair[9] of the r-th soliton equation in the hierarchy (Equation 2.4). Obviously, the zero curvature equations in Equation 2.5 are the compatibility conditions of the spatial and temporal matrix spectral problems

$$-i\phi_x = U\phi = U(u,\lambda)\phi, -i\phi_t = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi, \quad r \geq 0, \tag{2.6}$$

where ϕ is the matrix eigenfunction.

To show the Liouville integrability of the hierarchy (Equation 2.4), we normally establish a bi-Hamiltonian structure[10]:

$$u_t = K_r = J_1 \frac{\delta \tilde{H}_{r+1}}{\delta u} = J_2 \frac{\delta \tilde{H}_r}{\delta u}, \quad r \geq 1, \tag{2.7}$$

where J_1 and J_2 form a Hamiltonian pair and $\frac{\delta}{\delta u}$ denotes the variational derivative (see, eg, the study of Ma and Fuchssteiner[11]). Such Hamiltonian structures can be usually furnished under the help of the trace identity[6]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(W\frac{\partial U}{\partial \lambda})dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left[\lambda^{\gamma}\mathrm{tr}(W\frac{\partial U}{\partial u})\right], \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|,$$

or more generally, the variational identity[8]:

$$\frac{\delta}{\delta u}\int \langle W, \frac{\partial U}{\partial \lambda}\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left[\lambda^{\gamma}\langle W, \frac{\partial U}{\partial u}\rangle\right], \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|,$$

where $\langle \cdot, \cdot \rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra.[12] The bi-Hamiltonian structure ensures that there exist infinitely many commuting Lie symmetries $\{K_r\}_{r=0}^{\infty}$ and conserved quantities $\{\tilde{H}_r\}_{r=0}^{\infty}$:

$$[K_{r_1}, K_{r_2}] = K'_{r_1}[K_{r_2}] - K'_{r_2}[K_{r_1}] = 0,$$
$$\{\tilde{\mathcal{H}}_{r_1}, \tilde{\mathcal{H}}_{r_2}\}_J = \int \left(\frac{\delta \tilde{\mathcal{H}}_{r_1}}{\delta u}\right)^T J \frac{\delta \tilde{\mathcal{H}}_{r_2}}{\delta u} dx = 0,$$

where $r_1, r_2 \geq 0, J = J_1$ or J_2, and K' stands for the Gateaux derivative of K with respect to u:

$$K'(u)[S] = \frac{\partial}{\partial \varepsilon}|_{\varepsilon=0} K(u + \varepsilon S, u_x + \varepsilon S_x, \cdots).$$

It is known that for an evolution equation with a vector potential u, $\tilde{H} = \int H\, dx$ is a conserved functional iff $\frac{\delta \tilde{H}}{\delta u}$ is an adjoint symmetry,[13] and thus, a Hamiltonian structure links conserved functionals to adjoint symmetries and further symmetries. The existence of an adjoint symmetry is necessary for a totally nondegenerate system of differential equations to admit a conservation law, and a pair of a symmetry and an adjoint symmetry leads to a conservation law for whatever systems of differential equations.[14,15] When the underlying matrix loop algebra in the zero curvature formulation is simple, the associated zero curvature equations engender classical integrable hierarchies[16,17]; when semisimple, the associated zero curvature equations generate a collection of different integrable hierarchies and when non-semisimple, we get hierarchies of integrable couplings[18] that require extra care in presenting Hamiltonian structures.

2.2 | Multicomponent AKNS hierarchy

Let n be an arbitrary natural number. We consider the following matrix spectral problem

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \quad U = (U_{jl})_{(n+1)\times(n+1)} = \begin{bmatrix} \alpha_1\lambda & p \\ q & \alpha_2\lambda I_n \end{bmatrix}, \tag{2.8}$$

where α_1 and α_2 are real constants, λ is a spectral parameter and u is a $2n$-dimensional potential

$$u = (p, q^T)^T, \quad p = (p_1, p_2, \cdots, p_n), \quad q = (q_1, q_2, \cdots, q_n)^T. \tag{2.9}$$

A special case of $p_j = q_j = 0, 2 \le j \le n$, transforms the matrix spectral problem (Equation 2.8) into the standard AKNS matrix spectral problem,[19] and therefore, it is called a multicomponent AKNS matrix spectral problem and its associated hierarchy, a multicomponent AKNS integrable hierarchy. Because of the existence of a multiple eigenvalue of $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2 I_n)$, the matrix spectral problem (Equation 2.8) is degenerate.

To derive an associated multicomponent AKNS integrable hierarchy, we first solve the stationary zero curvature (Equation 2.2) corresponding to Equation 2.8, as suggested in the general zero curvature scheme. We look for a solution W of the form

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.10}$$

where a is a scalar, b^T and c are n-dimensional columns, and d is an $n \times n$ matrix. It is direct to show that the stationary zero curvature (Equation 2.2) is

$$\begin{cases} a_x = i(pc - bq), \\ b_x = i(\alpha\lambda b + pd - ap), \\ c_x = i(-\alpha\lambda c + qa - dq), \\ d_x = i(qb - cp), \end{cases} \tag{2.11}$$

where $\alpha = \alpha_1 - \alpha_2$. We take W as a formal series:

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{m=0}^{\infty} W_m \lambda^{-m}, \quad W_m = W_m(u) = \begin{bmatrix} a^{[m]} & b^{[m]} \\ c^{[m]} & d^{[m]} \end{bmatrix}, \quad m \ge 0, \tag{2.12}$$

where $b^{[m]}, c^{[m]}$, and $d^{[m]}$ are expressed as follows:

$$b^{[m]} = (b_1^{[m]}, b_2^{[m]}, \cdots, b_n^{[m]}), \quad c^{[m]} = (c_1^{[m]}, c_2^{[m]}, \cdots, c_n^{[m]})^T, \quad d^{[m]} = (d_{jl}^{[m]})_{n\times n}, \quad m \ge 0. \tag{2.13}$$

Then, the system (Equation 2.11) exactly presents the following recursion relations:

$$b^{[0]} = 0, \quad c^{[0]} = 0, \quad a_x^{[0]} = 0, \quad d_x^{[0]} = 0, \tag{2.14a}$$

$$b^{[m+1]} = \frac{1}{\alpha}(-ib_x^{[m]} - pd^{[m]} + a^{[m]}p), \quad m \ge 0, \tag{2.14b}$$

$$c^{[m+1]} = \frac{1}{\alpha}(ic_x^{[m]} + qa^{[m]} - d^{[m]}q), \quad m \ge 0, \tag{2.14c}$$

$$a_x^{[m]} = i(pc^{[m]} - b^{[m]}q), \quad d_x^{[m]} = i(qb^{[m]} - c^{[m]}p), \quad m \ge 1. \tag{2.14d}$$

