IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 165202 (13pp) doi:10.1088/1751-8113/43/16/165202

An extended Harry Dym hierarchy

Wen-Xiu Ma

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
and
College of Information Science and Technology, Shandong University of Science and Technology, Shandong 266510, People's Republic of China

E-mail: mawx@math.usf.edu

Received 22 October 2009, in final form 4 March 2010
Published 29 March 2010
Online at stacks.iop.org/JPhysA/43/165202

Abstract
An extended Harry Dym hierarchy is constructed by using eigenfunctions and adjoint eigenfunctions of the spectral problems of the Harry Dym hierarchy associated with the pseudo-differential operator $L = u\partial + u_0 + u_1\partial^{-1} + \cdots$. The corresponding Lax presentation possesses a self-consistent source involving squared eigenfunctions. The resulting extended Harry Dym hierarchy is reduced to the Harry Dym hierarchy with self-consistent sources under the n-reduction, $L^n = (L^n)_{\geqslant 2}$, and the k-constrained Harry Dym hierarchy under the k-constraint, $L^k = (L^k)_{\geqslant 2} + \sum_{i=1}^{N} q_i\partial^{-1}r_i\partial^2$. A few particular examples are computed, together with their Lax pairs.

PACS numbers: 02.30.Ik, 02.30.Jr
Mathematics Subject Classification: 37K05, 35Q53, 37K10, 35Q58

1. Introduction

Soliton equations with self-consistent sources, source soliton equations and constrained soliton equations extend integrable equations and arise in many fields of applied science such as mathematical physics, hydrodynamics, solid state physics, nonlinear optics and plasma physics [1–9]. The KP and mKP theories under symmetry constraints present the KdV equation, the KP equation, the NLS equation, the Boussinesq equation and the Davey–Stewartson equation with self-consistent sources as examples. There are different techniques to solve those nonlinear systems of equations (see, e.g., [2], [10–13]) and complexiton solutions besides soliton solutions can exist [14]. A key point in the study is to use non-Lie symmetries to separate, constrain or extend soliton equations within the standard Lax formulation (see, e.g., [15, 16]).

Recently, an extended KP hierarchy and an extended q-deformed KP hierarchy were generated through the dressing operator and the corresponding wavefunction [17, 18], and a generalized dressing approach for solving the extended KP hierarchy and the extended mKP hierarchy was proposed and Wronskian solutions were constructed [19]. This unifies soliton equations with self-consistent sources and constrained soliton equations, and extends integrable equations further in a quite systematic way. In particular, under the 2- and 3-reduction, the extended KP hierarchy is reduced to the KdV equations with self-consistent sources and the Boussinesq equations with self-consistent sources [17].

In this paper, motivated by the idea of using eigenfunctions and adjoint eigenfunctions in binary nonlinearization [20] and the extended KP hierarchy [17], we would like to extend the Harry Dym hierarchy of soliton equations, by directly introducing a sequence of Lax equations with new time variables τ_k. The Harry Dym hierarchy is the third hierarchy of soliton equations presented by the pseudo-differential operator technique, coming up after the first and second hierarchies—the KP and mKP hierarchies, respectively [21]. The resulting extended Harry Dym hierarchy will therefore provide us with another large class of integrable equations possessing many interesting good mathematical properties.

Let $\partial = \frac{\partial}{\partial x}$ be a differential operator. We consider the algebra g of pseudo-differential operators of the form

$$P = \sum_{i=-n}^{\infty} a_i \partial^{-i} = a_{-n}\partial^n + a_{-n+1}\partial^{n-1} + \cdots + a_0 + a_1\partial^{-1} + \cdots, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $a_i,\ i \geqslant -n$, are functions of x. Applying the Leibniz rule, an obvious induction allows us to prove a general Leibniz rule for commuting multiplication and differentiation operators:

