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# $N$-soliton solution and the Hirota condition of a (2+1)-dimensional combined equation

Wen-Xiu Ma*

*Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China*
*Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
*Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA*
*School of Mathematical and Statistical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa*

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**Abstract**

Within the Hirota bilinear formulation, we construct $N$-soliton solutions and analyze the Hirota $N$-soliton conditions in (2+1)-dimensions. A generalized algorithm to prove the Hirota conditions is presented by comparing degrees of the multivariate polynomials derived from the Hirota function in $N$ wave vectors, and two weight numbers are introduced for transforming the Hirota function to achieve homogeneity of the related polynomials. An application is developed for a general combined nonlinear equation, which provides a proof of existence of its $N$-soliton solutions. The considered model equation includes three integrable equations in (2+1)-dimensions: the (2+1)-dimensional KdV equation, the Kadomtsev–Petviashvili equation, and the (2+1)-dimensional Hirota–Satsuma–Ito equation, as specific examples.


*Keywords:* $N$-soliton solution; Hirota $N$-soliton condition; (2+1)-dimensional integrable equations

## 1. Introduction

$N$-soliton solutions are exact multiple wave solutions to nonlinear integrable equations [1,25]. Various significant solutions in mathematical physics, including breather, complexion, lump and rogue wave solutions, are special reductions of $N$-soliton solutions in different situations. Solitons superimposed in fibers can be applied to optical communications, which are faster, more secure, and more flexible [5]. It is well-known that the Hirota bilinear method is a standard and powerful technique to generate $N$-soliton solutions [11]. The innovative concept of bilinear derivatives is the key in the basic theory of exact solutions [22], and Hirota bilinear forms are the starting point to construct $N$-soliton solutions [11].

Hirota bilinear derivatives read [9]:

$$D_x^m f \cdot g = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (\partial_x^i f)(\partial_x^{m-i} g), \ \ m \geq 1, \tag{1.1}$$

* Correspondence to: Department of Mathematics and Statistics, University of South Florida, Tampa, FL, 33620, USA.
*E-mail address:* mawx@cas.usf.edu.

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and more generally, we have bilinear partial derivatives with multiple variables:

$$(D_x^m D_t^n f \cdot g)(x,t) = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x',t')|_{x'=x,t'=t},\ m,n \ge 0,\ m+n \ge 1. \tag{1.2}$$

The case of $f = g$ yields Hirota bilinear expressions:

$$D_x^{2m-1} f \cdot f = 0,\ D_x^{2m} f \cdot f = \sum_{i=0}^{2m} (-1)^{2m-i} \binom{2m}{i} (\partial_x^i f)(\partial_x^{2m-i} f),\ m \ge 1, \tag{1.3}$$

and bilinear partial derivative expressions:

$$D_x^m D_t^n f \cdot f = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} (\partial_x^i \partial_t^j f)(\partial_x^{m-i} \partial_t^{n-j} f),\ m,n \ge 0,\ m+n \ge 1. \tag{1.4}$$

By virtue of Hirota bilinear expressions, we can formulate Hirota bilinear equations. Take an even polynomial $P(x_1, x_2, \ldots, x_M)$ in $M$ variables, and assume that $P$ has no constant term, i.e.,

$$P(\mathbf{0}) = P(0, 0, \ldots, 0) = 0. \tag{1.5}$$

The corresponding Hirota bilinear equation reads

$$P(D_{x_1}, D_{x_2}, \ldots, D_{x_M}) f \cdot f = 0, \tag{1.6}$$

all terms of which are Hirota bilinear expressions. An important example is the bilinear Kadomtsev–Petviashvili equation

$$B(f) := (D_x^4 + D_x D_t + D_y^2) f \cdot f = 2(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2 + f_{xt} f - f_x f_t + f_{yy} f - f_y^2) = 0, \tag{1.7}$$

which is transformed into the nonlinear Kadomtsev–Petviashvili equation

$$N(u) := (u_t + 6 u u_x + u_{xxx})_x + u_{yy} = 0, \tag{1.8}$$

under the logarithmic derivative transformation $u = 2(\ln f)_{xx}$. The connection between the two equations is

$$N(u) = (B(f)/f^2)_{xx},\ u = 2(\ln f)_{xx}. \tag{1.9}$$

In this paper, we would like to construct $N$-soliton solutions and analyze the corresponding Hirota conditions. A generalized algorithm will be proposed for verifying the Hirota $N$-soliton conditions by comparing degrees of the multivariate polynomials derived from the Hirota function in $N$ wave vectors. An application will be made for a general (2+1)-dimensional combined bilinear equation associated with

$$P(x, y, t) = a_1(x^4 + xt) + a_2(x^3 y + yt) + a_3 x^2 + a_4 xy + a_5 y^2, \tag{1.10}$$

where $a_i$'s are arbitrary constants satisfying $a_1^2 + a_2^2 \ne 0$, thereby presenting a proof of existence of its $N$-soliton solutions. The considered model equation includes the three integrable equations in (2+1)-dimensions: the (2+1)-dimensional KdV equation, the Kadomtsev–Petviashvili equation and the (2+1)-dimensional Hirota–Satsuma–Ito equation, as specific examples.