Next, we choose the initial values:

$$a^{[0]} = \beta_1, \quad d^{[0]} = \beta_2 I_n, \tag{2.15}$$

where β_1, β_2 are arbitrary real constants and take constants of integration in Equation 2.14d to be zero, that is, require

$$W_m|_{u=0} = 0, \quad m \ge 1. \tag{2.16}$$

Then, with $a^{[0]}$ and $d^{[0]}$ given by Equation 2.15, all matrices $W_m, m \geq 1$ are uniquely determined. For example, a direct computation, based on Equation 2.14, generates that

$$b_j^{[1]} = \frac{\beta}{\alpha} p_j, \quad c_j^{[1]} = \frac{\beta}{\alpha} q_j, \quad a^{[1]} = 0, \quad d_{jl}^{[1]} = 0; \tag{2.17a}$$

$$b_j^{[2]} = -\frac{\beta}{\alpha^2} i p_{j,x}, \quad c_j^{[2]} = \frac{\beta}{\alpha^2} i q_{j,x}, \quad a^{[2]} = -\frac{\beta}{\alpha^2} pq, \quad d_{jl}^{[2]} = \frac{\beta}{\alpha^2} p_l q_j; \tag{2.17b}$$

$$b_j^{[3]} = -\frac{\beta}{\alpha^3}[p_{j,xx} + 2pqp_j], \quad c_j^{[3]} = -\frac{\beta}{\alpha^3}[q_{j,xx} + 2pqq_j], \tag{2.17c}$$

$$a^{[3]} = -\frac{\beta}{\alpha^3} i(pq_x - p_x q), \quad d_{jl}^{[3]} = -\frac{\beta}{\alpha^3} i(p_{l,x} q_j - p_l q_{j,x}); \tag{2.17d}$$

$$b_j^{[4]} = \frac{\beta}{\alpha^4} i[p_{j,xxx} + 3pqp_{j,x} + 3p_x qp_j], \tag{2.17e}$$

$$c_j^{[4]} = -\frac{\beta}{\alpha^4} i[q_{j,xxx} + 3pqq_{j,x} + 3pq_x q_j], \tag{2.17f}$$

$$a^{[4]} = \frac{\beta}{\alpha^4}[3(pq)^2 + pq_{xx} - p_x q_x + p_{xx} q], \tag{2.17g}$$

$$d_{jl}^{[4]} = -\frac{\beta}{\alpha^4}[3p_l pqq_j + p_{l,xx} q_j - p_{l,x} q_{j,x} + p_l q_{j,xx}]; \tag{2.17h}$$

where $\beta = \beta_1 - \beta_2$ and $1 \leq j, l \leq n$. Based on Equation 2.14d, we can obtain, from Equation 2.14b and 2.14c, a recursion relation for $b^{[m]}$ and $c^{[m]}$:

$$\begin{bmatrix} c^{[m+1]} \\ b^{[m+1]T} \end{bmatrix} = \Psi \begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix}, \quad m \geq 1, \tag{2.18}$$

where Ψ is a $2n \times 2n$ matrix operator

$$\Psi = \frac{i}{\alpha} \begin{bmatrix} (\partial + \sum_{j=1}^{n} q_j \partial^{-1} p_j) I_n + q\partial^{-1} p & -q\partial^{-1} q^T - (q\partial^{-1} q^T)^T \\ p^T \partial^{-1} p + (p^T \partial^{-1} p)^T & -(\partial + \sum_{j=1}^{n} p_j \partial^{-1} q_j) I_n - p^T \partial^{-1} q^T \end{bmatrix}. \tag{2.19}$$

To generate the multicomponent AKNS integrable hierarchy, we introduce the following Lax matrices

$$V^{[r]} = V^{[r]}(u, \lambda) = (V_{jl}^{[r]})_{(n+1)\times(n+1)} = (\lambda^r W)_+ = \sum_{m=0}^{r} W_m \lambda^{r-m}, \quad r \geq 0, \tag{2.20}$$

where the modification terms are taken as zero. The compatibility conditions of Equation 2.6, ie, the zero curvature equations in (Equation 2.5), engender the so-called multicomponent AKNS integrable hierarchy:

$$u_t = \begin{bmatrix} p^T \\ q \end{bmatrix}_t = K_r = i \begin{bmatrix} \alpha b^{[r+1]T} \\ -\alpha c^{[r+1]} \end{bmatrix}, \quad r \geq 0. \tag{2.21}$$

The first two nonlinear integrable systems in the above hierarchy (Equation 2.21) are as follows:

$$p_{j,t} = -\frac{\beta}{\alpha^2} i[p_{j,xx} + 2\left(\sum_{l=1}^{n} p_l q_l\right) p_j], \quad 1 \leq j \leq n, \tag{2.22a}$$

$$q_{j,t} = \frac{\beta}{\alpha^2} i[q_{j,xx} + 2\left(\sum_{l=1}^{n} p_l q_l\right) q_j], \quad 1 \leq j \leq n, \tag{2.22b}$$

and

$$p_{j,t} = -\frac{\beta}{\alpha^3}[p_{j,xxx} + 3(\sum_{l=1}^{n} p_l q_l)p_{j,x} + 3\left(\sum_{l=1}^{n} p_{l,x} q_l\right)p_j], \quad 1 \le j \le n, \tag{2.23a}$$

$$q_{j,t} = -\frac{\beta}{\alpha^3}[q_{j,xxx} + 3(\sum_{l=1}^{n} p_l q_l)q_{j,x} + 3\left(\sum_{l=1}^{n} p_l q_{l,x}\right)q_j], \quad 1 \le j \le n, \tag{2.23b}$$

which are the multicomponent versions of the AKNS systems of coupled nonlinear Schrödinger equations and coupled mKdV equations, respectively. When $n = 2$, under a special kind of symmetric reductions, the multicomponent AKNS systems (Equation 2.22) can be reduced to the Manokov system,[20] for which a decomposition into finite-dimensional integrable Hamiltonian systems was made in the study of Chen and Zhou,[21] while as the multicomponent AKNS systems (Equation 2.23) contain various systems of mKdV equations, for which there exist various kinds of integrable decompositions under symmetry constraints (see, eg, the studies of Ma and Yu and Zhou[22,23]).

