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Darboux Transformations for a Lax Integrable System in $2n$ Dimensions

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Abstract. A $2n$-dimensional Lax integrable system is proposed by a set of specific spectral problems. It contains Takasaki equations, the self-dual Yang–Mills equations and its integrable hierarchy as examples. An explicit formulation of Darboux transformations is established for this Lax integrable system. The Vandermonde and generalized Cauchy determinant formulas lead to a description for deriving explicit solutions and thus some rational and analytic solutions are obtained.

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1. Introduction

Darboux transformations provide us with a purely algebraic, powerful method to construct solutions for systems of nonlinear equations [1]. The key is to expose a kind of covariant properties that the corresponding spectral problems possess. There have been many tricks to do this for getting explicit solutions to various soliton equations including the KdV equation, KP equation, Davey–Stewartson equation, Veselov–Novikov equation, etc. (see, for example, [2–6] and references therein). Darboux transformations can also be applied to generating multi-soliton solutions to soliton equations and the Darboux covariance makes it possible to construct a series of exactly solvable systems of supersymmetric quantum mechanics [1].

In this Letter, we would like to establish a kind of Darboux transformation for a Lax integrable system in $2n$-dimensions, which we will introduce. The dimension reductions of this system contain some interesting and important equations as examples. Among them are Takasaki equations [7], the self-dual Yang–Mills (SDYM) equations, and the self-dual Yang–Mills hierarchy [8], etc. It is well-known that the SDYM equations (even the SDYM hierarchy), may be reduced to many integrable equations in $1+1$ dimensions and in $1+2$ dimensions, when certain symmetry conditions are imposed (see, for instance, [8, 9]). Therefore our Lax integrable system also includes a lot of integrable soliton equations in $1+1$ dimensions and in $1+2$ dimensions. Recently, considerable interest has been shown in the aspect of symmetry reductions of the SDYM equations (see [10, 11], a quite detailed list of the relevant references is included in [10]). This also increases, to a great extent, the validity of Ward's conjecture [12]: many (and perhaps all?) integrable equations may be derived from the SDYM gauge field equations or its generations by reduction.

This Letter is organized as follows. In Section 2, we derive a Lax integrable system starting from a set of specific spectral problems and display a few concrete systems of nonlinear equations. The corresponding Darboux transformations are established in Section 3 and an explicit description of a broad class of solutions is proposed by means of the resulting Darboux transformations. Section 4 contains some further discussions and two remarks, along with a more general system whose spectral problems include negative powers of a spectral parameter.

2. Lax Integrable System

Let the differential operators L_i, $1 \leqslant i \leqslant n$, be defended by

$$\begin{aligned} L_i &= L_i(\lambda) = \frac{\partial}{\partial p_i} - a_i(\lambda)\frac{\partial}{\partial x_i} \\ &= \frac{\partial}{\partial p_i} - \left(\sum_{k=0}^{M} a_{ik}\lambda^{k+1}\right)\frac{\partial}{\partial x_i}, \quad 1 \leqslant i \leqslant n, \end{aligned} \tag{2.1}$$

where the coefficients a_{ik}, $0 \leqslant k \leqslant M$, $1 \leqslant i \leqslant n$, are constants, and $p = (p_1, p_2, \ldots, p_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ or $\mathbb{R}^n$, are two vectors of independent variables. We consider a set of spectral problems

$$L_i(\lambda)\Psi = -A_i(\lambda)\Psi = -\left(\sum_{l=0}^{M} A_{il}\lambda^l\right)\Psi, \quad 1 \leqslant i \leqslant n, \tag{2.2}$$

where λ is a spectral parameter, Ψ is a $N \times N$ matrix of eigenfunctions, and A_{il}, $1 \leqslant i \leqslant n$, $0 \leqslant l \leqslant M$, are all $N \times N$ matrices of potential functions depending on p, x. This spectral problem contains the Takasaki case [7]: $M = 0$, $a_i(\lambda) = \lambda$, $1 \leqslant i \leqslant n$; the Gu and Zhou case [6]; $a_i(\lambda) = 0$, $1 \leqslant i \leqslant n$, $A_i(\lambda) = \lambda J_i + P_i$, $1 \leqslant i \leqslant n-1$, where J_i, P_i are the matrices given in [6]; and the Gu case [13]: $a_i(\lambda) = \lambda$, $1 \leqslant i \leqslant n-1$, $a_n(\lambda) = 0$, $A_i(\lambda) = A_{i0}$, $1 \leqslant i \leqslant n-1$. We assume that $n, N \geqslant 2$ in order to obtain nonlinear integrable systems (we shall explain this later). Noting that L_i, $1 \leqslant i \leqslant n$, are all linear operator, we can calculate that

$$\begin{aligned} &L_j(\lambda)L_i(\lambda)\Psi \\ &= L_j(\lambda)\left(-\sum_{l=0}^{M}\lambda^l A_{il}\Psi\right) \\ &= -\sum_{l=0}^{M}\lambda^l A_{il}L_j(\lambda)\Psi - \sum_{l=0}^{M}\lambda^l(L_j(\lambda)A_{il})\Psi \end{aligned}$$

$$= \sum_{l=0}^{M} \lambda^l A_{il} \sum_{k=0}^{M} \lambda^k A_{jk} \Psi - \sum_{l=0}^{M} \lambda^l \left(\frac{\partial A_{il}}{\partial p_j} - a_i(\lambda) \frac{\partial A_{il}}{\partial x_j} \right) \Psi$$

$$= \sum_{m=0}^{2M} \lambda^m \left(\sum_{\substack{k+l=m \\ 0 \leqslant k,l \leqslant M}} A_{ik} A_{jl} \right) \Psi - \sum_{m=0}^{M} \lambda^m \frac{\partial A_{im}}{\partial p_j} \Psi +$$

$$+ \sum_{m=0}^{2M} \lambda^{m+1} \left(\sum_{\substack{k+l=m \\ 0 \leqslant k,l \leqslant M}} a_{jk} \frac{\partial A_{il}}{\partial x_j} \right) \Psi.$$