$$\partial^m f = f\partial^m + \sum_{i=1}^{\infty} C_m^i f^{(i)}\partial^{m-i}, \qquad C_m^i = \frac{m(m-1)\cdots(m-i+1)}{i!}, \quad m \in \mathbb{Z}, \tag{1.2}$$

where $f^{(i)} = \partial^i f,\ i \geqslant 0$. We treat ∂^{-1} as the inverse of ∂ so that $\partial^{-1}\partial = \partial\partial^{-1} = 1$. The case of $m = 1$ of (1.2) gives

$$\partial^{-1} f = \sum_{i=0}^{\infty} (-1)^i f^{(i)}\partial^{-i-1}.$$

The adjoint pseudo-differential operator P^* of $P \in g$ is always helpful and defined by

$$P^* = \sum_{i=-n}^{\infty} (-1)^i \partial^{-i} a_i. \tag{1.3}$$

To formulate the Harry Dym hierarchy successfully, we will use the following decomposition of the algebra g:

$$P = P_+ + P_-,\ P_+ = P_{\geqslant 2} = \sum_{i=-n}^{-2} a_i\partial^{-i}, \qquad P_- = P_{\leqslant 1} = \sum_{i=-1}^{\infty} a_i\partial^{-i}. \tag{1.4}$$

Note that if $n < 2$, we have $P_+ = 0$. This decomposition gives rise to an r-matrix structure on the algebra g, which leads to Hamiltonian structures of the associated Harry Dym equations [21].

The Harry Dym hierarchy is associated with the following pseudo-differential operator:

$$L = u\partial + u_0 + u_1\partial^{-1} + \cdots, \tag{1.5}$$

and it is defined by the following Lax equations [21]:

$$L_{t_n} = [B_n, L], \quad n \geqslant 1, \tag{1.6}$$

where the differential operators B_n are taken as

$$B_n = L_+^n = (L^n)_+ = (L^n)_{\geqslant 2}, \quad n \geqslant 1. \tag{1.7}$$

This Harry Dym hierarchy belongs to a class of non-standard integrable equations [22]. The equations resulting from $L_y = [B_2, L]$ (setting $y = t_2$) are used to express all fields in terms of u and its derivatives. In particular, the equation defining the auxiliary field u_0 is as follows:

$$u_y = u^2 u_{xx} + 2u^2 u_{0,x}.$$

The Lax equations for $n \geqslant 3$ represent other equations in t_n, x and y for the field u. For example, the equation associated with $n = 3$ is

$$4u_{t_3} = u^3 u_{xxx} + \tfrac{1}{2}[u^2 \partial^{-1}(u_y/u^2)]_y, \tag{1.8}$$

which is reduced to the Harry Dym equation

$$4u_{t_3} = u^3 u_{xxx}, \tag{1.9}$$

upon dropping the y-dependence [23]. This Harry Dym equation is known to be integrable but does not pass Painlevé test. To compute the above (2+1)-dimensional Harry Dym equation, one needs

$$B_2 = u^2\partial^2, \qquad B_3 = u^3\partial^3 + 3u^2(u_x + u_0)\partial^2. \tag{1.10}$$

In this paper, we will introduce a sequence of Lax equations with new time variables τ_k and use eigenfunctions and adjoint eigenfunctions to introduce an extended Harry Dym hierarchy which amends the Harry Dym hierarchy. To our best knowledge, the resulting extended Harry Dym hierarchy is presented and analyzed here for the first time. The extended Harry Dym hierarchy has Lax presentations with Lax pairs involving extended Lax operators. It is pretty difficult for us to compute examples of the extended Harry Dym hierarchy, but we succeed in presenting a few examples, thanks to the role that a kind of non-standard adjoint operators play. We will also illustrate the extended Harry Dym hierarchy by the Harry Dym hierarchy with self-consistent sources and the k-constrained Harry Dym hierarchy. A few particular examples will be listed, together with their Lax pairs, and conclusions and remarks will be given in the final section.

