Original articles

\(N\)-soliton solution and the Hirota condition of a (2+1)-dimensional combined equation

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Abstract

Within the Hirota bilinear formulation, we construct \(N\)-soliton solutions and analyze the Hirota \(N\)-soliton conditions in (2+1)-dimensions. A generalized algorithm to prove the Hirota conditions is presented by comparing degrees of the multivariate polynomials derived from the Hirota function in \(N\) wave vectors, and two weight numbers are introduced for transforming the Hirota function to achieve homogeneity of the related polynomials. An application is developed for a general combined nonlinear equation, which provides a proof of existence of its \(N\)-soliton solutions. The considered model equation includes three integrable equations in (2+1)-dimensions: the (2+1)-dimensional KdV equation, the Kadomtsev–Petviashvili equation, and the (2+1)-dimensional Hirota–Satsuma–Ito equation, as specific examples.

Keywords: \(N\)-soliton solution; Hirota \(N\)-soliton condition; (2+1)-dimensional integrable equations

1. Introduction

\(N\)-soliton solutions are exact multiple wave solutions to nonlinear integrable equations [1,25]. Various significant solutions in mathematical physics, including breather, complexion, lump and rogue wave solutions, are special reductions of \(N\)-soliton solutions in different situations. Solitons superimposed in fibers can be applied to optical communications, which are faster, more secure, and more flexible [5]. It is well-known that the Hirota bilinear method is a standard and powerful technique to generate \(N\)-soliton solutions [11]. The innovative concept of bilinear derivatives is the key in the basic theory of exact solutions [22], and Hirota bilinear forms are the starting point to construct \(N\)-soliton solutions [11].

Hirota bilinear derivatives read [9]:

\[
D^m_x f \cdot g = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (\partial_x^i f)(\partial_x^{m-i} g), \quad m \geq 1,
\]

(1.1)

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and more generally, we have bilinear partial derivatives with multiple variables:

\[(D_x^m D_y^n f \cdot g)(x, t) = (\partial_x - \partial_y)^m(\partial_x - \partial_t)^n f(x, t)g(x', t')|_{x'=x, t'=t}, \quad m, n \geq 0, \quad m + n \geq 1.\]  

(1.2)

The case of \( f = g \) yields Hirota bilinear expressions:

\[D_x^{2m-1} f \cdot f = 0, \quad D_x^{2m} f \cdot f = \sum_{i=0}^{2m} (-1)^{2m-i} \binom{2m}{i} (\partial_x^i f)(\partial_x^{2m-i} f), \quad m \geq 1,\]

(1.3)

and bilinear partial derivative expressions:

\[D_x^m D_y^n f \cdot f = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} (\partial_x^i \partial_y^j f)(\partial_x^{m-i} \partial_y^{n-j} f), \quad m, n \geq 0, \quad m + n \geq 1.\]

(1.4)

By virtue of Hirota bilinear expressions, we can formulate Hirota bilinear equations. Take an even polynomial \( P(x_1, x_2, \ldots, x_M) \) in \( M \) variables, and assume that \( P \) has no constant term, i.e.,

\[P(0) = P(0, 0, \ldots, 0) = 0.\]

(1.5)

The corresponding Hirota bilinear equation reads

\[P(D_{x_1}, D_{x_2}, \ldots, D_{x_M}) f \cdot f = 0,\]

(1.6)

all terms of which are Hirota bilinear expressions. An important example is the bilinear Kadomtsev–Petviashvili equation

\[B(f) := (D_x^4 + D_y^2 D_t^2) f \cdot f = 2(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2 + f_x f_{xx} - f_x f_{x} + f_{yy} f - f_y^2) = 0,\]

(1.7)

which is transformed into the nonlinear Kadomtsev–Petviashvili equation

\[N(u) := (u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0,\]

(1.8)

under the logarithmic derivative transformation \( u = 2(\ln f)_{xx} \). The connection between the two equations is

\[N(u) = (B(f)/f^2)_{xx}, \quad u = 2(\ln f)_{xx}.\]

(1.9)

In this paper, we would like to construct \( N \)-soliton solutions and analyze the corresponding Hirota conditions. A generalized algorithm will be proposed for verifying the Hirota \( N \)-soliton conditions by comparing degrees of the multivariate polynomials derived from the Hirota function in \( N \) wave vectors. An application will be made for a general (2+1)-dimensional combined bilinear equation associated with

\[P(x, y, t) = a_1(x^4 + xt) + a_2(x^3 y + yt) + a_3 x^2 + a_4 xy + a_5 y^2,\]

(1.10)

where \( a_i \)'s are arbitrary constants satisfying \( a_1^2 + a_2^2 \neq 0 \), thereby presenting a proof of existence of its \( N \)-soliton solutions. The considered model equation includes the three integrable equations in (2+1)-dimensions: the (2+1)-dimensional KdV equation, the Kadomtsev–Petviashvili equation and the (2+1)-dimensional Hirota–Satsuma–Ito equation, as specific examples.