Therefore, we see that the compatibility conditions

$$L_j(\lambda) L_i(\lambda) \Psi = L_i(\lambda) L_j(\lambda) \Psi, \quad 1 \leqslant i, j \leqslant n, \tag{2.3}$$

are expressed as

$$\sum_{m=0}^{2M} \lambda^m \sum_{\substack{k+l=m \\ 0 \leqslant k,l \leqslant M}} A_{ik} A_{jl} - \sum_{m=0}^{M} \lambda^m \frac{\partial A_{im}}{\partial p_j} +$$

$$+ \sum_{m=0}^{2M} \lambda^{m+1} \sum_{\substack{k+l=m \\ 0 \leqslant k,l \leqslant M}} a_{jk} \frac{\partial A_{il}}{\partial x_j}$$

$$= \sum_{m=0}^{2M} \lambda^m \sum_{\substack{k+l=m \\ 0 \leqslant k,l \leqslant M}} A_{jk} A_{il} - \sum_{m=0}^{M} \lambda^m \frac{\partial A_{jm}}{\partial p_i} +$$

$$+ \sum_{m=0}^{2M} \lambda^{m+1} \sum_{\substack{k+l=m \\ 0 \leqslant k,l \leqslant M}} a_{ik} \frac{\partial A_{jl}}{\partial x_i}. \tag{2.4}$$

Equating coefficients of the terms λ^0; λ^m, $1 \leqslant m \leqslant M$; λ^m, $M+1 \leqslant m \leqslant 2M$ and λ^{2M+1} in the above equation, lead to the following Lax integrable system in $2n$-dimensions

$$[A_{i0}, A_{j0}] - \frac{\partial A_{i0}}{\partial p_j} + \frac{\partial A_{j0}}{\partial p_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.5}$$

$$\sum_{\substack{k+l=m\\0\leqslant k,l\leqslant M}} [A_{ik}, A_{jl}] - \frac{\partial A_{im}}{\partial p_j} + \frac{\partial A_{jm}}{\partial p_i} +$$

$$+ \sum_{\substack{k+l=m-1\\0\leqslant k,l\leqslant M}} \left(a_{jk}\frac{\partial A_{il}}{\partial x_j} - a_{ik}\frac{\partial A_{jl}}{\partial x_i} \right) = 0,$$

$$1 \leqslant m \leqslant M, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.6}$$

$$\sum_{\substack{k+l=m\\0\leqslant k,l\leqslant M}} [A_{ik}, A_{jl}] + \sum_{\substack{k+l=m-1\\0\leqslant k,l\leqslant M}} \left(a_{jk}\frac{\partial A_{il}}{\partial x_j} - a_{ik}\frac{\partial A_{jl}}{\partial x_i} \right) = 0,$$

$$M+1 \leqslant m \leqslant 2M, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.7}$$

$$a_{jM}\frac{\partial A_{iM}}{\partial x_j} - a_{iM}\frac{\partial A_{jM}}{\partial x_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.8}$$

where $[\cdot,\cdot]$ denotes the Lie bracket of matrix Lie algebras. Evidently for the case of $N = 1$, the commutators of the matrices A_{ik}, A_{jl} are all equal to zero and thus the above system is simplified into a linear system, which isn't what we need. Besides, if $n = 1$, the above system holds automatically and thus doesn't need any consideration. The Lax integrable system consisting of (2.5), (2.6), (2.7) and (2.8), includes some interesting and important systems of equations as examples. For instance, we have a few systems of equations as follows.

EXAMPLE 1. $M = 0$, $a_i(\lambda) = 1$, $1 \leqslant i \leqslant n$, correspond to the Takasaki case [7], which gives rise to the equations

$$[A_{i0}, A_{j0}] - \frac{\partial A_{i0}}{\partial p_j} + \frac{\partial A_{j0}}{\partial p_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.9}$$

$$\frac{\partial A_{i0}}{\partial x_j} - \frac{\partial A_{j0}}{\partial x_i} = 0, \quad 1 \leqslant i,\ j \leqslant n. \tag{2.10}$$

This system may be changed into

$$\frac{\partial}{\partial x_i}\left(\frac{\partial J}{\partial p_j}J^{-1}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial J}{\partial p_i}J^{-1}\right) = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.11}$$

or

$$\frac{\partial}{\partial x_i}\left(J^{-1}\frac{\partial J}{\partial p_j}\right) - \frac{\partial}{\partial x_j}\left(J^{-1}\frac{\partial J}{\partial p_i}\right) = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.12}$$

after making a transformation

$$A_{i0} = -\frac{\partial J}{\partial p_i} J^{-1} \quad \text{or} \quad A_{i0} = J^{-1}\frac{\partial J}{\partial p_i}, \quad 1 \leqslant i \leqslant n,$$

respectively. Equations (2.12) with $n = 2$ is a kind of version of the anti-self-dual Yang–Mills equations due to Pohlmeyer [14]. Its inverse scattering has been analyzed by Beals and Coifman [15] and it may be rewritten in the original Yang equation 'in R-gauge' [16], upon choosing

$$J = \frac{1}{u}\begin{bmatrix} 1 & w \\ v & u^2 + vw \end{bmatrix}.$$

EXAMPLE 2. Let $M = 1$, $n = 2$, $a_1(\lambda) = -\lambda$, $a_2(\lambda) = \lambda$. We obtain the equations

$$[A_{10}, A_{20}] - \frac{\partial A_{10}}{\partial p_2} + \frac{\partial A_{20}}{\partial p_1} = 0, \tag{2.13}$$