2. Extending the Harry Dym hierarchy

Let L and B_n be defined by (1.5) and (1.7), respectively. We introduce a set of N pairs of new functions, q_i and r_i, $1 \leqslant i \leqslant N$, by

$$q_{i,t_n} = B_n q_i, \qquad r_{i,t_n} = -\bar{B}_n r_i, \quad 1 \leqslant i \leqslant N, \tag{2.1}$$

where $\bar{B}_n$ is the non-standard adjoint operator:

$$\bar{B}_n = \partial^{-2} B_n^* \partial^2. \tag{2.2}$$

By $\bar{L}$, we denote the non-standard adjoint operator of L: $\bar{L} = \partial^{-2}L^*\partial^2$. We will see that this kind of adjoint operators fit well with the Lax equations associated with the pseudo-differential operator (1.5). Let us further define new pseudo-differential operators

$$\tilde{B}_k = B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2, \quad k \geqslant 1. \tag{2.3}$$

If the functions q_i's and r_i's also satisfy

$$Lq_i = \lambda_i q_i, \qquad \bar{L} r_i = \lambda_i r_i, \quad 1 \leqslant i \leqslant N,$$

then they are really eigenfunctions and adjoint eigenfunctions of the spectral problems:

$$L\phi = \lambda\phi, \qquad \phi_{t_n} = B_n\phi,$$

and

$$\bar{L}\phi = \lambda\phi, \qquad \phi_{t_n} = -\bar{B}_n\phi,$$

each of which yields the Lax equations (1.6). Now we introduce a sequence of τ_k-flows by

$$L_{\tau_k} = [\tilde{B}_k, L], \quad k \geqslant 1. \tag{2.4}$$

In view of the specific form (2.3), the τ_k-flows are well defined. To make the τ_k-flows compatible with the t_n-flows by (1.6), we need to compute their compatibility conditions.

Proposition 2.1. *Let B_n and $\bar{B}_n$ be defined by (1.7) and (2.2), respectively. Then*

$$[B_n, q\partial^{-1}r\partial^2]_- = (B_nq)\partial^{-1}r\partial^2 - q\partial^{-1}(\bar{B}_nr)\partial^2, \tag{2.5}$$

where $P_- = P_{\leqslant 1}$ is defined as in (1.4).

Proof. Without loss of generality, we restrict ourselves to a monomial $P = a\partial^m$, where $m \geqslant 2$. Clearly, we can have

$$[P, q\partial^{-1}r\partial^2]_- = aq^{(m)}\partial^{-1}r\partial^2 - (q\partial^{-1}r\partial^2a\partial^m)_-.$$

The second term above can be computed as follows:

$$\begin{aligned}
(q\partial^{-1}r\partial^2a\partial^m)_- &= (q\partial^{-1}(\partial r - r_x)\partial a\partial^m)_- \\
&= -(q\partial^{-1}r_x\partial a\partial^m)_- \\
&= -[q\partial^{-1}(\partial r_x - r_{xx})a\partial^m]_- \\
&= (q\partial^{-1}r_{xx}a\partial^m)_- \\
&= -[q\partial^{-1}(r_{xx}a)_x\partial^{m-1}]_- \\
&= \cdots\cdots \\
&= (-1)^{m-2}[q\partial^{-1}(r_{xx}a)^{(m-2)}\partial^2]_- \\
&= (-1)^{m-2}q\partial^{-1}(r_{xx}a)^{(m-2)}\partial^2 \\
&= q\partial^{-1}(\bar{P}r)\partial^2,
\end{aligned}$$

where $\bar{P} = \partial^{-2}P^*\partial^2 = (-1)^m\partial^{m-2}a\partial^2$ was used. Thus, the proposition is proved. □

We actually proved more than equality (2.5) itself. The operator B_n in (2.5) can be replaced with any differential operator with the least degree greater than 2.