## 2. Formulating $N$-soliton solutions and their conditions

Let $N \ge 1$ be an arbitrary integer. For a general Hirota bilinear equation (1.6), we construct its $N$-soliton solutions and analyze their sufficient and necessary conditions.

### 2.1. Bilinear formulation of soliton solutions

Let us assume that $N$ wave vectors are denoted by

$$\mathbf{k}_i = (k_{1,i}, k_{2,i}, \ldots, k_{M,i}),\ 1 \le i \le N, \tag{2.1}$$

where $k_{1,i}, k_{2,i}, \ldots, k_{M,i},\ 1 \le i \le N$, are constants to be determined. An $N$-soliton solution to the Hirota bilinear equation (1.6) is given by [8]:

$$f = \sum_{\mu=0,1} \exp\Big(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} a_{ij} \mu_i \mu_j\Big), \tag{2.2}$$

where $\mu=(\mu_1,\mu_2,\ldots,\mu_N)$, $\mu=0,1$ means that each $\mu_i$ takes values, either 0 or 1, the wave variables read

$$\eta_i=k_{1,i}x_1+k_{2,i}x_2+\cdots+k_{M,i}x_M+\eta_{i,0},\ \ 1\le i\le N, \tag{2.3}$$

$\eta_{i,0}$'s being arbitrary constants, and the phase shifts are determined by

$$\mathrm{e}^{a_{ij}}=A_{ij}:=-\frac{P(\mathbf{k}_i-\mathbf{k}_j)}{P(\mathbf{k}_i+\mathbf{k}_j)},\ \ 1\le i<j\le N. \tag{2.4}$$

Note that only the constants $\mathrm{e}^{a_{ij}}$'s, but not $a_{ij}$'s, are needed in the definition of $f$, (2.2).

Let us further introduce

$$H(\mathbf{k}_{i_1},\ldots,\mathbf{k}_{i_n})=\sum_{\sigma=\pm1}P(\sum_{r=1}^{n}\sigma_r\mathbf{k}_{i_r})\prod_{1\le r<s\le n}P(\sigma_r\mathbf{k}_{i_r}-\sigma_s\mathbf{k}_{i_s})\sigma_r\sigma_s,\ \ 1\le n\le N, \tag{2.5}$$

where $1\le i_1<\cdots<i_n\le N$, $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_n)$, and $\sigma=\pm1$ means that each $\sigma_r$ takes values, either 1 or $-1$. These functions are called the Hirota functions.

Observing the basic properties

$$P(D_{x_1},\ldots,D_{x_M})\mathrm{e}^{\eta_i}\cdot\mathrm{e}^{\eta_j}=P(\mathbf{k}_i-\mathbf{k}_j)\mathrm{e}^{\eta_i+\eta_j}, \tag{2.6}$$

and

$$P(D_{x_1},\ldots,D_{x_M})\mathrm{e}^{\eta_n}f\cdot\mathrm{e}^{\eta_n}g=\mathrm{e}^{2\eta_n}P(D_{x_1},\ldots,D_{x_M})f\cdot g, \tag{2.7}$$

where $\eta_i$, $\eta_j$ and $\eta_n$ are arbitrary wave variables defined by (2.3), we can have the following formulation [18,19].

**Theorem 2.1.** *Let the function $f$ be given by (2.2) and $\hat{\xi}$ denote that the term $\xi$ is not involved. Then we have*

$$\begin{aligned}&P(D_{x_1},\ldots,D_{x_M})f\cdot f\\&=(-1)^{\frac12N(N-1)}\frac{H(\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_N)}{\prod_{1\le i<j\le N}P(\mathbf{k}_i+\mathbf{k}_j)}e^{\eta_1+\eta_2+\cdots+\eta_N}\\&+\sum_{n=1}^{N-1}(-1)^{\frac12(N-n)(N-n-1)}\sum_{1\le i_1<\cdots<i_n\le N}\frac{H(\mathbf{k}_1,\ldots,\hat{\mathbf{k}}_{i_1},\ldots,\hat{\mathbf{k}}_{i_n},\ldots,\mathbf{k}_N)}{\prod_{\substack{1\le i<j\le N\\ i,j\notin\{i_1,\ldots,i_n\}}}P(\mathbf{k}_i+\mathbf{k}_j)}e^{\eta_1+\cdots+\hat{\eta}_{i_1}+\cdots+\hat{\eta}_{i_n}+\cdots+\eta_N}\\&+\sum_{n=1}^{N-1}\sum_{1\le i_1<\cdots<i_n\le N}e^{2(\eta_{i_1}+\cdots+\eta_{i_n}+\sum_{1\le r<s\le n}a_{i_ri_s})}P(D_{x_1},\ldots,D_{x_M})\tilde{f}_{i_1\cdots i_n}\cdot\tilde{f}_{i_1\cdots i_n}\end{aligned} \tag{2.8}$$