$$[A_{10}, A_{21}] + [A_{11}, A_{20}] - \frac{\partial A_{11}}{\partial p_2} + \frac{\partial A_{21}}{\partial p_1} + \frac{\partial A_{10}}{\partial x_2} + \frac{\partial A_{20}}{\partial x_1} = 0, \tag{2.14}$$

$$[A_{11}, A_{21}] + \frac{\partial A_{11}}{\partial x_2} + \frac{\partial A_{21}}{\partial x_1} = 0, \tag{2.15}$$

which yield the SDYM equations discussed in [17], upon making $A_{21} \to -A_{21}$. Here $A_{10}, A_{11}, A_{20}, A_{21}$ are all the Yang–Mills potentials. The corresponding inverse scattering problem was introduced by Belavin and Zakharov [18], many years ago. Recently, a scheme of symmetry reduction of the Lax pairs for the SDYM equations with respect to an arbitrary subgroup of their conformal group, has been described by Legarè and Popov [10] and, accordingly, the compatibility conditions of the reduced Lax pairs, lead to the SDYM equations reduced under the same symmetry group.

EXAMPLE 3. The case of $M = 1$, $a_i(\lambda) = \lambda$, $1 \leqslant i \leqslant n$, yields the equations

$$[A_{i0}, A_{j0}] - \frac{\partial A_{i0}}{\partial p_j} + \frac{\partial A_{j0}}{\partial p_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.16}$$

$$[A_{i0}, A_{j1}] + [A_{i1}, A_{j0}] - \frac{\partial A_{i1}}{\partial p_j} + \frac{\partial A_{j1}}{\partial p_i} + \\ + \frac{\partial A_{i0}}{\partial x_j} - \frac{\partial A_{j0}}{\partial x_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.17}$$

$$[A_{i1}, A_{j1}] + \frac{\partial A_{i1}}{\partial x_j} - \frac{\partial A_{j1}}{\partial x_i} = 0, \quad 1 \leqslant i,\ j \leqslant n. \tag{2.18}$$

In comparison with the SDYM equations (2.13), (2.14), (2.15), these equations may be referred to as the generalized SDYM equations.

EXAMPLE 4. Let $M = 1$, $a_i(\lambda) = \lambda^2$, $1 \leqslant i \leqslant n$. We obtain the equations

$$[A_{i0}, A_{j0}] - \frac{\partial A_{i0}}{\partial p_j} + \frac{\partial A_{j0}}{\partial p_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.19}$$

$$[A_{i0}, A_{j1}] + [A_{i1}, A_{j0}] - \frac{\partial A_{i1}}{\partial p_j} + \frac{\partial A_{j1}}{\partial p_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.20}$$

$$[A_{i1}, A_{j1}] + \frac{\partial A_{i0}}{\partial x_j} - \frac{\partial A_{j0}}{\partial x_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.21}$$

$$\frac{\partial A_{i1}}{\partial x_j} - \frac{\partial A_{j1}}{\partial x_i} = 0, \quad 1 \leqslant i,\ j \leqslant n. \tag{2.22}$$

EXAMPLE 5. The case of $M = 1$, $a_i(\lambda) = \lambda + \lambda^2$, $1 \leqslant i \leqslant n$, leads to the equations

$$[A_{i0}, A_{j0}] - \frac{\partial A_{i0}}{\partial p_j} + \frac{\partial A_{j0}}{\partial p_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.23}$$

$$[A_{i0}, A_{j1}] + [A_{i1}, A_{j0}] - [A_{i1}, A_{j1}] - \frac{\partial A_{i1}}{\partial p_j} + \frac{\partial A_{j1}}{\partial p_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.24}$$

$$[A_{i1}, A_{j1}] + \frac{\partial A_{i0}}{\partial x_j} - \frac{\partial A_{j0}}{\partial x_i} = 0, \quad 1 \leqslant i,\ j \leqslant n, \tag{2.25}$$

$$\frac{\partial A_{i1}}{\partial x_j} - \frac{\partial A_{j1}}{\partial x_i} = 0, \quad 1 \leqslant i,\ j \leqslant n. \tag{2.26}$$

The systems in Examples 4 and 5 are new to our knowledge. Obviously each system of Examples 3–5, is a generalization of the Takasaki system of (2.9), (2.10), since all corresponding reductions of potentials $A_{i1} = 0$, $1 \leqslant i \leqslant n$, lead to the Takasaki system.

EXAMPLE 6. Let $M \geqslant 2, n = 2, a_1(\lambda) = -\lambda, a_2(\lambda) = \lambda^M, A_1(\lambda) = A_{10}+A_{11}\lambda$. We obtain all equations but the SDYM equations (Example 2), in the SDYM integrable hierarchy [17]

$$[A_{10}, A_{20}] - \frac{\partial A_{10}}{\partial p_2} + \frac{\partial A_{20}}{\partial p_1} = 0, \tag{2.27}$$

$$[A_{10}, A_{21}] + [A_{11}, A_{20}] - \frac{\partial A_{11}}{\partial p_2} + \frac{\partial A_{21}}{\partial p_1} + \frac{\partial A_{20}}{\partial x_1} = 0, \tag{2.28}$$

$$[A_{10}, A_{2m}] + [A_{11}, A_{2,m-1}] + \frac{\partial A_{2m}}{\partial p_1} + \frac{\partial A_{2,m-1}}{\partial x_1} = 0,$$
$$2 \leqslant m \leqslant M-1, \tag{2.29}$$

$$[A_{10}, A_{2M}] + [A_{11}, A_{2,M-1}] + \frac{\partial A_{2M}}{\partial p_1} + \frac{\partial A_{2,M-1}}{\partial x_1} + \frac{\partial A_{10}}{\partial x_2} = 0, \tag{2.30}$$

$$[A_{11}, A_{2M}] + \frac{\partial A_{11}}{\partial x_2} + \frac{\partial A_{2M}}{\partial x_1} = 0. \tag{2.31}$$