Theorem 2.1. *Let $n, k \geqslant 1$, and B_n and $\tilde{B}_k$ be defined by (1.7) and (2.3), respectively. Assume that a set of pairs of functions q_i, r_i, $1 \leqslant i \leqslant N$, are determined by (2.1). Then two flows defined by the Lax equations*

$$L_{t_n} = [B_n, L], \qquad L_{\tau_k} = [\tilde{B}_k, L] \tag{2.6}$$

commute if and only if the pseudo-differential zero curvature equation

$$B_{n,\tau_k} - \tilde{B}_{k,t_n} + [B_n, \tilde{B}_k] = 0 \tag{2.7}$$

holds.

Proof. Obviously, the zero curvature equation (2.7) implies that

$$[B_{n,\tau_k} - \tilde{B}_{k,t_n} + [B_n, \tilde{B}_k], L] = 0.$$

This tells us that $L_{t_n,\tau_k} = L_{\tau_k,t_n}$, and thus, two Lax equations in (2.6) are compatible with each other.

In what follows, we show that the zero curvature equation (2.7) is necessary to guarantee the compatibility of the Lax equations in (2.6).

For brevity, we only focus on the case of $N = 1$, and denote q_1 and r_1 by q and r, respectively. Based on the Lax equations in (2.6), and noting that

$$\left[q\partial^{-1}r\partial^2, L_-^n\right]_+ = 0,$$

we can compute that

$$\begin{aligned} B_{k,t_n} &= \left(L^k_{t_n}\right)_+ = [B_n, L^k]_+ \\ &= \left[L^k, L^n_-\right]_+ = \left[B_k, L^n_-\right]_+, \end{aligned} \tag{2.8}$$

$$\begin{aligned} B_{n,\tau_k} &= \left(L^n_{\tau_k}\right)_+ \\ &= [B_k + q\partial^{-1}r\partial^2, L^n]_+ \\ &= \left[B_k + q\partial^{-1}r\partial^2, L^n_+\right]_+ + \left[B_k + q\partial^{-1}r\partial^2, L^n_-\right]_+ \\ &= \left[B_k + q\partial^{-1}r\partial^2, L^n_+\right] - \left[B_k + q\partial^{-1}r\partial^2, L^n_+\right]_- + \left[B_k, L^n_-\right]_+ \\ &= [B_k + q\partial^{-1}r\partial^2, B_n] - [q\partial^{-1}r\partial^2, B_n]_- + \left[B_k, L^n_-\right]_+ \\ &= [B_k + q\partial^{-1}r\partial^2, B_n] + (B_nq)\partial^{-1}r\partial^2 - q\partial^{-1}(\bar{B}_nr)\partial^2 + B_{k,t_n} \\ &= [B_k + q\partial^{-1}r\partial^2, B_n] + (B_k + q\partial^{-1}r\partial^2)_{t_n}, \end{aligned} \tag{2.9}$$

the last two equalities of which used proposition 2.1, (2.8) and (2.1). This completes the proof of the theorem. □

We remark that the commutativity of the t_n-flow by (1.6) and the τ_k-flow by (2.4) means that one defines a symmetry of the other. Under the constraint of (2.1), this commutativity equivalently requires the pseudo-differential zero curvature equation (2.7), which guarantees that by (2.4), we can define a sequence of the τ_k-flows compatible with the t_n-flows. Thus, equation (2.7) brings us to a hierarchy of important nonlinear equations extending the Harry Dym hierarchy (1.6).

Definition 2.1. *The extended Harry Dym hierarchy consists of all systems of equations*

$$B_{n,\tau_k} - \left(B_k + \sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right)_{t_n} + \left[B_n, B_k + \sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right] = 0, \tag{2.10a}$$

$$q_{i,t_n} = B_nq_i, \qquad r_{i,t_n} = -\bar{B}_nr_i, \quad 1 \leqslant i \leqslant N, \tag{2.10b}$$

where $n, k, N \geqslant 1$.