*with*

$$\tilde{f}_{i_1\cdots i_n}=\sum_{\tilde{\mu}_{i_1\cdots i_n}=0,1}exp(\sum_{\substack{1\le i\le N\\ i\notin\{i_1,\ldots,i_n\}}}\mu_i\tilde{\eta}_i+\sum_{\substack{1\le i<j\le N\\ i,j\notin\{i_1,\ldots,i_n\}}}a_{ij}\mu_i\mu_j),\ \ \tilde{\eta}_i=\eta_i+\sum_{r=1}^{n}a_{ii_r}, \tag{2.9}$$

*where $\tilde{\mu}_{i_1\cdots i_n}=(\mu_1,\ldots,\hat{\mu}_{i_1}\cdots,\hat{\mu}_{i_n},\ldots,\mu_N)$ and $\tilde{\mu}_{i_1\cdots i_n}=0,1$ means that each $\mu_i$ in $\tilde{\mu}_{i_1\cdots i_n}$ takes values, either 0 or 1.*

Based on this formulation, we readily know that the Hirota bilinear equation (1.6) possesses an $N$-soliton solution (2.2) if and only if the following condition

$$H(\mathbf{k}_{i_1},\ldots,\mathbf{k}_{i_n})=0,\ \ 1\le i_1<\cdots<i_n\le N,\ \ 1\le n\le N, \tag{2.10}$$

is satisfied. This is called the Hirota condition for an $N$-soliton solution, or simply, the $N$-soliton condition (see [10], p165). On account of the even property of $P$, the case of (2.10) with $n=1$ presents the dispersion relations

$$P(\mathbf{k}_i)=P(k_{1,i},k_{2,i},\ldots,k_{M,i})=0,\ \ 1\le i\le N. \tag{2.11}$$

There exist very few studies (see, e.g., [23,27]) about the Hirota $N$-soliton condition, because of its complexity involved in the Hirota functions.

### 2.2. Illustrative examples

The one-soliton condition is exactly the dispersion relation: $P(\mathbf{k}_1) = 0$, which means that $f = 1 + \mathrm{e}^{\eta_1}$ is a solution if $P(\mathbf{k}_1) = 0$.

Besides the dispersion relations $P(\mathbf{k}_1) = P(\mathbf{k}_2) = 0$, the two-soliton condition requires

$$2(P(\mathbf{k}_1 + \mathbf{k}_2)P(\mathbf{k}_1 - \mathbf{k}_2) - P(\mathbf{k}_1 - \mathbf{k}_2)P(\mathbf{k}_1 + \mathbf{k}_2)) = 0, \tag{2.12}$$

which is an identity. Therefore, a Hirota bilinear equation always has a two-soliton solution:

$$f = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + A_{12}\mathrm{e}^{\eta_1+\eta_2}, \tag{2.13}$$

where $P(\mathbf{k}_1) = P(\mathbf{k}_2) = 0$.

Upon taking $N = 3$, it is easy to see that the three-soliton condition [6,7] requires

$$\sum_{\sigma_1,\sigma_2,\sigma_3=\pm1} P(\sigma_1\mathbf{k}_1 + \sigma_2\mathbf{k}_2 + \sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1 - \sigma_2\mathbf{k}_2) \times P(\sigma_2\mathbf{k}_2 - \sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1 - \sigma_3\mathbf{k}_3) = 0, \tag{2.14}$$

in addition to the dispersion relations $P(\mathbf{k}_1) = P(\mathbf{k}_2) = P(\mathbf{k}_3) = 0$. Clearly, this is equivalent to

$$\sum_{(\sigma_1,\sigma_2,\sigma_3)\in S} P(\sigma_1\mathbf{k}_1 + \sigma_2\mathbf{k}_2 + \sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1 - \sigma_2\mathbf{k}_2) \times P(\sigma_2\mathbf{k}_2 - \sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1 - \sigma_3\mathbf{k}_3) = 0, \tag{2.15}$$

where $S = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$. Obviously, the corresponding three-soliton solution reads

$$f = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_3} + A_{12}\mathrm{e}^{\eta_1+\eta_2} + A_{13}\mathrm{e}^{\eta_1+\eta_3} + A_{23}\mathrm{e}^{\eta_2+\eta_3} + A_{123}\mathrm{e}^{\eta_1+\eta_2+\eta_3}, \quad A_{123} = A_{12}A_{13}A_{23}. \tag{2.16}$$

It is generally accepted that the three-soliton condition implies the general $N$-soliton condition, without proof of its accuracy.