Note that when $M = 2$, Equation (2.29) doesn't appear. Recursion operators, symmetries and conservation laws associated with this hierarchy have been considered by Ablowitz *et al.* [8]. In fact, it follows from (2.29), that we have a recursion relation

$$A_{2m} = -\left(\mathrm{ad}_{A_{10}} + \frac{\partial}{\partial p_1}\right)^{-1}\left(\mathrm{ad}_{A_{11}} + \frac{\partial}{\partial x_1}\right) A_{2,m-1} = RA_{2,m-1},$$

where R is exactly a recursion operator of the SDYM equations. The above four-dimensional SDYM hierarchy can also be considered as a 'universal' integrable hierarchy. Upon appropriate reduction and suitable choice of gauge group, it can produce virtually all well-known hierarchies of soliton equations in $1+1$ or $1+2$ dimensions [8].

3. Darboux Transformations

We recall the spectral problems (2.2)

$$L_i(\lambda)\Psi = \left(\frac{\partial}{\partial p_i} - a_i(\lambda)\frac{\partial}{\partial x_i}\right)\Psi = -A_i(\lambda)\Psi, \quad 1 \leqslant i \leqslant n.$$

The Darboux transformation problem means we need to obtain a new set of

$$\tilde{\Psi},\ \tilde{A}_i(\lambda) = \sum_{l=0}^{M} \tilde{A}_{il}\lambda^l, \quad 1 \leqslant i \leqslant n, \tag{3.1}$$

from an old set of

$$\Psi, A_i(\lambda) = \sum_{l=0}^{M} A_{il}\lambda^l, \quad 1 \leqslant i \leqslant n,$$

so that the spectral problems (2.2) hold covariantly. The purpose of this section is just to give an answer to this problem. We shall prove that for certain α to be determined,

$$\tilde{\Psi} = (\lambda I + \alpha)\Psi, \; I = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{N}) \tag{3.2}$$

will give a practicable new matrix of eigenfunctions. Of course, this matrix α has to satisfy some conditions.

For the time being, let us observe what kind of conditions they should be. Because we have

$$\begin{aligned} L_i(\lambda)\tilde{\Psi} &= L_i(\lambda)[(\lambda I + \alpha)\Psi] \\ &= (L_i(\lambda)\alpha)\Psi + (\lambda I + \alpha)L_i(\lambda)\Psi \\ &= (L_i(\lambda)\alpha)\Psi - (\lambda I + \alpha)A_i(\lambda)\Psi \\ &= \left(\frac{\partial \alpha}{\partial p_i} - \sum_{k=0}^{M} a_{ik}\lambda^{k+1}\frac{\partial \alpha}{\partial x_i}\right)\Psi - \\ &\quad -(\lambda I + \alpha)\left(\sum_{l=0}^{M} A_{il}\lambda^l\right)\Psi, \quad 1 \leqslant i \leqslant n, \end{aligned}$$

and

$$-\tilde{A}_i(\lambda)\tilde{\Psi} = -\left(\sum_{l=0}^{M} \tilde{A}_{il}\lambda^l\right)(\lambda I + \alpha)\Psi, \quad 1 \leqslant i \leqslant n;$$

a balance of the coefficients of the powers λ^0; λ^m, $1 \leqslant m \leqslant M$ and λ^{M+1} of $L_i(\lambda)\tilde{\Psi} = -\tilde{A}_i(\lambda)\tilde{\Psi}$, $1 \leqslant i \leqslant n$, yields that for $1 \leqslant i \leqslant n$,

$$\begin{aligned} &\frac{\partial \alpha}{\partial p_i} - \alpha A_{i0} = -\tilde{A}_{i0}\alpha, \\ &-a_{i,m-1}\frac{\partial \alpha}{\partial x_i} - A_{i,m-1} - \alpha A_{im} = -\tilde{A}_{i,m-1} - \tilde{A}_{im}\alpha, \quad 1 \leqslant m \leqslant M, \\ &-a_{iM}\frac{\partial \alpha}{\partial x_i} - A_{iM} = -\tilde{A}_{iM}. \end{aligned}$$

From these equalities, we get the condition for α

$$\frac{\partial \alpha}{\partial p_i} = \alpha A_{i0} - \tilde{A}_{i0}\alpha, \quad 1 \leqslant i \leqslant n, \tag{3.3}$$

and at the same time a recursion formula to determine $\tilde{A}_{im}$

$$\begin{aligned} \tilde{A}_{iM} &= A_{iM} + a_{iM}\frac{\partial\alpha}{\partial x_i}, \quad 1 \leqslant i \leqslant n, \\ \tilde{A}_{i,m-1} &= A_{i,m-1} + a_{i,m-1}\frac{\partial\alpha}{\partial x_i} + \alpha A_{im} - \tilde{A}_{im}\alpha, \\ &\qquad 1 \leqslant m \leqslant M,\ 1 \leqslant i \leqslant n. \end{aligned} \tag{3.4}$$

We observe (3.4) and (3.3) a little more. Notice that for $1 \leqslant r \leqslant M$, $1 \leqslant i \leqslant n$, we may make a further calculation

$$\begin{aligned} \tilde{A}_{i,r-1} &= A_{i,r-1} + a_{i,r-1}\frac{\partial\alpha}{\partial x_i} + \alpha A_{ir} + \tilde{A}_{ir}(-\alpha) \\ &= A_{i,r-1} + a_{i,r-1}\frac{\partial\alpha}{\partial x_i} + \alpha A_{ir} + \\ &\quad + \left(A_{ir} + a_{ir}\frac{\partial\alpha}{\partial x_i} + \alpha A_{i,r+1}\right)(-\alpha) + \tilde{A}_{i,r+1}(-\alpha)^2 \\ &\quad \cdots\cdots \\ &= \sum_{k=0}^{M-r}\left(A_{i,r+k-1} + a_{i,r+k-1}\frac{\partial\alpha}{\partial x_i} + \alpha A_{i,r+k}\right)(-\alpha)^k + \\ &\quad + \tilde{A}_{iM}(-\alpha)^{M-r+1}. \end{aligned} \tag{3.5}$$