The multicomponent AKNS integrable hierarchy (Equation 2.21) possesses a bi-Hamiltonian structure,[13,24] which can be generated through the trace identity,[6] or more generally, the variational identity.[8] In fact, we have

$$-i\,\mathrm{tr}(W\frac{\partial U}{\partial \lambda}) = \alpha_1 a + \alpha_2 \mathrm{tr}(d) = \sum_{m=0}^{\infty}(\alpha_1 a^{[m]} + \alpha_2 \sum_{j=1}^{n} d_{jj}^{[m]})\lambda^{-m},$$

and

$$-i\,\mathrm{tr}(W\frac{\partial U}{\partial u}) = \begin{bmatrix} c \\ b^T \end{bmatrix} = \sum_{m\ge 0} G_{m-1}\lambda^{-m}.$$

Plugging these into the trace identity and checking the case of $m = 2$ tells $\gamma = 0$ in the trace identity, and thus, we have

$$\frac{\delta \tilde{H}_m}{\delta u} = iG_{m-1}, \quad \tilde{H}_m = -\frac{i}{m}\int (\alpha_1 a^{[m+1]} + \alpha_2 \sum_{j=1}^{n} d_{jj}^{[m+1]})dx, \quad G_{m-1} = \begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix}, \quad m \ge 1. \tag{2.24}$$

This tells the following bi-Hamiltonian structure for the multicomponent AKNS systems (Equation 2.21):

$$u_t = K_r = J_1 G_r = J_1 \frac{\delta \tilde{H}_{r+1}}{\delta u} = J_2 \frac{\delta \tilde{H}_r}{\delta u}, \quad r \ge 1, \tag{2.25}$$

where the Hamiltonian pair $(J_1, J_2 = J_1\Psi)$ is given by

$$J_1 = \begin{bmatrix} 0 & \alpha I_n \\ -\alpha I_n & 0 \end{bmatrix}, \tag{2.26a}$$

$$J_2 = i\begin{bmatrix} p^T\partial^{-1}p + (p^T\partial^{-1}p)^T & -(\partial + \sum_{j=1}^{n} p_j\partial^{-1}q_j)I_n - p^T\partial^{-1}q^T \\ -(\partial + \sum_{j=1}^{n} p_j\partial^{-1}q_j)I_n - q\partial^{-1}p & q\partial^{-1}q^T + (q\partial^{-1}q^T)^T \end{bmatrix}. \tag{2.26b}$$

Thus, the operator $\Phi = \Psi^\dagger = J_2 J_1^{-1}$ gives a recursion operator for the whole hierarchy (Equation 2.21). Adjoint symmetry constraints (or equivalently symmetry constraints) decompose each multicomponent AKNS system into two commuting finite-dimensional Liouville integrable Hamiltonian systems.[13,24] In the next section, we will concentrate on the multicomponent mKdV system (Equation 2.23).

3 | RIEMANN-HILBERT PROBLEMS

The matrix spectral problems of the multicomponent mKdV system (Equation 2.23) are

$$-i\phi_x = U\phi = U(u,\lambda)\phi, -i\phi_t = V^{[3]}\phi = V^{[3]}(u,\lambda)\phi, \tag{3.1}$$

where the Lax pair reads

$$U = \lambda\Lambda + P, \quad V^{[3]} = \lambda^3\Omega + Q, \tag{3.2}$$

with $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2 I_n)$, $\Omega = \mathrm{diag}(\beta_1, \beta_2 I_n)$, and

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} a^{[1]}\lambda^2 + a^{[2]}\lambda + a^{[3]} & b^{[1]}\lambda^2 + b^{[2]}\lambda + b^{[3]} \\ c^{[1]}\lambda^2 + c^{[2]}\lambda + c^{[3]} & d^{[1]}\lambda^2 + d^{[2]}\lambda + d^{[3]} \end{bmatrix}. \tag{3.3}$$

Here, u, p, q are defined by Equation 2.9, and $a^{[m]}, b^{[m]}, c^{[m]}, d^{[m]}$, $1 \le m \le 3$, are determined in Equation 2.17.

In what follows, we discuss the scattering and inverse scattering for the multicomponent mKdV system (Equation 2.23) using the Riemann-Hilbert approach[1] (see also the studies of Gerdjikov and Doktorov and Leble[25,26]). The results will lay the groundwork for soliton solutions in the following section. Assume that all the potentials rapidly vanish when $x \to \pm\infty$ or $t \to \pm\infty$. In order to facilitate the expression, we also assume that

$$\alpha = \alpha_1 - \alpha_2 < 0, \quad \beta = \beta_1 - \beta_2 < 0. \tag{3.4}$$

From the matrix spectral problems in Equation 3.1, we note, under Equation 3.8, that when $x, t \to \pm\infty$, we have the asymptotic behavior: $\phi \sim \mathrm{e}^{i\lambda\Lambda x + i\lambda^3\Omega t}$. Therefore, if we make the variable transformation

$$\phi = \psi E_g, \quad E_g = \mathrm{e}^{i\lambda\Lambda x + i\lambda^3\Omega t},$$

then we can have the canonical normalization $\psi \to I_{n+1}$, when $x, t \to \pm\infty$. Once setting $\check{P} = iP$ and $\check{Q} = iQ$, the equivalent pair of matrix spectral problems to Equation 3.1 reads

$$\psi_x = i\lambda[\Lambda, \psi] + \check{P}\psi, \tag{3.5}$$

$$\psi_t = i\lambda^3[\Omega, \psi] + \check{Q}\psi, \tag{3.6}$$

Applying a generalized Liouville's formula,[27] we can have

$$\det \psi = 1 \tag{3.7}$$

because of $\mathrm{tr}(\check{P}) = \mathrm{tr}(\check{Q}) = 0$.