If a sufficient Hirota $N$-soliton condition (see [20], p951):

$$P(\mathbf{k}_i - \mathbf{k}_j) = 0, \quad 1 \le i < j \le N, \tag{2.17}$$

is satisfied, we arrive at a resonant $N$-soliton solution:

$$f = 1 + c_1\mathrm{e}^{\eta_1} + c_2\mathrm{e}^{\eta_2} + \cdots + c_N\mathrm{e}^{\eta_N}, \tag{2.18}$$

where $c_i$'s are arbitrary constants. Note that all wave vectors $\mathbf{k}_i$'s associated with resonant solutions form a vector sub-space in $\mathbb{R}^M$ [21], p7178.

### 2.3. Basic properties of the Hirota functions

In order to prove the Hirota $N$-soliton conditions, we usually need to factor out as many common factors out of the Hirota function $H(\mathbf{k}_1, \ldots, \mathbf{k}_N)$ as possible. To this end, we will use the following two theorems. The first one is an automatic consequence of the definition of the Hirota functions.

**Theorem 2.2.** *The Hirota functions defined by (2.5) are even and symmetric functions in the involved wave vectors.*

When taking $\mathbf{k}_2 = \pm\mathbf{k}_1$, we immediately have

$$P(\sigma_i\mathbf{k}_i - \mathbf{k}_2)P(\sigma_i\mathbf{k}_i \pm \mathbf{k}_1) = P(\mathbf{k}_i - \mathbf{k}_1)P(\mathbf{k}_i + \mathbf{k}_1) \tag{2.19}$$

in both cases of $\sigma_i = \pm1$, based on the even property of the polynomial $P$. Utilizing this basic property, we can derive the following consequence [19], with a careful computation.

**Theorem 2.3.** *If* $\mathbf{k}_2 = \pm\mathbf{k}_1$, *then we have*

$$H(\mathbf{k}_1,\ldots,\mathbf{k}_N) = 2H(\mathbf{k}_3,\ldots,\mathbf{k}_N)P(2\mathbf{k}_1)\prod_{i=3}^{N} P(\mathbf{k}_i-\mathbf{k}_1)P(\mathbf{k}_i+\mathbf{k}_1), \tag{2.20}$$

*where* $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ *and* $H(\mathbf{k}_3,\ldots,\mathbf{k}_N)$ *are two Hirota functions defined in* (2.5).

This specific theorem will be used to factor out common factors out of the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$, in order to verify the Hirota $N$-soliton conditions.

## 3. A generalized algorithm and its applications

### 3.1. A generalized algorithm

Let us focus on the (2+1)-dimensional case and state the corresponding $N$ wave vectors as

$$\mathbf{k}_i = (k_i, l_i, -\omega_i),\ \ 1 \le i \le N. \tag{3.1}$$

We also assume that the dispersion relations (2.11) determine all frequencies in terms of wave numbers $k_i, l_i$: $\omega_i = \omega(k_i, l_i),\ \ 1 \le i \le N$. In this way, $P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j)$ becomes functions of $k_i, l_i$ and $k_j, l_j$ only.

First, we suppose that under the substitution

$$l_i = \gamma k_i^{w_1} l_i^{w_2},\ \ 1 \le i \le N, \tag{3.2}$$

for some integer weights $w_1$ and $w_2$ and a nonzero constant coefficient $\gamma$, the two functions $P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j)$ and $P(\sigma_1\mathbf{k}_1 + \cdots + \sigma_N\mathbf{k}_N)$ are simplified into rational forms as follows:

$$P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j) = \frac{\sigma_i\sigma_j k_i k_j Q_1(k_i, l_i, k_j, l_j, \sigma_i, \sigma_j)}{Q_2(k_i, l_i, k_j, l_j)}, \tag{3.3}$$

and

$$P(\sigma_1\mathbf{k}_1 + \cdots + \sigma_N\mathbf{k}_N) = \frac{Q_3(k_1, l_1, \ldots, k_N, l_N, \sigma_1, \ldots, \sigma_N)}{Q_4(k_1, l_1, \ldots, k_N, l_N)}, \tag{3.4}$$

where $Q_1$, $Q_2$, $Q_3$ and $Q_4$ are polynomial functions in the indicated variables.