Therefore (3.4) becomes

$$\begin{aligned} \tilde{A}_{i,r-1} &= \sum_{k=0}^{M-r+1}\left(A_{i,r+k-1} + a_{i,r+k-1}\frac{\partial\alpha}{\partial x_i}\right)(-\alpha)^k + \\ &\quad + \sum_{k=0}^{M-r}\alpha A_{i,r+k}(-\alpha)^k, \quad 1 \leqslant r \leqslant M, \quad 1 \leqslant i \leqslant n, \end{aligned} \tag{3.6}$$

$$\tilde{A}_{iM} = A_{iM} + a_{iM}\frac{\partial\alpha}{\partial x_i}, \quad 1 \leqslant i \leqslant n,$$

and further, the condition (3.3) reads as

$$\begin{aligned} \frac{\partial\alpha}{\partial p_i} &= \alpha A_{i0} - \tilde{A}_{i0}\alpha \\ &= \alpha A_{i0} + \sum_{k=0}^{M} A_{ik}(-\alpha)^{k+1} + \sum_{k=0}^{M} a_{ik}\frac{\partial\alpha}{\partial x_i}(-\alpha)^{k+1} + \end{aligned}$$

$$+\sum_{k=1}^{M}\alpha A_{ik}(-\alpha)^k$$

$$= \sum_{k=0}^{M} A_{ik}(-\alpha)^{k+1} + \sum_{k=0}^{M} a_{ik}\frac{\partial\alpha}{\partial x_i}(-\alpha)^{k+1} +$$

$$+\sum_{k=0}^{M}\alpha A_{ik}(-\alpha)^k, \quad 1 \leqslant i \leqslant n. \tag{3.7}$$

This is the final condition to restrict the matrix α, in the construction of new matrices of eigenfunctions according to (3.2). The following result can provide us with such a kind of useful matrices.

THEOREM 3.1. *Let h_s, $1 \leqslant s \leqslant N$, be N-dimensional column eigenvectors corresponding to the spectral parameters $\lambda_1, \lambda_2, \ldots, \lambda_N$, respectively, that is to say, that the N-dimensional column eigenvectors h_s, $1 \leqslant s \leqslant N$, satisfy*

$$\frac{\partial h_s}{\partial p_i} = a_i(\lambda_s)\frac{\partial h_s}{\partial x_i} - A_i(\lambda_s)h_s, \quad 1 \leqslant s \leqslant N,\ 1 \leqslant i \leqslant n. \tag{3.8}$$

Assume that the determinant of the matrix $H = [h_1, h_2, \ldots, h_N]$ is nonzero. Then the matrix defined by

$$\alpha = -H\Lambda H^{-1}, \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N), \tag{3.9}$$

satisfies the condition (3.7). *Therefore, we have a Darboux transformation $\tilde{\Psi} = (\lambda I + \alpha)\Psi$ and a new solution $\tilde{A}_{il}$, $1 \leqslant i \leqslant n$, $0 \leqslant l \leqslant M$, defined by* (3.6).

Proof. Noting that

$$\frac{\partial H^{-1}}{\partial y_i} = -H^{-1}\frac{\partial H}{\partial y_i}H^{-1}, \quad y_i = p_i \text{ or } x_i,\ 1 \leqslant i \leqslant n,$$

we have for any $1 \leqslant i \leqslant n$

$$\frac{\partial\alpha}{\partial p_i} = -\frac{\partial H}{\partial p_i}\Lambda H^{-1} + H\Lambda H^{-1}\frac{\partial H}{\partial p_i}H^{-1},$$

$$\frac{\partial\alpha}{\partial x_i} = -\frac{\partial H}{\partial x_i}\Lambda H^{-1} + H\Lambda H^{-1}\frac{\partial H}{\partial x_i}H^{-1}.$$

On the other hand, from (3.8) we obtain

$$\frac{\partial H}{\partial p_i} = \sum_{k=0}^{M} a_{ik}\frac{\partial H}{\partial x_i}\Lambda^{k+1} - [A_i(\lambda_1)h_1, \ldots, A_i(\lambda_N)h_N]$$

$$= \sum_{k=0}^{M} a_{ik}\frac{\partial H}{\partial x_i}\Lambda^{k+1} - \sum_{l=0}^{M} A_{il}H\Lambda^l, \quad 1 \leqslant i \leqslant n.$$