2. Formulating \( N \)-soliton solutions and their conditions

Let \( N \geq 1 \) be an arbitrary integer. For a general Hirota bilinear equation (1.6), we construct its \( N \)-soliton solutions and analyze their sufficient and necessary conditions.

2.1. Bilinear formulation of soliton solutions

Let us assume that \( N \) wave vectors are denoted by

\[k_i = (k_{1,i}, k_{2,i}, \ldots, k_{M,i}), \quad 1 \leq i \leq N,\]

(2.1)

where \( k_{1,i}, k_{2,i}, \ldots, k_{M,i}, \quad 1 \leq i \leq N, \) are constants to be determined. An \( N \)-soliton solution to the Hirota bilinear equation (1.6) is given by \[8\]:

\[f = \sum_{\mu=0}^{N} \exp\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} a_{ij} \mu_i \mu_j\right),\]

(2.2)
where \( \mu = (\mu_1, \mu_2, \ldots, \mu_N) \), \( \mu = 0, 1 \) means that each \( \mu_i \) takes values, either 0 or 1, the wave variables read
\[
\eta_i = k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{M,i}x_M + \eta_{i,0}, \quad 1 \leq i \leq N,
\]
(2.3)
\( \eta_{i,0} \)'s being arbitrary constants, and the phase shifts are determined by
\[
e^{\sigma_{ij}} = A_{ij} := -\frac{P(k_i - k_j)}{P(k_i + k_j)}, \quad 1 \leq i < j \leq N.
\]
(2.4)
Note that only the constants \( e^{\sigma_{ij}} \)'s, but not \( a_{ij} \)'s, are needed in the definition of \( f \), (2.2).

Let us further introduce
\[
H(k_1, \ldots, k_n) = \sum_{\sigma} P\left( \sum_{r=1}^{n} \sigma_r k_r \right) \prod_{1 \leq r < s \leq n} P(\sigma_r k_r - \sigma_s k_s) \sigma_r \sigma_s, \quad 1 \leq n \leq N,
\]
(2.5)
where \( 1 \leq i_1 < \cdots < i_n \leq N \), \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \), and \( \sigma = \pm 1 \) means that each \( \sigma_r \) takes values, either 1 or \(-1\). These functions are called the Hirota functions.

Observing the basic properties
\[
P(D_{x_1}, \ldots, D_{x_M}) e^{\eta_1} \cdot e^{\eta_2} = P(k_i - k_j) e^{\eta_1 + \eta_2},
\]
(2.6)
and
\[
P(D_{x_1}, \ldots, D_{x_M}) e^{\eta_1} f \cdot e^{\eta_2} g = e^{\eta_1} P(D_{x_1}, \ldots, D_{x_M}) f \cdot g,
\]
(2.7)
where \( \eta_1, \eta_2 \) and \( \eta_n \) are arbitrary wave variables defined by (2.3), we can have the following formulation [18,19].