Second, note that Theorem 2.3 implies that under the induction assumption, the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ will be zero, if there exist two equal wave vectors $\mathbf{k}_i = \mathbf{k}_j$ for some pair $1 \le i < j \le N$. It also follows from the symmetric property in Theorem 2.2 that under the transforms in (3.2), $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ is still even with respect to $k_i, l_i$ $1 \le i \le N$, while $w_1 + w_2$ is odd, and it is even only with respect to $k_i,\ 1 \le i \le N$, while $w_1$ is odd. Therefore, in both cases, we can simplify the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ into the following form:

$$H(\mathbf{k}_1,\ldots,\mathbf{k}_N) = (k_i^2 - k_j^2)^2 g_{ij} + (l_i - l_j)^2 h_{ij},\ \text{ for each pair } 1 \le i < j \le N, \tag{3.5}$$

where $g_{ij}$ and $h_{ij}$ are rational functions of the wave numbers $k_n, l_n,\ 1 \le n \le N$. Then we see that the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ can be expressed as

$$H(\mathbf{k}_1,\ldots,\mathbf{k}_N) = \frac{\prod_{1\le i<j\le N} k_i^2 k_j^2 [\prod_{1\le i<j\le N}(k_i^2-k_j^2)^2 g + \prod_{1\le i<j\le N}(l_i-l_j)^2 h]}{Q_4(k_1, l_1, \ldots, k_N, l_N)\prod_{1\le i<j\le N} Q_2(k_i, l_i, k_j, l_j)} \tag{3.6}$$

under the substitution (3.2), where $g$ and $h$ are polynomials of the wave numbers $k_n, l_n,\ 1 \le n \le N$, and $g$ can be nonzero when $H(\mathbf{k}_1,\ldots,\mathbf{k}_N) \ne 0$.

Now, if $H(\mathbf{k}_1,\ldots,\mathbf{k}_N) \ne 0$, we readily know that the degree of the polynomial

$$\begin{aligned}\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N) &:= H(\mathbf{k}_1,\ldots,\mathbf{k}_N) Q_4(k_1, l_1, \ldots, k_N, l_N) \prod_{1\le i<j\le N} Q_2(k_i, l_i, k_j, l_j) \\ &= \prod_{1\le i<j\le N} k_i^2 k_j^2 [\prod_{1\le i<j\le N}(k_i^2-k_j^2)^2 g + \prod_{1\le i<j\le N}(l_i-l_j)^2 h]\end{aligned} \tag{3.7}$$

is at least $2N(N-1) + 2N(N-1) = 4N(N-1)$. From (3.3), (3.4) and (3.7), we have

$$\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N) = \sum_{\sigma=\pm1} Q_3(k_1, l_1, \ldots, k_N, l_N, \sigma_1, \ldots, \sigma_N) \prod_{1\le i<j\le N} \sigma_i\sigma_j k_i k_j Q_1(k_i, l_i, k_j, l_j, \sigma_i, \sigma_j), \tag{3.8}$$

and so the degree of the polynomial on the left-hand side should not be less than $4N(N-1)$. Otherwise, we will have $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=0$, which is what we want to prove. Therefore, the final task is to compute $Q_1$ and $Q_3$ and check if the degree of the polynomial $\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ is less than $4N(N-1)$, based on (3.8). If we can make a contradiction, then we verify the Hirota $N$-soliton conditions, thereby proving the existence of $N$-soliton solutions.

### 3.2. Application to a (2+1)-dimensional combined equation

We consider a (2+1)-dimensional combined bilinear equation associated with a polynomial $P$ given by (1.10). We restate $P$ here for ease of reference:

$$P(x,y,t)=a_1(x^4+xt)+a_2(x^3y+yt)+a_3x^2+a_4xy+a_5y^2, \tag{3.9}$$

where $a_i$'s are arbitrary constants, which satisfy $a_1^2+a_2^2\neq 0$ to guarantee the nonlinearity of the corresponding model equation. The resulting (2+1)-dimensional combined bilinear equation is

$$\begin{aligned}B(f) &:= [a_1(D_x^4+D_xD_t)+a_2(D_x^3D_y+D_yD_t)+a_3D_x^2+a_4D_xD_y+a_5D_y^2]f\cdot f\\ &= 2[a_1(f_{xxxx}f-4f_{xxx}f_x+3f_{xx}^2+f_{xt}f-f_xf_t)\\ &\quad +a_2(f_{xxxy}f-3f_{xxy}f_x+3f_{xy}f_{xx}-f_yf_{xxx}+f_{yt}f-f_yf_t)\\ &\quad +a_3(f_{xx}f-f_x^2)+a_4(f_{xy}f-f_xf_y)+a_5(f_{yy}f-f_y^2)]=0.\end{aligned} \tag{3.10}$$