In this way, we can compute that

$$\begin{aligned}
\frac{\partial\alpha}{\partial p_i} &= -\frac{\partial H}{\partial p_i}\Lambda H^{-1} + H\Lambda H^{-1}\frac{\partial H}{\partial p_i}H^{-1} \\
&= -\sum_{k=0}^{M} a_{ik}\frac{\partial H}{\partial x_i}\Lambda^{k+2}H^{-1} + \sum_{l=0}^{M} A_{il}H\Lambda^{l+1}H^{-1} + \\
&\quad + \sum_{k=0}^{M} a_{ik}H\Lambda H^{-1}\frac{\partial H}{\partial x_i}\Lambda^{k+1}H^{-1} - H\Lambda H^{-1}\sum_{l=0}^{M} A_{il}H\Lambda^{l}H^{-1} \\
&= -\sum_{k=0}^{M} a_{ik}\frac{\partial H}{\partial x_i}H^{-1}(-\alpha)^{k+2} + \sum_{l=0}^{M} A_{il}(-\alpha)^{l+1} + \\
&\quad + \sum_{k=0}^{M} a_{ik}H\Lambda H^{-1}\frac{\partial H}{\partial x_i}H^{-1}\alpha^{k+1} + \alpha\sum_{l=0}^{M} A_{il}(-\alpha)^{l} \\
&= \sum_{k=0}^{M} a_{ik}\left(-\frac{\partial H}{\partial x_i}\Lambda H^{-1} + H\Lambda H^{-1}\frac{\partial H}{\partial x_i}H^{-1}\right)(-\alpha)^{k+1} + \\
&\quad + \sum_{l=0}^{M} A_{il}(-\alpha)^{l+1} + \alpha\sum_{l=0}^{M} A_{il}(-\alpha)^{l} \\
&= \sum_{k=0}^{M} a_k\frac{\partial\alpha}{\partial x_i}(-\alpha)^{k+1} + \sum_{l=0}^{M} A_{il}(-\alpha)^{l+1} + \alpha\sum_{l=0}^{M} A_{il}(-\alpha)^{l}.
\end{aligned}$$

This shows that the matrix α defined by (3.9) indeed satisfies the condition (3.7). The rest of the proof is evident. The proof is completed. □

We mention that we need a requirement that there are at least two different parameters in the set $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, while making Darboux transformation determined by Theorem 3.1. Otherwise we only get an original solution, i.e. not a new solution, since α becomes a unit matrix up to a constant factor. Equation (3.8) for h_s is linear and hence there is no problem in solving it. So Darboux transformations (3.2) can always engender new explicit solutions, once one solution is found. Furthermore, from the solution induced by Darboux transformation, we may make Darboux transformation once more and obtain another new solution. This process can be done continually and usually it may yield a series of multi-soliton solutions [6, 13].

4. Rational and Analytic Solutions

We make Darboux transformation starting from a special initial solution: zero solution $A_{il} = 0$, $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant M$. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be arbitrary parameters so that at least two of them are different. In this way, we obtain a new solution

$$\begin{aligned} \tilde{A}_{i,r-1} &= \sum_{k=0}^{M-r+1} a_{i,r+k-1} \frac{\partial \alpha}{\partial x_i} (-\alpha)^k, \quad 1 \leqslant r \leqslant M, \quad 1 \leqslant i \leqslant n, \\ \tilde{A}_{iM} &= a_{iM} \frac{\partial \alpha}{\partial x_i}, \quad 1 \leqslant i \leqslant n, \end{aligned} \tag{4.1}$$

where the matrix α is defined (3.9).

Evidently, it is crucial to construct an invertible matrix $H = [h_1, h_2, \ldots, h_N]$. We shall utilize two special determinants to generate the kind of matrices that we need. The one is the Vandermonde determinant formula

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_N \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{N-1} & a_2^{N-1} & \ldots & a_N^{N-1} \end{vmatrix} = \prod_{i>j} (a_i - a_j), \tag{4.2}$$

and the other is a generalized Cauchy determinant formula developed by Constantinescu [19]

$$\begin{aligned} &(-1)^{N(N-1)/2} \begin{vmatrix} \Delta_{11} & \ldots & \Delta_{1n_2} \\ \vdots & \ddots & \vdots \\ \Delta_{n_1 1} & \ldots & \Delta_{n_1 n_2} \end{vmatrix} \\ &\quad = \frac{\prod_{1 \leqslant i < j \leqslant N_1} (a_i - a_j)^{n_1^2} \prod_{1 \leqslant i < j \leqslant N_2} (b_i - b_j)^{n_2^2}}{\prod_{1 \leqslant i \leqslant N_1,\, 1 \leqslant j \leqslant N_2} (a_i - b_j)^{n_1 n_2}}, \end{aligned} \tag{4.3}$$

where $n_1 N_1 = n_2 N_2 = N$ and the matrices Δ_{kl}, $1 \leqslant k \leqslant n_1$, $1 \leqslant l \leqslant n_2$, are defined by

$$\Delta_{kl} = \binom{k+l-2}{k-1} \begin{bmatrix} \dfrac{1}{(a_1 - b_1)^{k+l-1}} & \ldots & \dfrac{1}{(a_1 - b_{N_2})^{k+l-1}} \\ \vdots & \ddots & \vdots \\ \dfrac{1}{(a_{N_1} - b_1)^{k+l-1}} & \ldots & \dfrac{1}{(a_{N_1} - b_{N_2})^{k+l-1}} \end{bmatrix}. \tag{4.4}$$

When $N_1 = N_2 = N$ (this moment $n_1 = n_2 = 1$), the above generalized Cauchy determinant formula is reduced to a Cauchy determinant formula.

Let $i_1, i_2, \dots, i_N$ be N integers and $n_1 N_1 = n_2 N_2 = N$. For simplicity, we accept

$$a(\lambda_i)\cdot p + x = (a_1(\lambda_i)p_1 + x_1, \dots, a_n(\lambda_i)p_n + x_n), \quad 1 \leqslant i \leqslant N. \tag{4.5}$$

We choose the first class of the matrices H as follows

$$H_1 = \begin{bmatrix} f_1^{i_1} & f_2^{i_2} & \cdots & f_N^{i_N} \\ f_1^{i_1+1} & f_2^{i_2+1} & \cdots & f_N^{i_N+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{i_1+N-1} & f_2^{i_2+N-1} & \cdots & f_N^{i_N+N-1} \end{bmatrix}, \tag{4.6}$$

with the functions $f_i = f_i(a(\lambda_i)\cdot p + x)$, $1 \leqslant i \leqslant N$. This is a little more general Vandermonde matrix and hence we have

$$\det H_1 = f_1^{i_1} f_2^{i_2} \dots f_N^{i_N} \prod_{i>j} (f_i - f_j). \tag{4.7}$$