Let us now formulate an associated Riemann-Hilbert problem with the variable x, under the integrable conditions:

$$\int_{-\infty}^{\infty} |x|^m \sum_{j=1}^{n} (|p_j| + |q_j|)\, dx < \infty, \quad m = 0, 1. \tag{3.8}$$

In the scattering problem, we first introduce the two matrix solutions $\psi^{\pm}(x, \lambda)$ of Equation 3.5 with the asymptotic conditions

$$\psi^{\pm} \to I_{n+1}, \quad \text{when} \quad x \to \pm\infty, \tag{3.9}$$

respectively. The above superscripts refers to which end of the x-axis the boundary conditions are required for. By Equation 3.7, we see that $\det \psi^{\pm} = 1$ for all $x \in \mathbb{R}$. Since

$$\phi^{\pm} = \psi^{\pm} E, \quad E = \mathrm{e}^{i\lambda\Lambda x} \tag{3.10}$$

are both matrix solutions of Equation 3.1, they are linearly dependent, and as a result of the fact, one has

$$\psi^- E = \psi^+ E S(\lambda), \quad \lambda \in \mathbb{R}, \tag{3.11}$$

where $S(\lambda) = (s_{jl})_{(n+1)\times(n+1)}$is the scattering matrix. Note that $\det S(\lambda) = 1$ because of $\det \psi^{\pm} = 1$.

Through the method of variation in parameters, we can turn the x-part of Equation 3.1) into the following Volterra integral equations for $\psi^{\pm 1}$:

$$\psi^-(\lambda, x) = I_{n+1} + \int_{-\infty}^{x} \mathrm{e}^{i\lambda\Lambda(x-y)} \check{P}(y) \psi^-(\lambda, y) \mathrm{e}^{i\lambda\Lambda(y-x)}\, dy, \tag{3.12}$$

$$\psi^+(\lambda, x) = I_{n+1} - \int_{x}^{\infty} \mathrm{e}^{i\lambda\Lambda(x-y)} \check{P}(y) \psi^+(\lambda, y) \mathrm{e}^{i\lambda\Lambda(y-x)}\, dy, \tag{3.13}$$

where the boundary condition (Equation 3.9) has been used. Therefore, under the conditions (Equation 3.8), the theory of Volterra integral equations tells that the eigenfunctions $\psi^{\pm}$ exist and allow analytical continuations off the real axis $\lambda \in \mathbb{R}$ as long as the integrals on their right hand sides converge. Based on the diagonal form of Λ and the first assumption in Equation 3.4, we can directly find that the integral equation for the first column of ψ^- contains only the exponential factor $e^{-i\alpha\lambda(x-y)}$, which decays because of $y < x$ in the integral, when λ is in the upper half-plane $\mathbb{C}^+$, and the integral equation for the last n columns of ψ^+ contains only the exponential factor $e^{i\alpha\lambda(x-y)}$, which also decays because of $y > x$ in the integral, when λ is in the upper half-plane $\mathbb{C}^+$. Thus, these $n+1$ columns can be analytically continued to the closed upper half-plane. In a similar manner, we can find that the last n columns of ψ^- and the first column of ψ^+ can be analytically continued to the closed lower half-plane.

First, if we express

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \cdots, \psi_{n+1}^{\pm}), \tag{3.14}$$

that is, $\psi_j^{\pm}$ stands for the jth column of $\phi^{\pm}$ $(1 \leq j \leq n+1)$, then the matrix solution

$$P^+ = P^+(x,\lambda) = (\psi_1^-, \psi_2^+, \cdots, \psi_{n+1}^+) = \psi^- H_1 + \psi^+ H_2 \tag{3.15}$$

is analytic in $\lambda \in \mathbb{C}^+$, and the matrix solution

$$(\psi_1^+, \psi_2^-, \cdots, \psi_{n+1}^-) = \psi^+ H_1 + \psi^- H_2 \tag{3.16}$$

is analytic in $\lambda \in \mathbb{C}^-$, where

$$H_1 = \mathrm{diag}(1, \underbrace{0, \cdots, 0}_{n}), \quad H_2 = \mathrm{diag}(0, \underbrace{1, \cdots, 1}_{n}). \tag{3.17}$$

Moreover, from the Volterra integral (Equations 3.12 and 3.13), we see that

$$P^+(x,\lambda) \to I_{n+1}, \quad \text{when} \quad \lambda \in \mathbb{C}^+ \to \infty, \tag{3.18}$$

and

$$(\psi_1^+, \psi_2^-, \cdots, \psi_{n+1}^-) \to I_{n+1}, \quad \text{when} \quad \lambda \in \mathbb{C}^- \to \infty. \tag{3.19}$$

Secondly, we construct the analytic counterpart of P^+ in the lower half-plane $\mathbb{C}^-$ from the adjoint counterparts of the matrix spectral problems. The adjoint equation of the x-part of Equation 3.1 and the adjoint equation of Equation 3.5 read as

$$i\tilde{\phi}_x = \tilde{\phi} U, \tag{3.20}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi} P. \tag{3.21}$$

Note that the inverse matrices $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and $\tilde{\psi}^{\pm} = (\psi^{\pm})^{-1}$ solve these two adjoint equations, respectively. Upon expressing $\tilde{\psi}^{\pm}$ as follows:

$$\tilde{\psi}^{\pm} = (\tilde{\psi}^{\pm,1}, \tilde{\psi}^{\pm,2}, \cdots, \tilde{\psi}^{\pm,n+1})^T, \tag{3.22}$$

that is, $\tilde{\psi}^{\pm,j}$ stands for the jth row of $\tilde{\psi}^{\pm}$ $(1 \leq j \leq n+1)$, we can verify by similar arguments that the adjoint matrix solution of Equation 3.21,

$$P^- = (\tilde{\psi}^{-,1}, \tilde{\psi}^{+,2}, \cdots, \tilde{\psi}^{+,n+1})^T = H_1\tilde{\psi}^- + H_2\tilde{\psi}^+ = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1}, \tag{3.23}$$

is analytic for $\lambda \in \mathbb{C}^-$, and the other matrix solution of Equation 3.21,

$$(\tilde{\psi}^{+,1}, \tilde{\psi}^{-,2}, \cdots, \tilde{\psi}^{-,n+1})^T = H_1\tilde{\psi}^+ + H_2\tilde{\psi}^- = H_1(\psi^+)^{-1} + H_2(\psi^-)^{-1}, \tag{3.24}$$

is analytic for $\lambda \in \mathbb{C}^+$. In the same way, we find that

$$P^-(x,\lambda) \to I_{n+1}, \quad \text{when} \quad \lambda \in \mathbb{C}^- \to \infty, \tag{3.25}$$

and

$$(\tilde{\psi}^{+,1}, \tilde{\psi}^{-,2}, \cdots, \tilde{\psi}^{-,n+1})^T \to I_{n+1}, \quad \text{when} \quad \lambda \in \mathbb{C}^+ \to \infty. \tag{3.26}$$