Equivalently, we can replace (2.10*a*) with (2.6), thanks to theorem 2.1. Under (2.10*b*), a Lax representation for (2.10*a*) is given by

$$\psi_{t_n} = B_n\psi, \qquad \psi_{\tau_k} = \left(B_k + \sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right)\psi, \tag{2.11}$$

and by proposition 2.1, the zero curvature equation (2.10*a*) is equivalent to

$$B_{n,\tau_k} - B_{k,t_n} + [B_n, B_k] + \left[B_n, \sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right]_+ = 0, \tag{2.12}$$

which is a differential operator equation. This is like the triple L–A–B representation of integrable equations, which has beautiful algebraic structures [24]. Taking $q_i = 0, 1 \leqslant i \leqslant N$, or $r_i = 0, 1 \leqslant i \leqslant N$, (2.10$a$) reduces to the Harry Dym hierarchy.

In order to compute examples of the extended Harry Dym hierarchy (2.10), we list some useful formulas in the following proposition.

Proposition 2.2. *The following equalities hold:*

$$[a\partial^2, q\partial^{-1}r\partial^2]_+ = [2a(qr)_x - a_x qr]\partial^2, \tag{2.13}$$

$$[a\partial^3, q\partial^{-1}r\partial^2]_+ = 3a(q_{xx}r + q_x r_x)\partial^2 + [3a(qr)_x - a_x qr]\partial^3, \tag{2.14}$$

where $P_+ = P_{\geqslant 2}$ is defined as in (1.4).

The proof of this proposition is just a direct computation, and the proposition itself shows the non-symmetric feature of the eigenfunctions q_i and the adjoint eigenfunctions r_i. We are now going to present two examples of the extended Harry Dym hierarchy (2.10).

Example 2.1. We consider one case: $n = 2$ and $k = 3$, and set $t_2 = y,\ \tau_3 = t$ and $v = u_0$. Then the extended Harry Dym system of equations (2.10) becomes

$$B_{2,t} - B_{3,y} + [B_2, B_3] + \left[B_2, \sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right]_+ = 0 \tag{2.15a}$$

$$q_{i,y} = B_2 q_i, \qquad r_{i,y} = -\bar{B}_2 r_i, \quad 1 \leqslant i \leqslant N. \tag{2.15b}$$

Based on proposition 2.2, the above system reads

$$\begin{aligned}2uu_t - 3[u^2(u_x + v)]_y + u^3(uu_{xxx} + 6u_x u_{xx} + 3uv_{xx} + 12u_x v_x)\\ + 2u^2\sum_{i=1}^N (q_i r_i)_x - 2uu_x \sum_{i=1}^N q_i r_i = 0,\end{aligned} \tag{2.16a}$$

$$u_y - u^2(u_{xx} + 2v_x) = 0, \tag{2.16b}$$

$$q_{i,y} = u^2 q_{i,xx}, \qquad r_{i,y} = -u^2 r_{i,xx}, \quad 1 \leqslant i \leqslant N. \tag{2.16c}$$

Under (2.16c), this (2+1)-dimensional system has the Lax representation

$$\psi_y = u^2\partial^2\psi,\ \psi_t = \left[u^3\partial^3 + 3u^2(u_x + v)\partial^2 + \sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right]\psi.$$

Example 2.2. We consider another case: $n = 3$ and $k = 2$, and set $t_3 = t,\ \tau_2 = y$ and $v = u_0$. Now the extended Harry Dym system of equations (2.10) becomes

$$B_{3,y} - B_{2,t} + [B_3, B_2] + \left[B_3, \sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right]_+ = 0 \tag{2.17a}$$

$$q_{i,t} = B_3 q_i, \qquad r_{i,t} = -\bar{B}_3 r_i, \quad 1 \leqslant i \leqslant N. \tag{2.17b}$$