We obtain $\alpha_1 = -H_1 \Lambda_1 H_1^{-1}$, where $\Lambda_1 = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$, provided that $\det H_1 \neq 0$. We choose the second class of the matrices H as follows

$$H_2 = \begin{bmatrix} \Delta_{11}(f,g,\mu) & \cdots & \Delta_{1n_2}(f,g,\mu) \\ \vdots & \ddots & \vdots \\ \Delta_{1n_1}(f,g,\mu) & \cdots & \Delta_{n_1 n_2}(f,g,\mu) \end{bmatrix}, \tag{4.8}$$

where $\Delta_{kl}(f,g,\mu)$, $1 \leqslant k \leqslant n_1$, $1 \leqslant l \leqslant n_2$, are given by

$$\Delta_{kl}(f,g,\mu) = \binom{k+l-2}{k-1} \begin{bmatrix} \dfrac{f_1^{i_{(l-1)N_2+1}}}{(\mu_1 - g_1)^{k+l-1}} & \cdots & \dfrac{f_{N_2}^{i_{lN_2}}}{(\mu_1 - g_{N_2})^{k+l-1}} \\ \vdots & \ddots & \vdots \\ \dfrac{f_1^{i_{(l-1)N_2+1}}}{(\mu_{N_1} - g_1)^{k+l-1}} & \cdots & \dfrac{f_{N_2}^{i_{lN_2}}}{(\mu_{N_1} - g_{N_2})^{k+l-1}} \end{bmatrix}, \tag{4.9}$$

with the constants μ_i, $1 \leqslant i \leqslant N_1$, and the functions

$$f_i = f_i(a(\lambda_i)\cdot p + x), \qquad g_i = g_i(a(\lambda_i)\cdot p + x), \quad 1 \leqslant i \leqslant N_2.$$

The determinant of this matrix H_2 can be computed by the generalized Cauchy determinant formula and eventually we obtain

$$\det H_2 = (-1)^{N(N-1)/2} f_1^{\Sigma_{l=1}^{n_2} i_{(l-1)N_2+1}} \cdots f_{N_2}^{\Sigma_{l=1}^{n_2} i_{lN_2}} \times \\ \times \frac{\prod_{1\leqslant i<j\leqslant N_1}(\mu_i-\mu_j)^{n_1^2}\prod_{1\leqslant i<j\leqslant N_2}(g_i-g_j)^{n_2^2}}{\prod_{1\leqslant i\leqslant N_1, 1\leqslant j\leqslant N_2}(\mu_i-g_j)^{n_1n_2}}. \tag{4.10}$$

In this way, we have $\alpha_2 = -H_2\Lambda_2H_2^{-1}$, where

$$\Lambda_2 = \mathrm{diag}(\lambda_1, \underbrace{\ldots, \lambda_{N_2}; \ldots; \lambda_1, \ldots}_{n_2}, \lambda_{N_2}),$$

provided that $\det H_2 \neq 0$.

In what follows, we restrict our consideration within the real field, but the case of the complex field is completely similar.

(1) *Rational function solutions*

(1.1) Let $f_1, \ldots, f_N\colon \mathbb{R}^n \to \mathbb{R}$ be nonzero distinct rational functions. At this moment, we have $f_i(a(\lambda_i)\cdot p+x) \not\equiv 0$, $1 \leqslant i \leqslant N$, and $f_i(a(\lambda_i)\cdot p+x) \not\equiv f_j(a(\lambda_j)\cdot p+x)$, $i \neq j$. Otherwise we have $f_i(x) \equiv 0$, $1 \leqslant i \leqslant N$, and $f_i(x) \equiv f_j(x)$, $i \neq j$, by setting $p = 0$, which contradicts the original hypothesis. It follows from (4.7) that in this case, $\det H_1$ is a nonzero rational function, with the independent variables p, $x \in \mathbb{R}^n$ and hence we can take a special matrix $\alpha = \alpha_1 = -H_1\Lambda_1H_1^{-1}$, whose elements are all rational functions of p, $x \in \mathbb{R}^n$. Further, we can obtain nonzero rational function solutions by (4.1).

(1.2) Let μ_i, $1 \leqslant i \leqslant N_1$, be distinct real numbers, $f_i\colon \mathbb{R}^n \to \mathbb{R}$, $1 \leqslant i \leqslant N_2$, be nonzero rational functions, and $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $1 \leqslant i \leqslant N_2$, be nonzero distinct rational functions, so that $\prod_{1\leqslant i\leqslant N_1, 1\leqslant j\leqslant N_2}(\mu_i - g_j) \not\equiv 0$. It follows from (4.10) that in this case, $\det H_2$ is a nonzero rational function with the independent variables p, $x \in \mathbb{R}^n$ and then we can choose a special matrix $\alpha = \alpha_2 = -H_2\Lambda_2H_2^{-1}$, whose elements are all rational functions of p, $x \in \mathbb{R}^n$. In this way, we can obtain a class of nonzero rational function solutions defined by (4.1).

(2) *Analytic function solutions*

(2.1) Let $\gamma_1, \ldots, \gamma_N$ be real numbers and h_i, $1 \leqslant i \leqslant N$, be any analytic functions to satisfy $|h_i(y)| \leqslant B$, $1 \leqslant i \leqslant N$, $y \in \mathbb{R}^n$. There are a lot of functions of this kind. For example, $\sin q(y)$, $\cos q(y)$, $\tanh q(y)$, $\mathrm{sech}\, q(y)$, where $q\colon \mathbb{R}^n \to \mathbb{R}$ is any analytic function. We choose

$$f_i = f_i(a(\lambda_i)\cdot p + x) = h_i(a(\lambda_i)\cdot p + x) + \gamma_i, \quad 1 \leqslant i \leqslant N.$$

At this moment, $\det H_1$ has no zero points with respect to $(p, x) \in \mathbb{R}^{2n}$ while

$$|\gamma_i| > B, \quad 1 \leqslant i \leqslant N, \quad |\gamma_i - \gamma_j| > 2B, \quad 1 \leqslant i < j \leqslant N. \tag{4.11}$$

Therefore under the condition (4.11), we may take a special matrix $\alpha = \alpha_1 = -H_1\Lambda_1 H_1^{-1}$, whose elements are all analytic functions of p, $x \in \mathbb{R}^n$. This can result in nonzero analytic function solutions by (4.1).