**Theorem 2.1.** Let the function \( f \) be given by (2.2) and \( \xi \) denote that the term \( \xi \) is not involved. Then we have
\[
P(D_{x_1}, \ldots, D_{x_M}) f \cdot f
= (-1)^{\frac{1}{2}(N(N-1))} \frac{e^{\eta_1 + \eta_2 + \cdots + \eta_N}}{P(k_i + k_j)}
\]
\[
\times \sum_{n=1}^{N-1} (-1)^{\frac{1}{2}(N-n)(N-n-1)} \sum_{1 \leq i_1 < \cdots < i_n \leq N} \frac{H(k_1, \ldots, \hat{k}_{i_1}, \ldots, \hat{k}_{i_n}, \ldots, k_N)}{P(k_i + k_j)}
\]
\[
\times \sum_{n=1}^{N-1} \sum_{1 \leq i_1 < \cdots < i_n \leq N} e^{\eta_1 + \cdots + \eta_n + \sum_{1 \leq r < s \leq n} \eta_{i_r} \eta_{i_s}} P(D_{x_1}, \ldots, D_{x_M}) f_{i_1} \cdots f_{i_n}
\]
(2.8)
with
\[
f_{i_1} \cdots f_{i_n} = \sum_{\tilde{\mu}_{i_1} \cdots \tilde{\mu}_{i_n} = 0,1} \exp(\sum_{1 \leq i \leq N} \mu_i \tilde{\eta}_i + \sum_{1 \leq j < s \leq N} \eta_{i_j} \eta_{i_s})
\]
\[
\tilde{\eta}_i = \eta_i + \sum_{r=1}^{n} a_{i_r},
\]
(2.9)
where \( \tilde{\mu}_{i_1} \cdots \tilde{\mu}_{i_n} = (\mu_1, \ldots, \tilde{\mu}_{i_1} \cdots, \tilde{\mu}_{i_n}, \ldots, \mu_N) \) and \( \tilde{\mu}_{i_1} \cdots \tilde{\mu}_{i_n} = 0,1 \) means that each \( \mu_i \) in \( \tilde{\mu}_{i_1} \cdots \tilde{\mu}_{i_n} \) takes values, either 0 or 1.

Based on this formulation, we readily know that the Hirota bilinear equation (1.6) possesses an \( N \)-soliton solution (2.2) if and only if the following condition
\[
H(k_1, \ldots, k_n) = 0, \quad 1 \leq i_1 < \cdots < i_n \leq N, \quad 1 \leq n \leq N,
\]
(2.10)
is satisfied. This is called the Hirota condition for an \( N \)-soliton solution, or simply, the \( N \)-soliton condition (see [10, p165]. On account of the even property of \( P \), the case of (2.10) with \( n = 1 \) presents the dispersion relations
\[
P(k_i) = P(k_{i_1}, k_{i_2}, \ldots, k_{M,i}) = 0, \quad 1 \leq i \leq N.
\]
(2.11)
There exist very few studies (see, e.g., [23,27]) about the Hirota \( N \)-soliton condition, because of its complexity involved in the Hirota functions.
2.2. Illustrative examples

The one-soliton condition is exactly the dispersion relation: \( P(k_1) = 0 \), which means that \( f = 1 + e^{\eta_1} \) is a solution if \( P(k_1) = 0 \).

Besides the dispersion relations \( P(k_1) = P(k_2) = 0 \), the two-soliton condition requires

\[
2(P(k_1 + k_2)P(k_1 - k_2) - P(k_1 - k_2)P(k_1 + k_2)) = 0,
\]

which is an identity. Therefore, a Hirota bilinear equation always has a two-soliton solution:

\[
f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2},
\]

where \( P(k_1) = P(k_2) = 0 \).

Upon taking \( N = 3 \), it is easy to see that the three-soliton condition \([6,7]\) requires

\[
\sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} P(\sigma_1 k_1 + \sigma_2 k_2 + \sigma_3 k_3)P(\sigma_1 k_1 - \sigma_2 k_2) \\
\times P(\sigma_2 k_2 - \sigma_3 k_3)P(\sigma_1 k_1 - \sigma_3 k_3) = 0,
\]

in addition to the dispersion relations \( P(k_1) = P(k_2) = P(k_3) = 0 \). Clearly, this is equivalent to

\[
\sum_{(\sigma_1,\sigma_2,\sigma_3) \in S} P(\sigma_1 k_1 + \sigma_2 k_2 + \sigma_3 k_3)P(\sigma_1 k_1 - \sigma_2 k_2) \\
\times P(\sigma_2 k_2 - \sigma_3 k_3)P(\sigma_1 k_1 - \sigma_3 k_3) = 0,
\]

where \( S = \{(1, 1, 1), (1, -1, -1), (1, -1, 1), (-1, 1, 1)\} \). Obviously, the corresponding three-soliton solution reads

\[
f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} \\
+ A_{23}e^{\eta_2+\eta_3} + A_{123}e^{\eta_1+\eta_2+\eta_3}, \quad A_{123} = A_{12}A_{13}A_{23}.
\]

It is generally accepted that the three-soliton condition implies the general \( N \)-soliton condition, without proof of its accuracy.