Now, we have constructed the two matrix functions, P^+ and P^-, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. Defining

$$G^+(x,\lambda) = \lim_{\mu\in\mathbb{C}^+,\mu\to\lambda} P^+(x,\mu), \quad (G^-)^{-1}(x,\lambda) = \lim_{\mu\in\mathbb{C}^-,\mu\to\lambda} P^-(x,\mu), \quad \lambda\in\mathbb{R}, \tag{3.27}$$

we can directly show that on the real line, the two matrix functions G^+ and G^- are related by

$$G^+(x,\lambda) = G^-(x,\lambda)G(x,\lambda), \quad \lambda\in\mathbb{R}, \tag{3.28}$$

where by Equation 3.11,

$$\begin{aligned} G(x,\lambda) &= E(H_1 + H_2S(\lambda))(H_1 + S^{-1}(\lambda)H_2)E^{-1} \\ &= E\begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} & \cdots & \hat{s}_{1,n+1} \\ s_{21} & 1 & 0 & \cdots & 0 \\ s_{31} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ s_{n+1,1} & 0 & \cdots & 0 & 1 \end{bmatrix}E^{-1} \end{aligned} \tag{3.29}$$

with $S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{jl})_{(n+1)\times(n+1)}$. Equations 3.28 and 3.29 are exactly the associated matrix Riemann-Hilbert problems that we would like to build for the multicomponent mKdV system (Equation 2.23). The asymptotic properties

$$P^{\pm}(x,\lambda) \to I_{n+1}, \quad \text{when} \quad \lambda\in\mathbb{C}^{\pm}\to\infty, \tag{3.30}$$

provide the canonical normalization conditions

$$G^{\pm}(x,\lambda) \to I_{n+1}, \quad \text{when} \quad \lambda\in\mathbb{R}, \quad \lambda\to\pm\infty, \tag{3.31}$$

for the presented Riemann-Hilbert problems.

To complete the direct scattering transform, let us take the derivative of Equation 3.11 with time t and use the vanishing conditions of the potentials at infinity of t. This way, we can verify that the scattering matrix S satisfies

$$S_t = i\lambda^3[\Omega, S], \tag{3.32}$$

which tells the time evolution of the time-dependent scattering coefficients:

$$\begin{cases} s_{12} = s_{12}(0,\lambda)\mathrm{e}^{i\beta\lambda^3 t}, \quad s_{13} = s_{13}(0,\lambda)\mathrm{e}^{i\beta\lambda^3 t}, \quad \cdots, \quad s_{1,n+1} = s_{1,n+1}(0,\lambda)\mathrm{e}^{i\beta\lambda^3 t}, \\ s_{21} = s_{21}(0,\lambda)\mathrm{e}^{-i\beta\lambda^3 t}, \quad s_{31} = s_{31}(0,\lambda)\mathrm{e}^{-i\beta\lambda^3 t}, \quad \cdots, \quad s_{n+1,1} = s_{n+1,1}(0,\lambda)\mathrm{e}^{-i\beta\lambda^3 t}, \end{cases}$$

and all other scattering coefficients are independent of the time variable t.

4 | SOLITON SOLUTIONS

It is known that the Riemann-Hilbert problems with zeros generate soliton solutions and can be solved by transforming into the ones without zeros.[1] The uniqueness of solutions to each associated Riemann-Hilbert problem, defined by Equations 3.28 and 3.29, does not hold unless the zeros of $\det P^{\pm}$ in the upper and lower half planes are specified and the structures of $\ker P^{\pm}$ at these zeros are determined.[28,29]

Based on $\det\psi^{\pm} = 1$, it follows from the definitions of $P^{\pm}$ and the scattering relation between ψ^+ and ψ^- that

$$\det P^+(x,\lambda) = s_{11}(\lambda), \quad \det P^-(x,\lambda) = \hat{s}_{11}(\lambda), \tag{4.1}$$

where, because of $\det S = 1$, we have

$$\hat{s}_{11} = (S^{-1})_{11} = \begin{vmatrix} s_{22} & s_{23} & \cdots & s_{2,n+1} \\ s_{32} & s_{33} & \cdots & s_{3,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+1,2} & s_{n+1,3} & \cdots & s_{n+1,n+1} \end{vmatrix}.$$

Let N be another arbitrary natural number and assume that s_{11} has N zeros $\{\lambda_k\in\mathbb{C}^+, \quad 1\le k\le N\}$, and $\hat{s}_{11}$ has N zeros $\{\hat{\lambda}_k\in\mathbb{C}^-, \quad 1\le k\le N\}$. To generate soliton solutions, we also assume that all these zeros, λ_k and $\hat{\lambda}_k$, $\quad 1\le k\le N$, are simple. Therefore, each of $\ker P^+(\lambda_k)$, $1\le k\le N$, contains only a single basis column vector, denoted by v_k, $1\le k\le N$; and each of $\ker P^-(\hat{\lambda}_k)$, $1\le k\le N$, a single basis row vector, denoted by $\hat{v}_k$, $1\le k\le N$:

$$P^+(\lambda_k)v_k = 0, \quad \hat{v}_kP^-(\hat{\lambda}_k) = 0, \quad 1\le k\le N. \tag{4.2}$$

The Riemann-Hilbert problems, by Equations 3.28 and 3.29, with the canonical normalization conditions in Equation 3.31 and the zero structures in Equation 4.2 can be solved explicitly,[1,30] and thus, one can readily work out the potential matrix P as follows. Note that P^+ is a solution to the matrix spectral problem (Equation 3.5). Therefore, as long as we expand P^+ at large λ as

$$P^+(x,\lambda) = I_{n+1} + \frac{1}{\lambda}P_1^+(x) + \mathrm{O}(\frac{1}{\lambda^2}), \quad \lambda \to \infty, \tag{4.3}$$

plugging this series expansion into Equation 3.5 and comparing O(1) terms yield

$$\check{P} = -i[\Lambda, P_1^+]. \tag{4.4}$$

This equivalently presents the potential matrix:

$$P = -[\Lambda, P_1^+] = \begin{bmatrix} 0 & -\alpha(P_1^+)_{12} & -\alpha(P_1^+)_{13} & \cdots & -\alpha(P_1^+)_{1,n+1} \\ \alpha(P_1^+)_{21} & 0 & 0 & \cdots & 0 \\ \alpha(P_1^+)_{31} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha(P_1^+)_{n+1,1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{4.5}$$

where $P_1^+ = ((P_1^+)_{jl})_{(n+1)\times(n+1)}$. That is to say, the $2n$ potentials p_i and q_i, $1 \le i \le n$, can be computed as follows:

$$p_i = -\alpha(P_1^+)_{1,i+1}, \quad q_i = \alpha(P_1^+)_{i+1,1}, \quad 1 \le i \le n. \tag{4.6}$$