(2.2) Let μ_i, $1 \leqslant i \leqslant N_1$, be distinct real numbers, $f_i: \mathbb{R}^n \to \mathbb{R}$, $1 \leqslant i \leqslant N_2$, be nonnegative or nonpositive analytic functions, and $h_i: \mathbb{R}^n \to \mathbb{R}$, $1 \leqslant i \leqslant N_2$, be analytic functions to satisfy $|h_i(y)| \leqslant B$, $1 \leqslant i \leqslant N_2$. We choose

$$g_i = g_i(a(\lambda_i) \cdot p + x) = h_i(a(\lambda_i) \cdot p + x) + \gamma_i, \quad 1 \leqslant i \leqslant N_2,$$

where γ_i, $1 \leqslant i \leqslant N_2$, are all real numbers, so that

$$|\gamma_i - \gamma_j| > 2B, \quad 1 \leqslant i < j \leqslant N_2,$$

$$|\gamma_i| > B + \max_{1 \leqslant i \leqslant N_1} |\mu_i|, \quad 1 \leqslant i \leqslant N_2.$$

In this way, $\det H_2$ has no zero points with respect to $(p, x) \in \mathbb{R}^{2n}$. Therefore, we can take a special matrix $\alpha = \alpha_2 = -H_2\Lambda_2 H_2^{-1}$, whose elements are all analytic functions of p, $x \in \mathbb{R}^n$. This may yield a class of nonzero analytic function solutions, by means of (4.1).

5. Discussions and Remarks

From the mathematical point of view, it is very interesting to generate more general integrable systems. Such an example may be introduced for the Lax integrable system in Section 2. We display that more general system here

$$\sum_{\substack{k+l=m \\ -M \leqslant k,l \leqslant M}} [A_{ik}, A_{jl}] + \sum_{\substack{k+l=m \\ -M \leqslant k,l \leqslant M}} \left(a_{jk}\frac{\partial A_{il}}{\partial x_j} - a_{ik}\frac{\partial A_{jl}}{\partial x_i} \right) = 0,$$

$$-2M \leqslant m \leqslant -M-1, \quad M+1 \leqslant m \leqslant 2M,$$

$$\sum_{\substack{k+l=m \\ -M \leqslant k,l \leqslant M}} [A_{ik}, A_{jl}] - \frac{\partial A_{im}}{\partial p_j} + \frac{\partial A_{jm}}{\partial p_i} +$$

$$+ \sum_{\substack{k+l=m \\ -M \leqslant k,l \leqslant M}} \left(a_{jk}\frac{\partial A_{il}}{\partial x_j} - a_{ik}\frac{\partial A_{jl}}{\partial x_i} \right) = 0, \quad -M \leqslant m \leqslant M,$$

where $1 \leqslant i, j \leqslant n$. It is in agreement with the compatibility conditions of the following spectral problems, with negative powers of the spectral parameter

$$\left[\left(\frac{\partial}{\partial p_i} - \left(\sum_{k=-M}^{M} a_{ik}\lambda^k \right) \frac{\partial}{\partial x_i} \right) \right] \Psi$$

$$= - \left(\sum_{l=-M}^{M} A_{il}\lambda^l \right) \Psi, \quad 1 \leqslant i \leqslant n,$$

where $M \geqslant 0$. A more general Lax set of spectral problems

$$\left[\left(\frac{\partial}{\partial p_i}-\left(\sum_{k=-M'}^{M''} a_{ik}\lambda^k\right)\frac{\partial}{\partial x_i}\right)\right]\Psi = -\left(\sum_{l=-M'}^{M''} A_{il}\lambda^l\right)\Psi, \quad 1 \leqslant i \leqslant n,$$

where M', $M'' \geqslant 0$, may arise as a reduction of this Lax set, with $M = \max(M', M'')$ and vice versa. The Lax integrable system displayed above includes the generalized self-dual Yang–Mills flows in [20]. It seems more reasonable that it is considered as a 'universal' integrable system, whereas it is too general to lose some concrete characteristics. It would be valuable to know whether its more reductions have physical interpretations. We may also construct a class of generalized chiral field equations similar to the one in [21]. Essentially, the deduction is the same but we should note particular properties.

We remark that there has been a huge class of higher-dimensional nonlinear integrable equations, which can be solved through the nonlocal Riemann–Hilbert method proposed by Zakharov and Manakov [22]. These kinds of equations are connected with the Lax representations defined by higher-order differential operators, with respect to some independent variables. However, our initial Lax integrable system (2.5–2.8), is derived by the spectral problems involving higher-order powers of the parameter λ, other than higher-order differential operators. Therefore, there isn't a direct relation between two classes of resulting integrable equations, although there exist the same reductions of them, for instance, KP equation. On the other hand, our explicit solutions presented in Section 4, are quite broad because many arbitrary functions are involved in the construction of solutions and thus it is difficult to discuss their properties. But we should be able to study some important properties of sub-classes of the obtained solutions, for example, localization property, the existence of the instanton like solutions and the soliton interactions, etc., as in the work by Zhou [23]. In particular, some careful consideration about specific reductions deserves further investigation.

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