If a sufficient Hirota \( N \)-soliton condition (see \([20], p951\)):

\[
P(k_i - k_j) = 0, \quad 1 \leq i < j \leq N,
\]

is satisfied, we arrive at a resonant \( N \)-soliton solution:

\[
f = 1 + c_1 e^{\eta_1} + c_2 e^{\eta_2} + \cdots + c_N e^{\eta_N},
\]

where \( c_i \)'s are arbitrary constants. Note that all wave vectors \( k_i \)'s associated with resonant solutions form a vector sub-space in \( \mathbb{R}^M \) \([21], p7178\).

2.3. Basic properties of the Hirota functions

In order to prove the Hirota \( N \)-soliton conditions, we usually need to factor out as many common factors out of the Hirota function \( H(k_1, \ldots, k_N) \) as possible. To this end, we will use the following two theorems. The first one is an automatic consequence of the definition of the Hirota functions.

**Theorem 2.2.** The Hirota functions defined by (2.5) are even and symmetric functions in the involved wave vectors.

When taking \( k_2 = \pm k_1 \), we immediately have

\[
P(\sigma_i k_i - k_2)P(\sigma_i k_i \pm k_1) = P(k_i - k_1)P(k_i + k_1)
\]

in both cases of \( \sigma_i = \pm 1 \), based on the even property of the polynomial \( P \). Utilizing this basic property, we can derive the following consequence \([19]\), with a careful computation.
Theorem 2.3. If $k_2 = \pm k_1$, then we have

$$H(k_1, \ldots, k_N) = 2H(k_3, \ldots, k_N)P(2k_1)\prod_{j=3}^{N} P(k_j - k_i)P(k_i + k_1),$$

(2.20)

where $H(k_1, \ldots, k_N)$ and $H(k_3, \ldots, k_N)$ are two Hirota functions defined in (2.5).

This specific theorem will be used to factor out common factors out of the Hirota function $H(k_1, \ldots, k_N)$, in order to verify the Hirota $N$-soliton conditions.

3. A generalized algorithm and its applications

3.1. A generalized algorithm

Let us focus on the (2+1)-dimensional case and state the corresponding $N$ wave vectors as

$$k_i = (k_i, l_i, -\omega_i), \quad 1 \leq i \leq N.$$ (3.1)

We also assume that the dispersion relations (2.11) determine all frequencies in terms of wave numbers $k_l, l_i$: $\omega_i = \omega(k_i, l_i), \quad 1 \leq i \leq N$. In this way, $P(\sigma_i k_i - \sigma_j k_j)$ becomes functions of $k_i, l_i$ and $k_j, l_j$ only.

First, we suppose that under the substitution

$$l_i = \gamma k_i w_1 l_i w_2, \quad 1 \leq i \leq N,$$

(3.2)

for some integer weights $w_1$ and $w_2$ and a nonzero constant coefficient $\gamma$, the two functions $P(\sigma_i k_i - \sigma_j k_j)$ and $P(\sigma_i k_i + \cdots + \sigma_N k_N)$ are simplified into rational forms as follows:

$$P(\sigma_i k_i - \sigma_j k_j) = \frac{\sigma_i \sigma_j k_i k_j Q_1(k_1, l_1, k_j, l_j, \sigma_i, \sigma_j)}{Q_2(k_i, l_i, k_j, l_j)},$$

(3.3)

and

$$P(\sigma_i k_i + \cdots + \sigma_N k_N) = \frac{Q_3(k_1, l_1, \ldots, k_N, l_N, \sigma_1, \ldots, \sigma_N)}{Q_4(k_1, l_1, \ldots, k_N, l_N)},$$

(3.4)

where $Q_1, Q_2, Q_3$ and $Q_4$ are polynomial functions in the indicated variables.

Second, note that Theorem 2.3 implies that under the induction assumption, the Hirota function $H(k_1, \ldots, k_N)$ will be zero, if there exist two equal wave vectors $k_i = k_j$ for some pair $1 \leq i < j \leq N$. It also follows from the symmetric property in Theorem 2.2 that under the transforms in (3.2), $H(k_1, \ldots, k_N)$ is still even with respect to $k_i, l_i \quad 1 \leq i \leq N$, while $w_1 + w_2$ is odd, and it is even only with respect to $k_j, 1 \leq i \leq N$, while $w_1$ is odd. Therefore, in both cases, we can simplify the Hirota function $H(k_1, \ldots, k_N)$ into the following form:

$$H(k_1, \ldots, k_N) = (k_i^2 - k_j^2)^2 g_{ij} + (l_i - l_j)^2 h_{ij}, \quad \text{for each pair} \quad 1 \leq i < j \leq N,$$