To compute soliton solutions, we take $G = I_{n+1}$ in each Riemann-Hilbert problem (Equation 3.28). This can be achieved if we assume that $s_{i1} = \hat{s}_{1i} = 0, \quad 2 \le i \le n+1$, which means that no reflection exists in the scattering problem. The solutions to this special Riemann-Hilbert problem can be presented through (see, eg, previous studies[1,30]):

$$P^+(\lambda) = I_{n+1} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l}, \quad P^-(\lambda) = I_{n+1} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_l}, \tag{4.7}$$

where $M = (m_{kl})_{N\times N}$ is a square matrix whose entries are determined by

$$m_{kl} = \frac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, \quad 1 \le k, l \le N. \tag{4.8}$$

Since the zeros λ_k and $\hat{\lambda}_k$ are constants, ie, space and time independent, we can easily work out the spatial and temporal evolutions for the vectors, $v_k(x,t)$ and $\hat{v}_k(x,t)$, $1 \le k \le N$, in the kernels. For example, let us evaluate the x-derivative of both sides of the first set of equations in Equation 4.2. By using Equation 3.5 first and then again the first set of equations in Equation 4.2, we can arrive at

$$P^+(x,\lambda_k)\left(\frac{dv_k}{dx} - i\lambda_k \Lambda v_k\right) = 0, \quad 1 \le k \le N. \tag{4.9}$$

This implies that for each $1 \le k \le N$, $\frac{dv_k}{dx} - i\lambda_k\Lambda v_k$ is in the kernel of $P^+(x,\lambda_k)$ and so a constant multiple of v_k. Without loss of generality, we assume

$$\frac{dv_k}{dx} = i\lambda_k \Lambda v_k, \quad 1 \le k \le N. \tag{4.10}$$

The time dependence of v_k:

$$\frac{dv_k}{dt} = i\lambda_k^3 \Omega v_k, \quad 1 \le k \le N, \tag{4.11}$$

can be obtained similarly through an application of the t-part of the matrix spectral problem, Equation 3.6. To sum up, we can have

$$v_k(x,t) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^3 \Omega t} w_k, \quad 1 \le k \le N, \tag{4.12}$$

$$\hat{v}_k(x,t) = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^3 \Omega t}, \quad 1 \le k \le N, \tag{4.13}$$

where w_k and $\hat{w}_k$, $1 \le k \le N$, are arbitrary constant column and row vectors, respectively.

Finally, from the solutions in Equation 4.7, we have

$$P_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{4.14}$$

and thus further through the presentations in Equation 4.6, obtain the N-soliton solution to the multicomponent mKdV system (Equation 2.23):

$$p_j = \alpha \sum_{k,l=1}^{N} v_{k,1} (M^{-1})_{kl} \hat{v}_{l,j+1}, \quad q_j = -\alpha \sum_{k,l=1}^{N} v_{k,j+1} (M^{-1})_{kl} \hat{v}_{l,1}, \quad 1 \le j \le n, \tag{4.15}$$

where $v_k = (v_{k,1}, v_{k,2}, \cdots, v_{k,n+1})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \cdots, \hat{v}_{k,n+1})$, $1 \le k \le N$, are defined by Equations 4.12 and 4.13, respectively.

5 | SPECIFIC REDUCTION

Let us take a specific reduction for the potential matrix P:

$$P^\dagger = CPC^{-1}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & \Sigma \end{bmatrix}, \tag{5.1}$$

where $\dagger$ stands for the Hermitian transpose of a matrix and Σ is a constant $n \times n$ Hermitian symmetric matrix: $\Sigma^\dagger = \Sigma$. Below, we assume that $\bar{z}$ denotes the complex conjugate of a complex quantity z, and $A^\dagger(\bar{\lambda}) = (A(\lambda))^\dagger$ and $A^{-1}(\bar{\lambda}) = (A(\bar{\lambda}))^{-1}$ for a matrix $A(\lambda)$ depending on the spectral parameter λ.

If $\psi(\lambda)$ is a matrix eigenfunction of the spectral problem in Equation 3.5, then in addition to a known matrix adjoint eigenfunction $C\psi^{-1}(\bar{\lambda})$, we have another matrix adjoint eigenfunction

$$\tilde{\psi}(\bar{\lambda}) = \psi^\dagger(\bar{\lambda})C, \tag{5.2}$$

associated with an eigenvalue $\bar{\lambda}$, ie, $\psi^\dagger(\bar{\lambda})C$ solves the adjoint spectral problem in Equation 3.21 with λ replaced with $\bar{\lambda}$. Therefore, upon observing the asymptotic properties for $\psi^\pm$ at infinity of λ (see, eg, Equation 3.30), the uniqueness of solutions tells

$$\psi^\dagger(\bar{\lambda}) = C\psi^{-1}(\bar{\lambda})C^{-1}, \quad \psi = \psi^\pm. \tag{5.3}$$