This is equivalent to the (2+1)-dimensional combined nonlinear equation:

$$N(u,v):=a_1(u_t+6uu_x+u_{xxx})+a_2[v_t+3(uv)_x+v_{xxx}]+a_3u_x+a_4v_x+a_5v_y=0, \tag{3.11}$$

where $u_y=v_x$, and the direct link is

$$N(u,v)=(B(f)/f^2)_x, \tag{3.12}$$

under the logarithmic derivative transformations

$$u=2(\ln f)_{xx},\ \ v=2(\ln f)_{xy}. \tag{3.13}$$

Therefore, if $f$ solves the bilinear equation (3.10), then $u$ and $v$ defined by (3.13) solve the corresponding nonlinear equation (3.11).

Obviously, we can work out that

$$\omega_i=k_i^3+\frac{a_3+a_4\gamma+a_5\gamma^2}{a_1+a_2\gamma}k_i,\ \ 1\le i\le N, \tag{3.14}$$

and

$$Q_1=-3(a_1+a_2\gamma)(\sigma_ik_i-\sigma_jk_j)^2,\ \ \deg Q_3=4,\ \ Q_2=1,\ \ Q_4=1, \tag{3.15}$$

under the substitution (3.2) with $w_1=1$ and $w_2=0$. Therefore, if $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)\neq 0$, then based on (3.8), the degree of the polynomial $\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ (=$H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$) is $2N(N-1)+4=2N^2-2N+4$, which could not be greater than $4N(N-1)$ when $N\ge 3$. Then it follows that $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=0,\ N\ge 1$, since the nonlinear model equation (3.11) passes the three-soliton test, which just needs a direct computation (this is important, though omitted).

### 3.3. Specific reductions

#### 3.3.1. (2+1)-dimensional KdV equation

The case of $a_2=1$ and all other zero coefficients presents

$$P(x,y,t)=x^3y+yt, \tag{3.16}$$

with which the (2+1)-dimensional KdV equation is associated. The corresponding (2+1)-dimensional bilinear KdV equation reads

$$B(f):=D_y(D_t+D_x^3)f\cdot f=2(f_{yt}f-f_yf_t+f_{xxxy}f-3f_{xxy}f_x+3f_{xy}f_{xx}-f_yf_{xxx})=0, \tag{3.17}$$

which is equivalent to the (2+1)-dimensional KdV equation [3]:

$$N(u,v) := v_t + 3(uv)_x + v_{xxx} = 0, \tag{3.18}$$

where $u_y = v_x$, under the logarithmic derivative transformations in (3.13).

We can also have a direct verification for the Hirota $N$-soliton condition under a different selection of weights: $w_1 = 0$ and $w_2 = 1$, and $\gamma = 1$ [18]. In this case, we have

$$\omega_i = k_i^3,\ \ 1 \le i \le N, \tag{3.19}$$

and

$$Q_1 = -3(\sigma_i k_i - \sigma_j k_j)(\sigma_i l_i - \sigma_j l_j),\ \deg Q_3 = 4,\ \ Q_2 = 1,\ \ Q_4 = 1. \tag{3.20}$$

Therefore, if $H(\mathbf{k}_1,\ldots,\mathbf{k}_N) \neq 0$, then based on (3.8), the degree of the polynomial $\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ $(= H(\mathbf{k}_1,\ldots,\mathbf{k}_N))$ is $2N(N-1)+4 = 2N^2 - 2N + 4$, which could not be greater than $4N(N-1)$ when $N \ge 3$. Thus, $H(\mathbf{k}_1,\ldots,\mathbf{k}_N) = 0,\ \ N \ge 1$.

It is known that the spatial symmetric version of the (2+1)-dimensional KdV equation (3.18):

$$v_t + 3(uv)_x + 3(vw)_y + v_{xxx} + v_{yyy} = 0, \tag{3.21}$$

where $u_y = v_x$ and $v_y = w_x$, has been discussed (see [24], p707 and [28], p589), and its inverse scattering transform and algebro-geometric solutions have been announced in [24,28], respectively. However, this (2+1)-dimensional symmetric equation does not pass the three-soliton test.