(3.5)

where $g_{ij}$ and $h_{ij}$ are rational functions of the wave numbers $k_n, l_n, \quad 1 \leq n \leq N$. Then we see that the Hirota function $H(k_1, \ldots, k_N)$ can be expressed as

$$H(k_1, \ldots, k_N) = \frac{\prod_{1 \leq i < j \leq N} k_i^2 k_j^2 \prod_{1 \leq i < j \leq N} (k_i^2 - k_j^2)^2 g + \prod_{1 \leq i < j \leq N} (l_i - l_j)^2 h}{Q_4(k_1, l_1, \ldots, k_N, l_N) \prod_{1 \leq i < j \leq N} Q_2(k_i, l_i, k_j, l_j)}$$

(3.6)

under the substitution (3.2), where $g$ and $h$ are polynomials of the wave numbers $k_n, l_n, \quad 1 \leq n \leq N$, and $g$ can be nonzero when $H(k_1, \ldots, k_N) \neq 0$.

Now, if $H(k_1, \ldots, k_N) \neq 0$, we readily know that the degree of the polynomial

$$\tilde{H}(k_1, \ldots, k_N) := H(k_1, \ldots, k_N)Q_4(k_1, l_1, \ldots, k_N, l_N) \prod_{1 \leq i < j \leq N} Q_2(k_i, l_i, k_j, l_j) = \prod_{1 \leq i < j \leq N} k_i^2 k_j^2 \prod_{1 \leq i < j \leq N} (k_i^2 - k_j^2)^2 g + \prod_{1 \leq i < j \leq N} (l_i - l_j)^2 h$$

(3.7)

is at least $2N(N - 1) + 2N(N - 1) = 4N(N - 1)$. From (3.3), (3.4) and (3.7), we have

$$\tilde{H}(k_1, \ldots, k_N) = \sum_{\sigma = \pm 1} Q_3(k_1, l_1, \ldots, k_N, l_N, \sigma_1, \ldots, \sigma_N) \prod_{1 \leq i < j \leq N} \sigma_i \sigma_j k_i k_j Q_1(k_i, l_i, k_j, l_j, \sigma_i, \sigma_j),$$

(3.8)
and so the degree of the polynomial on the left-hand side should not be less than $4N(N - 1)$. Otherwise, we will have $H(k_1, \ldots, k_N) = 0$, which is what we want to prove. Therefore, the final task is to compute $Q_1$ and $Q_3$ and check if the degree of the polynomial $\hat{H}(k_1, \ldots, k_N)$ is less than $4N(N - 1)$, based on (3.8). If we can make a contradiction, then we verify the Hirota $N$-soliton conditions, thereby proving the existence of $N$-soliton solutions.

### 3.2. Application to a (2+1)-dimensional combined equation

We consider a (2+1)-dimensional combined bilinear equation associated with a polynomial $P$ given by (1.10). We restate $P$ here for ease of reference:

$$P(x, y, t) = a_1(x^4 + xt) + a_2(x^3y + yt) + a_3x^2 + a_4xy + a_5y^2,$$

(3.9)

where $a_i$'s are arbitrary constants, which satisfy $a_1^2 + a_2^2 \neq 0$ to guarantee the nonlinearity of the corresponding model equation. The resulting (2+1)-dimensional combined bilinear equation is

$$B(f) := \{a_1(D_x^4 + D_x D_y) + a_2(D_y^3 D_x + D_y D_t) + a_3 D_x^2 + a_4 D_x D_y + a_5 D_y^2\} f \cdot f$$

$$= 2[a_1(f_{xxxx} f - 4f_{xxx} f_x + 3f_{xx}^2 + f_{xt} f - f_x f_t)$$

$$+ a_2(f_{xxyy} f - 3f_{xyy} f_x + 3f_{xy} f_{xx} - f_x f_{xxx} + f_{xt} f - f_x f_t)$$

$$+ a_3(f_{xx} - f_x^2) + a_4(f_{xy} f - f_x f_y) + a_5(f_{yy} f - f_y^2)] = 0.$$  

(3.10)

This is equivalent to the (2+1)-dimensional combined nonlinear equation:

$$N(u, v) := a_1(u_t + 6uu_x + u_{xxx}) + a_2[v_t + 3(uv)_x + v_{xxx}] + a_3u_x + a_4v_x + a_5v_y = 0,$$

(3.11)

where $u_y = v_x$, and the direct link is

$$N(u, v) = (B(f)/f^2)_x,$$

(3.12)

under the logarithmic derivative transformations

$$u = 2(\ln f)_{xx}, \ v = 2(\ln f)_{xy}.$$  

(3.13)

Therefore, if $f$ solves the bilinear equation (3.10), then $u$ and $v$ defined by (3.13) solve the corresponding nonlinear equation (3.11).