Further from the definitions of $P^\pm$ in Equations 3.15 and 3.23, we see that the two matrix solutions $P^\pm$ satisfy the involution relation

$$(P^+)^\dagger(\bar{\lambda}) = CP^-(\bar{\lambda})C^{-1}. \tag{5.4}$$

and from the definition of the scattering matrix $S(\lambda)$ in Equation 3.11, we see that the scattering matrix $S(\lambda)$ has the involution property

$$S^\dagger(\bar{\lambda}) = CS^{-1}(\bar{\lambda})C^{-1}. \tag{5.5}$$

Thanks to Equation 5.4, we have $s_{11}(\bar{\lambda}) = \hat{s}_{11}(\lambda)$ through Equation 4.1, and thus, the zeros of $\det P^\pm$ satisfy the involution relation:

$$\hat{\lambda}_k = \bar{\lambda}_k, \quad 1 \le k \le N. \tag{5.6}$$

To obtain involution eigenvectors v_k and $\hat{v}_k$, we check the Hermitian transpose of the first set of equations in Equation 4.2:

$$0 = v_k^\dagger (P^+(\lambda_k))^\dagger = v_k^\dagger CP^-(\hat{\lambda}_k)C^{-1}, \quad 1 \le k \le N,$$

where we have used Equations 5.4 and 5.6. Therefore, we can take

$$\hat{v}_k = v_k^\dagger C, \quad 1 \le k \le N, \tag{5.7}$$

as the solutions to the second set of equations in Equation 4.2. It follows that we have the following involution eigenvectors:

$$v_k(x,t) = e^{i\lambda_k \Lambda x + i\lambda_k^3 \Omega t} w_k, \quad \hat{v}_k(x,t) = w_k^{\dagger} e^{-i\bar{\lambda}_k \Lambda x - i\bar{\lambda}_k^3 \Omega t} C, \quad 1 \le k \le N, \tag{5.8}$$

where w_k are arbitrary constant column vectors as before.

To make an appropriate reduction which works for the t-part of the Lax pair, we take

$$\Sigma = \frac{1}{\sigma} I_n, \quad \sigma \in \mathbb{R}. \tag{5.9}$$

At this moment, we have

$$q = \sigma p^{\dagger}. \tag{5.10}$$

The vector function c in Equation 2.11 under such reductions could be taken as follows:

$$c = \sigma b^{\dagger}. \tag{5.11}$$

Those three reduction relations guarantee that $\bar{a} = a$ and $d^{\dagger} = d$, where a and d satisfy Equation 2.11, and that

$$(V^{[3]})^{\dagger}(\bar{\lambda}) = C V^{[3]}(\bar{\lambda}) C^{-1}, \quad \text{i.e.,} \quad Q^{\dagger}(\bar{\lambda}) = C Q(\bar{\lambda}) C^{-1}, \tag{5.12}$$

where $V^{[3]}$ and Q are defined in Equations 3.2 (or 2.20) and (3.3), respectively. Therefore, the reduction with Equation 5.9 works perfectly well for the x-part and the t-part of the matrix spectral problems (Equation 3.1). The multicomponent mKdV system (Equation 2.23) is reduced to

$$p_{j,t} = -\frac{\beta}{\alpha^3}[p_{j,xxx} + 3\sigma(\sum_{l=1}^{n} |p_l|^2) p_{j,x} + 3\sigma \left(\sum_{l=1}^{n} p_{l,x} \bar{p}_l \right) p_j], \quad 1 \le j \le n. \tag{5.13}$$

In order to compute the reduced N-soliton solutions, we check if there is an involution property for P_1^+ determined in Equation 4.14 to satisfy. By using Equation 5.6 and 5.7, a direct computation can show that

$$(P_1^+)^{\dagger} = -C P_1^+ C^{-1}. \tag{5.14}$$

Therefore, the potential matrix P determined through Equation 4.5 satisfies the reduction relation (Equation 5.1) with Σ defined by Equation 5.9. This way, the reduced N-soliton solution of the mKdV system in Equation 2.23 gives the N-soliton solution to the reduced multicomponent mKdV system (Equation 5.13):

$$p_j = \alpha \sum_{k,l=1}^{N} v_{k,1} (M^{-1})_{kl} \hat{v}_{l,j+1}, \quad 1 \le j \le n, \tag{5.15}$$

where $v_k = (v_{k,1}, v_{k,2}, \cdots, v_{k,n+1})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \cdots, \hat{v}_{k,n+1})$, $1 \le k \le N$, are given by Equation 5.8.

Particularly, when $n = 2$, taking $\alpha_1 = \beta_1 = 2, \alpha_2 = \beta_2 = 1$ and $\sigma = 1$ leads to a special mKdV system

$$p_{j,t} = -[p_{j,xxx} + 3(|p_1|^2 + |p_2|^2) p_{j,x} + 3(p_{1,x}\bar{p}_1 + p_{2,x}\bar{p}_2) p_j], \quad j = 1, 2. \tag{5.16}$$

Further taking

$$\begin{cases} \lambda_1 = i, \quad \lambda_2 = 2i, \quad \lambda_3 = 3i, \quad \hat{\lambda}_1 = -i, \quad \hat{\lambda}_2 = -2i, \quad \hat{\lambda}_3 = -3i, \\ w_1 = (2, i, 5)^T, \quad w_2 = (2i, 3, 2)^T, \quad w_3 = (2, 1, i)^T, \\ \hat{w}_1 = (2, -i, 5), \quad \hat{w}_2 = (-2i, 3, 2), \quad \hat{w}_3 = (2, 1, -i), \end{cases} \tag{5.17}$$

we obtain one three-soliton solution to the special mKdV system in Equation 5.16:

$$p_1 = \frac{f_1}{g}, \quad p_2 = \frac{f_2}{g}, \tag{5.18}$$

where

$$\begin{aligned}
f_1 =&(27360+2400i)e^{-20x+134t}-(450-18000i)e^{-19x+127t}-(5184-12240i)e^{-18x+108t}\\
&-(6480-7830i)e^{-17x+101t}-(1440-1440i)e^{-16x+82t}-(17280-33780i)e^{-15x+99t}\\
&-(1440-6240i)e^{-14x+80t}+(1512+1440i)e^{-13x+73t}+120ie^{-21x+117t}\\
&+15120ie^{-19x+115t}+24e^{-23x+143t}+360e^{-22x+136t}-2400e^{-17x+89t},\\
f_2 =&(6720-3600i)e^{-20x+134t}+(3600-17250i)e^{-19x+127t}+(10080+1440i)e^{-18x+108t}\\
&+(13230-6480i)e^{-17x+101t}-(960+2160i)e^{-16x+82t}+(74640-30240i)e^{-15x+99t}\\
&-(6720+22320i)e^{-14x+80t}+(2880-6060i)e^{-13x+73t}+120ie^{-23x+143t}\\
&-1200ie^{-17x+89t}+240e^{-22x+136t}+120e^{-21x+117t}+8640e^{-19x+115t},\\
g =&200e^{-18x+90t}+3042e^{-14x+74t}+576e^{-19x+109t}+4e^{-24x+144t}+2925e^{-20x+128t}\\
&+6720e^{-15x+81t}+12960e^{-16x+100t}+480e^{-21x+135t}+9000e^{-16x+88t}\\
&+4058e^{-12x+72t}+8064e^{-17x+107t}+936e^{-22x+142t}+54250e^{-18x+126t}.
\end{aligned}$$

Two three-dimensional plots, contour plots, and x-curves of this set of solutions are made in Figure 1 and Figure 2.