### 3.3.2. The Kadomtsev–Petviashvili equation

The case of $a_1 = a_5 = 1$ and all other zero coefficients tells

$$P(x,y,t) = x^4 + xt + y^2, \tag{3.22}$$

with which the Kadomtsev–Petviashvili equation is associated. The corresponding bilinear Kadomtsev–Petviashvili equation reads

$$\begin{aligned} B(f) &:= (D_x^4 + D_x D_t + D_y^2) f \cdot f \\ &= 2(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2 + f_{xt} f - f_x f_t + f_{yy} f - f_y^2) = 0, \end{aligned} \tag{3.23}$$

which is equivalent to the Kadomtsev–Petviashvili equation [13]:

$$N(u) := u_t + 6uu_x + u_{xxx} + v_y = 0, \tag{3.24}$$

where $u_y = v_x$, under the logarithmic derivative transformations in (3.13).

We can also give a direct proof for the Hirota $N$-soliton condition under a different selection of weights: $w_1 = w_2 = 1$, and $\gamma = 1$ [18]. In this case, we have

$$\omega_i = k_i^3 + l_i^2 k_i,\ \ 1 \le i \le N, \tag{3.25}$$

and

$$Q_1 = -3(\sigma_i k_i - \sigma_j k_j)^2 + (l_i - l_j)^2,\ \deg Q_3 = 4,\ \ Q_2 = 1,\ \ Q_4 = 1. \tag{3.26}$$

Therefore, if $H(\mathbf{k}_1,\ldots,\mathbf{k}_N) \neq 0$, then based on (3.8), the degree of the polynomial $\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ $(= H(\mathbf{k}_1,\ldots,\mathbf{k}_N))$ is $2N(N-1)+4 = 2N^2 - 2N + 4$, which could not be greater than $4N(N-1)$ when $N \ge 3$. It thus follows that $H(\mathbf{k}_1,\ldots,\mathbf{k}_N) = 0,\ \ N \ge 1$.

We point out that the $N$-soliton solutions of the Kadomtsev–Petviashvili equation have been also presented in [2,26] and the quasiperiodic multiphase solutions of the Kadomtsev–Petviashvili equation can be decomposed into finite-dimensional canonical Hamiltonian systems [4].

*3.3.3. (2+1)-dimensional Hirota–Satsuma–Ito equation*

The case of $a_2 = a_3 = 1$ and all other zero coefficients gives

$$P(x, y, t) = x^3 t + yt + x^2, \tag{3.27}$$

under an exchange of $y$ and $t$, with which the Hirota–Satsuma–Ito equation is associated. The corresponding (2+1)-dimensional bilinear Hirota–Satsuma–Ito equation reads

$$\begin{aligned} B(f) &:= (D_x^3 D_t + D_y D_t + D_x^2) f \cdot f \\ &= 2(f_{xxxt} f - 3 f_{xxt} f_x + 3 f_{xt} f_{xx} - f_t f_{xxx} + f_{yt} f - f_y f_t + f_{xx} f - f_x^2) = 0, \end{aligned} \tag{3.28}$$

which is equivalent to the (2+1)-dimensional nonlinear Hirota–Satsuma–Ito equation:

$$N(u) := v_y + 3(uv)_x + v_{xxx} + u_x = 0, \tag{3.29}$$

where $u_t = v_x$, under the logarithmic derivative transformations in (3.13). If we take a potential form $v = \tilde{v}_t$, then the above equation becomes the original (2+1)-dimensional Hirota–Satsuma–Ito equation [7]:

$$\tilde{v}_{xx} + \tilde{v}_{ty} + 3(\tilde{v}_t \tilde{v}_x)_x + \tilde{v}_{txxx} = 0. \tag{3.30}$$

If this equation does not depend on $y$, it reduces to the Hirota–Satsuma equation in (1+1)-dimensions [12].

We can similarly present a direct verification for the Hirota $N$-soliton condition under a different selection of weights: $w_1 = 1$ and $w_2 = 2$, and $\gamma = 1$ [18]. In this case, we can directly get

$$\omega_i = \frac{k_i}{k_i^2 + l_i^2}, \ 1 \le i \le N, \tag{3.31}$$

and

$$\begin{cases} Q_1 = (k_i^2 + k_j^2)^2 + (l_i^2 - l_j^2)^2 + 2k_i^2 k_j^2 + 2k_i^2 l_i^2 + 2k_j^2 l_j^2 + k_i^2 l_j^2 + k_j^2 l_i^2 \\ \qquad -3\sigma_i k_i \sigma_j k_j (k_i^2 + k_j^2 + l_i^2 + l_j^2), \\ \deg Q_3 = 2(N+1), \ \ Q_2 = (k_i^2 + l_i^2)(k_j^2 + l_j^2), \ \ Q_4 = \prod_{i=1}^N (k_i^2 + l_i^2). \end{cases} \tag{3.32}$$

Therefore, if $H(\mathbf{k}_1, \ldots, \mathbf{k}_N) \neq 0$, then based on (3.8), the degree of the polynomial $\tilde{H}(\mathbf{k}_1, \ldots, \mathbf{k}_N)$ is $3N(N-1) + 2(N+1) = 3N^2 - N + 2$, which could not be greater than $4N(N-1)$ when $N \ge 4$. It then follows that $H(\mathbf{k}_1, \ldots, \mathbf{k}_N) = 0, \ N \ge 1$.