Obviously, we can work out that

$$\omega_i = k_i^3 + \frac{a_3 + a_4y + a_5y^2}{a_1 + a_2y} k_i, \ 1 \leq i \leq N,$$

(3.14)

and

$$Q_1 = -3(a_1 + a_2y)(\sigma_i k_i - \sigma_j k_j)^2, \ \text{deg} \ Q_3 = 4, \ Q_2 = 1, \ Q_4 = 1,$$

(3.15)

under the substitution (3.2) with $w_1 = 1$ and $w_2 = 0$. Therefore, if $H(k_1, \ldots, k_N) \neq 0$, then based on (3.8), the degree of the polynomial $\hat{H}(k_1, \ldots, k_N) (=H(k_1, \ldots, k_N))$ is $2N(N - 1) + 4 = 2N^2 - 2N + 4$, which could not be greater than $4N(N - 1)$ when $N \geq 3$. Then it follows that $H(k_1, \ldots, k_N) = 0, \ N \geq 1$, since the nonlinear model equation (3.11) passes the three-soliton test, which just needs a direct computation (this is important, though omitted).

### 3.3. Specific reductions

#### 3.3.1. (2+1)-dimensional KdV equation

The case of $a_2 = 1$ and all other zero coefficients presents

$$P(x, y, t) = x^3 y + yt,$$

(3.16)

with which the (2+1)-dimensional KdV equation is associated. The corresponding (2+1)-dimensional bilinear KdV equation reads

$$B(f) := D_y(D_t + D_x^3) f \cdot f = 2(f_{yt} f - f_y f_t + f_{xxy} f - 3f_{xyy} f_x + 3f_{xy} f_{xx} - f_y f_{xxx}) = 0,$$

(3.17)
which is equivalent to the (2+1)-dimensional KdV equation [3]:

\[ N(u, v) \equiv v_t + 3(uv)_x + v_{xxx} = 0, \]  

(3.18)

where \( u_y = v_x \), under the logarithmic derivative transformations in (3.13).

We can also have a direct verification for the Hirota \( N \)-soliton condition under a different selection of weights: \( w_1 = 0 \) and \( w_2 = 1 \), and \( \gamma = 1 \) [18]. In this case, we have

\[ \omega_i = k_i^3, \quad 1 \leq i \leq N, \]  

(3.19)

and

\[ Q_1 = -3(\sigma_i k_i - \sigma_j k_j)(\sigma_i l_i - \sigma_j l_j), \quad \text{deg} Q_3 = 4, \quad Q_2 = 1, \quad Q_4 = 1. \]  

(3.20)

Therefore, if \( H(k_1, \ldots, k_N) \neq 0 \), then based on (3.8), the degree of the polynomial \( \tilde{H}(k_1, \ldots, k_N) \) (= \( H(k_1, \ldots, k_N) \)) is \( 2N(N - 1) + 4 = 2N^2 - 2N + 4 \), which could not be greater than \( 4N(N - 1) \) when \( N \geq 3 \). Thus, \( H(k_1, \ldots, k_N) = 0, \quad N \geq 1 \).

It is known that the spatial symmetric version of the (2+1)-dimensional KdV equation (3.18):

\[ v_t + 3(uv)_x + 3(vu)_y + v_{xxx} + v_{yyy} = 0, \]  

(3.21)

where \( u_y = v_x \) and \( v_y = w_x \), has been discussed (see [24], p707 and [28], p589), and its inverse scattering transform and algebro-geometric solutions have been announced in [24, 28], respectively. However, this (2+1)-dimensional symmetric equation does not pass the three-soliton test.