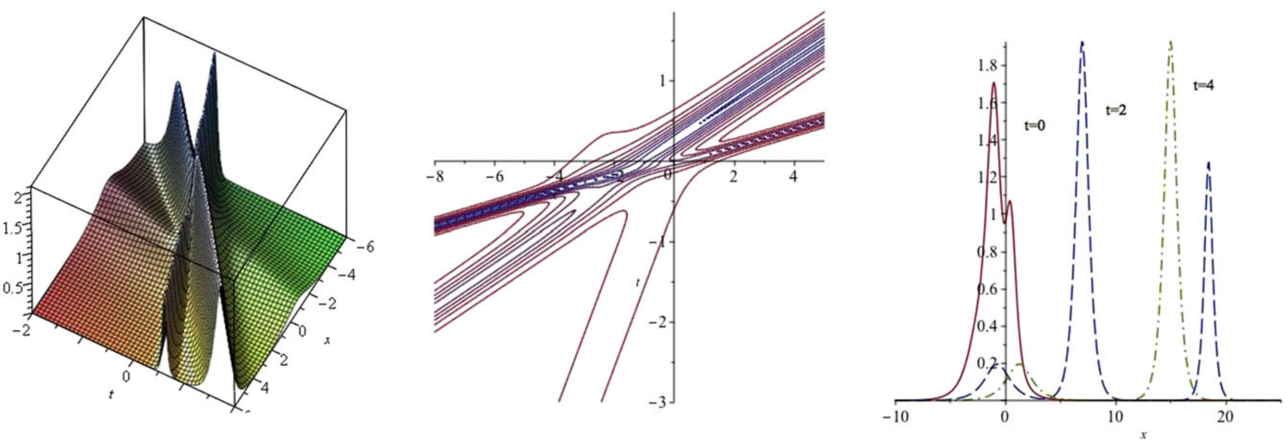

FIGURE 1 Profiles of $|p_1|$: 3d plot (left), contour plot (middle), and x-curves (right) [Colour figure can be viewed at wileyonlinelibrary.com]

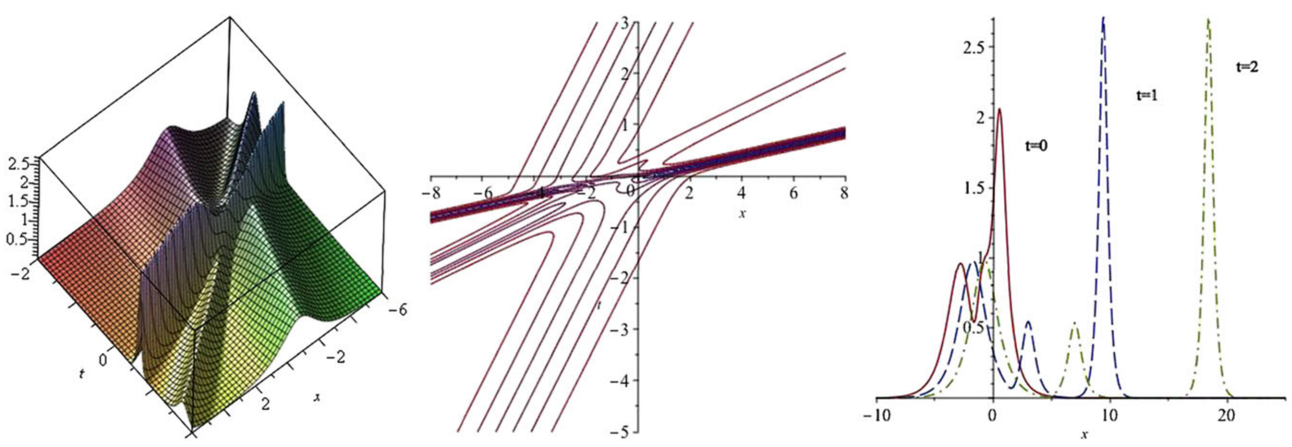

FIGURE 2 Profiles of $|p_2|$: 3d plot (left), contour plot (middle), and x-curves (right) [Colour figure can be viewed at wileyonlinelibrary.com]

6 | CONCLUDING REMARKS

The paper aims at formulating Riemann-Hilbert problems associated with matrix spectral problems to compute soliton solutions of integrable equations. One of the important steps is to introduce a kind of equivalent matrix spectral problems, so that the existence of analytical eigenfunctions in the upper or lower half plane can be guaranteed. We considered an $(n+1)\times(n+1)$ degenerate AKNS spatial matrix spectral problem and generated its integrable hierarchy possessing a bi-Hamiltonian structure. Taking the multicomponent mKdV system an example, we built its Riemann-Hilbert problems and presented an explicit formula for jump matrices in the resulting Riemann-Hilbert problems. Upon taking the identity jump matrix, we computed soliton solutions to the considered multicomponent mKdV system. A specific reduction was successfully made and the corresponding N-soliton solutions were generated for the reduced multicomponent mKdV system.

The Riemann-Hilbert approach is very effective in generating soliton solutions (see also, eg, the studies of Wang, Zhang and Yang, Xiao and Fan, Geng and Wu, and Yang[3-5,31]). It has been recently generalized to solve initial-boundary value problems of integrable equations on the half line and the finite interval.[32,33] Many other approaches to soliton solutions are available in the field of integrable equations, among which are the Hirota direct method,[34] the generalized bilinear technique,[35] the Wronskian technique,[36,37] and the Darboux transformation.[38] It would be interesting to find links between those different approaches.

We point out that it would be also interesting to explore other kinds of exact solutions to integrable equations, including positon and complexiton solutions,[39,40] lump solutions,[41-43] and algebro-geometric solutions,[44,45] by using Riemann-Hilbert problems. It is hoped that our results could be helpful in recognizing those exact solutions from the perspective of Riemann-Hilbert techniques.

About coupled mKdV systems, there are many other studies such as integrable couplings,[46,47] super hierarchies,[48] and fractional analogous equations.[49] Therefore, another important topic for further study is to formulate Riemann-Hilbert problems for solving those generalized integrable counterparts.

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ORCID

Wen-Xiu Ma https://orcid.org/0000-0001-5309-1493

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