The $N$-soliton solution of the (2+1)-dimensional Hirota–Satsuma–Ito equation can be reduced to diverse other interesting solutions such as breather, lump and rogue wave solutions and their interaction solutions [14,30].

## 4. Concluding remarks

We have discussed the Hirota $N$-soliton conditions for a combined bilinear differential equation in (2+1)-dimensions, and shown the existence of its $N$-soliton solutions. The general model equation includes three (2+1)-dimensional integrable equations: the (2+1)-dimensional KdV equation, the Kadomtsev–Petviashvili equation and the (2+1)-dimensional Hirota–Satsuma–Ito equation, as specific examples. It would always be intriguing to explore new examples of bilinear equations in (2+1)-dimensions, which possess $N$-soliton solutions. Along with the presented generalized efficient algorithm, symbolic computations would be extremely helpful in determining such bilinear (and then nonlinear) equations in (2+1)-dimensions.

There are various generalized bilinear derivatives which allow us to deal with bilinear differential equations, including odd-order ones (not as in the Hirota case). The $D_{p,x}$-operators are particular examples [15]:

$$D_{p,x}^m D_{p,t}^n f \cdot g = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \alpha_p^{i+j} (\partial_x^{m-i} \partial_t^{n-j} f)(\partial_x^i \partial_t^j g), \ \ m, n \ge 0, \ \ m + n \ge 1, \tag{4.1}$$

where the powers of $\alpha_p$ determine the corresponding signs as follows:

$$\alpha_p^i = (-1)^{r(i)}, \ \ i \equiv r(i) \bmod p, \ \ i \ge 0, \tag{4.2}$$

with $0 \le r(i) < p$. In particular, the patterns of those signs for $i = 1, 2, 3, \ldots$ read

$$p = 3: \ -, +, +, -, +, +, \ldots;$$

$$p = 5: \ -, +, -, +, +, -, +, -, +, +, \ldots;$$

$$p = 7: \ -, +, -, +, -, +, +, -, +, -, +, -, +, +, \ldots.$$

Two such simple generalized bilinear derivatives are $D_{3,x}$ and $D_{5,x}$, associated with the two smallest odd prime numbers: $p = 3, 5$. The cases of $p = 2k$, $k \in \mathbb{N}$, are exactly the same as the Hirota case. The corresponding generalized bilinear expressions can exhibit new characteristics, indeed. For example, we have

$$D_{3,x}^2 D_{3,t} f \cdot f = 2 f_{xxt} f, \ \ D_{3,x}^3 D_{3,t} f \cdot f = 6 f_{xx} f_{xt}, \tag{4.3}$$

which are totally different from the Hirota derivatives. Of course, we can have many other generalized bilinear derivatives such as $D_{9,x}$ and $D_{15,x}$.

We point out that resonant $N$-solitons have been analyzed for generalized bilinear equations [16] and trilinear equations [17]. When a multivariate polynomial $P$ satisfies

$$P(\mathbf{k}_i + \alpha_p \mathbf{k}_j) + P(\mathbf{k}_j + \alpha_p \mathbf{k}_i) = 0, \ \ 1 \le i \le j \le N, \tag{4.4}$$

where $\mathbf{k}_i$'s are wave vectors defined earlier, the corresponding generalized bilinear equation

$$P(D_{p,x_1}, \ldots, D_{p,x_M}) f \cdot f = 0 \tag{4.5}$$

can possess the resonant $N$-soliton solution [16]:

$$f = 1 + c_1 \mathrm{e}^{\eta_1} + c_2 \mathrm{e}^{\eta_2} + \cdots + c_N \mathrm{e}^{\eta_N} \tag{4.6}$$

where $\eta_i$'s are the wave variables defined previously and $c_i$'s are arbitrary constants.

We are interested in searching for concrete examples of generalized bilinear equations, which possess $N$-soliton solutions. There are various basic questions in the corresponding theory. Those include how to formulate a generalized $N$-soliton condition; and how to identify generalized bilinear equations, for example,

$$P(D_{3,x}, D_{3,t}) = 0, \ \ P(D_{3,x}, D_{3,y}, D_{3,t}) = 0,$$

in both (1+1)-dimensions and (2+1)-dimensions, which have $N$-soliton solutions. All related studies will be helpful in improving our understanding of bilinear partial differential equations and their associated nonlinear wave phenomena [29].

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