3.3.2. The Kadomtsev–Petviashvili equation

The case of \( a_1 = a_5 = 1 \) and all other zero coefficients tells

\[ P(x, y, t) = x^4 + xt + y^2, \]  

(3.22)

with which the Kadomtsev–Petviashvili equation is associated. The corresponding bilinear Kadomtsev–Petviashvili equation reads

\[ B(f) \equiv (D^4_x + D_x D_t + D^2_y)f \cdot f \]
\[ = 2(f_{xxxx}f - 4f_{xxx}f_x + 3f^2_{xx} + 3f_{xx}f_x f_t + f_{xy}f - f_{yy}f_y - f_{y}f_y) = 0, \]  

(3.23)

which is equivalent to the Kadomtsev–Petviashvili equation [13]:

\[ N(u) \equiv u_t + 6uu_x + u_{xxx} + v_y = 0, \]  

(3.24)

where \( u_y = v_x \), under the logarithmic derivative transformations in (3.13).

We can also give a direct proof for the Hirota \( N \)-soliton condition under a different selection of weights: \( w_1 = w_2 = 1 \), and \( \gamma = 1 \) [18]. In this case, we have

\[ \omega_i = k_i^3 + l_i^2 k_i, \quad 1 \leq i \leq N, \]  

(3.25)

and

\[ Q_1 = -3(\sigma_i k_i - \sigma_j k_j)^2 + (l_i - l_j)^2, \quad \text{deg} Q_3 = 4, \quad Q_2 = 1, \quad Q_4 = 1. \]  

(3.26)

Therefore, if \( H(k_1, \ldots, k_N) \neq 0 \), then based on (3.8), the degree of the polynomial \( \tilde{H}(k_1, \ldots, k_N) \) (= \( H(k_1, \ldots, k_N) \)) is \( 2N(N - 1) + 4 = 2N^2 - 2N + 4 \), which could not be greater than \( 4N(N - 1) \) when \( N \geq 3 \). It thus follows that \( H(k_1, \ldots, k_N) = 0, \quad N \geq 1 \).

We point out that the \( N \)-soliton solutions of the Kadomtsev–Petviashvili equation have been also presented in [2, 26] and the quasiperiodic multiphase solutions of the Kadomtsev–Petviashvili equation can be decomposed into finite-dimensional canonical Hamiltonian systems [4].
3.3.3. (2+1)-dimensional Hirota–Satsuma–Ito equation

The case of $a_2 = a_3 = 1$ and all other zero coefficients gives

$$P(x, y, t) = x^3 t + y t + x^2,$$

under an exchange of $y$ and $t$, with which the Hirota–Satsuma–Ito equation is associated. The corresponding (2+1)-dimensional bilinear Hirota–Satsuma–Ito equation reads

$$B(f) := (D_y D_t + D_x D_t + D_z D_t) f \cdot f
= 2(f_{xxx} f - 3 f_{xx} f_x + 3 f_x f_{xx} - f_{t} f_x f - f_x f_t f + f_x^2 f - f_{xx}^2) = 0,$$  

which is equivalent to the (2+1)-dimensional nonlinear Hirota–Satsuma–Ito equation:

$$N(u) := v_y + 3( uv)_x + v_{xxx} + u_x = 0,$$

where $u_t = v_x$, under the logarithmic derivative transformations in (3.13). If we take a potential form $v = \tilde{v}_t$, then the above equation becomes the original (2+1)-dimensional Hirota–Satsuma–Ito equation [7]:

$$\tilde{v}_{xx} + \tilde{v}_{yy} + 3(\tilde{v}_t \tilde{v}_x)_x + \tilde{v}_{xxxx} = 0.$$  

If this equation does not depend on $y$, it reduces to the Hirota–Satsuma equation in (1+1)-dimensions [12].

We can similarly present a direct condition for the Hirota $N$-soliton under a different selection of weights: $w_1 = 1$ and $w_2 = 2$, and $\gamma = 1$ [18]. In this case, we can directly get

$$\omega_i = \frac{k_i}{k_i^2 + l_i^2}, \quad 1 \leq i \leq N,$$

and

$$\begin{align*}
Q_1 &= (k_i^2 + k_i^2)^2 + (l_i^2 - l_i^2)^2 + 2k_i^2 k_i^2 + 2k_i^2 l_i^2 + k_i^2 l_i^2 + k_i^2 l_i^2
-3\sigma_i \sigma_j k_i k_j - 3 \sigma_i \sigma_j l_i l_j, \\
\text{deg } Q_2 &= 2(N+1), \quad Q_2 = (k_i^2 + l_i^2)(k_i^2 + l_i^2), \quad Q_4 = \prod_{i=1}^N (k_i^2 + l_i^2).
\end{align*}$$

Therefore, if $H(k_1, \ldots, k_N) \neq 0$, then based on (3.8), the degree of the polynomial $\tilde{H}(k_1, \ldots, k_N)$ is $3N(N - 1) + 2(N + 1) = 3N^2 - N + 2$, which could not be greater than $4N(N - 1)$ when $N \geq 4$. It then follows that $H(k_1, \ldots, k_N) = 0$, $N \geq 1$.

The $N$-soliton solution of the (2+1)-dimensional Hirota–Satsuma–Ito equation can be reduced to diverse other interesting solutions such as breather, lump and rogue wave solutions and their interaction solutions [14,30].

4. Concluding remarks

We have discussed the Hirota $N$-soliton conditions for a combined bilinear differential equation in (2+1)-dimensions, and shown the existence of its $N$-soliton solutions. The general model equation includes three (2+1)-dimensional integrable equations: the (2+1)-dimensional KdV equation, the Kadomtsev–Petviashvili equation and the (2+1)-dimensional Hirota–Satsuma–Ito equation, as specific examples. It would always be intriguing to explore new examples of bilinear equations in (2+1)-dimensions, which possess $N$-soliton solutions. Along with the presented generalized efficient algorithm, symbolic computations would be extremely helpful in determining such bilinear (and then nonlinear) equations in (2+1)-dimensions.

There are various generalized bilinear derivatives which allow us to deal with bilinear differential equations, including odd-order ones (not as in the Hirota case). The $D_{p,i}$-operators are particular examples [15]:

$$D^m_{p,i} D^n_{p,i} f \cdot g = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \alpha_p^{i+j} (\partial_x^{m-i} \partial_t^{n-j} f)(\partial_x^{i} \partial_t^{j} g), \quad m, n \geq 0, \quad m + n \geq 1,$$

where the powers of $\alpha_p$ determine the corresponding signs as follows:

$$\alpha_p^i = (-1)^{c(i), \quad i \equiv r(i) \text{ mod } p, \quad i \geq 0,$$

with $0 \leq r(i) < p$. In particular, the patterns of those signs for $i = 1, 2, 3, \ldots$ read
\[ p = 3 : \ -, +, +, +, +, +, \ldots; \]
\[ p = 5 : \ -, +, +, +, +, +, +, \ldots; \]
\[ p = 7 : \ -, +, +, +, +, +, +, +, +, +, \ldots. \]

Two such simple generalized bilinear derivatives are \( D_{3,x} \) and \( D_{3,y} \), associated with the two smallest odd prime numbers: \( p = 3, 5 \). The cases of \( p = 2k, k \in \mathbb{N} \), are exactly the same as the Hirota case. The corresponding generalized bilinear expressions can exhibit new characteristics, indeed. For example, we have

\[
D_{3,x}^2 D_{3,t} f \cdot f = 2 f_{xt} f, \quad D_{3,x}^3 D_{3,t} f \cdot f = 6 f_{xxt} f_{xt},
\]

which are totally different from the Hirota derivatives. Of course, we can have many other generalized bilinear derivatives such as \( D_{3,x} \) and \( D_{15,x} \).

We point out that resonant \( N \)-solitons have been analyzed for generalized bilinear equations [16] and trilinear equations [17]. When a multivariate polynomial \( P \) satisfies

\[
P(k_i + \alpha_p k_j) + P(k_j + \alpha_p k_i) = 0, \quad 1 \leq i \leq j \leq N,
\]

where \( k_i \)'s are wave vectors defined earlier, the corresponding generalized bilinear equation

\[
P(D_{p,x_1}, \ldots, D_{p,x_M}) f \cdot f = 0
\]

can possess the resonant \( N \)-soliton solution [16]:

\[
f = 1 + c_1 e^{\eta_1} + c_2 e^{\eta_2} + \cdots + c_N e^{\eta_N}
\]

where \( \eta_i \)'s are the wave variables defined previously and \( c_i \)'s are arbitrary constants.

We are interested in searching for concrete examples of generalized bilinear equations, which possess \( N \)-soliton solutions. There are various basic questions in the corresponding theory. Those include how to formulate a generalized \( N \)-soliton condition; and how to identify generalized bilinear equations, for example,

\[
P(D_{3,x}, D_{3,y}) = 0, \quad P(D_{3,x}, D_{3,y}, D_{3,t}) = 0,
\]

in both (1+1)-dimensions and (2+1)-dimensions, which have \( N \)-soliton solutions. All related studies will be helpful in improving our understanding of bilinear partial differential equations and their associated nonlinear wave phenomena [29].

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