Based on proposition 2.2, the above system reads

$$3[u^2(u_x+v)]_y - 2uu_t - u^3(uu_{xxx} + 6u_xu_{xx} + 3uv_{xx} + 12u_xv_x) + 6u^2(u_x+v)\sum_{i=1}^N (q_ir_i)_x$$
$$- 3[u^2(u_x+v)]_x \sum_{i=1}^N q_ir_i + 3u^3 \sum_{i=1}^N (q_{i,xx}r_i + q_{i,x}r_{i,x}) = 0, \qquad (2.18a)$$

$$u_y - u^2(u_{xx}+2v_x) + u\sum_{i=1}^N (q_ir_i)_x - u_x \sum_{i=1}^N q_ir_i = 0, \qquad (2.18b)$$

$$q_{i,t} = u^3 q_{i,xxx} + 3u^2(u_x+v)q_{i,xx}, \quad 1 \leqslant i \leqslant N, \qquad (2.18c)$$

$$r_{i,t} = u^3 r_{i,xxx} + 3u^2u_xr_{i,xx} - 3u^2(u_x+v)r_{i,xx}, \quad 1 \leqslant i \leqslant N. \qquad (2.18d)$$

Under (2.18d), this (2+1)-dimensional system has the Lax representation

$$\psi_t = [u^3\partial^3 + 3u^2(u_x+v)\partial^2]\psi, \qquad \psi_y = \left(u^2\partial^2 + \sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right)\psi.$$

3. Reductions of the extended Harry Dym hierarchy

3.1. The n-reduction

For each $n \geqslant 1$, the n-reduction is defined by

$$L^n = B_n, \qquad \text{i.e. } L^n_- = (L^n)_- = 0. \qquad (3.1)$$

Moreover, the eigenfunctions and adjoint eigenfunctions are required to satisfy

$$B_nq_i = \lambda_{n,i}q_i, \qquad \bar{B}_nr_i = \lambda_{n,i}r_i, \quad 1 \leqslant i \leqslant N, \qquad (3.2)$$

where $\lambda_{n,i}$, $1 \leqslant i \leqslant N$, are constants. This can be achieved if

$$Lq_i = \lambda_iq_i, \qquad \bar{L}r_i = \lambda_ir_i, \quad 1 \leqslant i \leqslant N, \qquad (3.3)$$

which exactly tells us that all q_i's and r_i's are eigenfunctions and adjoin eigenfunctions of the corresponding spectral problems. Further based on proposition 2.1, we see that the submanifold determined by the n-reduction (3.1) is invariant under the τ_k-flow:

$$\begin{aligned}
\left(L^n_-\right)_{\tau_k} &= [B_k, L^n]_- + \sum_{i=1}^N [q_i\partial^{-1}r_i\partial^2, L^n]_- \\
&= [B_k, L^n_-]_- + \sum_{i=1}^N [q_i\partial^{-1}r_i\partial^2, L^n_+]_- + \sum_{i=1}^N [q_i\partial^{-1}r_i\partial^2, L^n_-]_- \\
&= \sum_{i=1}^N [q_i\partial^{-1}r_i\partial^2, B_n]_- \\
&= -\sum_{i=1}^N [(B_nq_i)\partial^{-1}r_i\partial^2 - q_i\partial^{-1}(\bar{B}_nr_i)\partial^2] \\
&= -\sum_{i=1}^N [(\lambda_{n,i}q_i)\partial^{-1}r_i\partial^2 - q_i\partial^{-1}(\lambda_{n,i}r_i)\partial^2] = 0.
\end{aligned}$$

Therefore, when constraining the extended Harry Dym flows on this invariant sub-manifold, we can drop the t_n dependence from (2.10*a*) and reduce the extended Harry Dym hierarchy (2.10) to the following hierarchy:

$$\begin{aligned} B_{n,\tau_k} &= \left[(B_n)_+^{\frac{k}{n}} + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2, B_n \right] \\ &= \left[(B_n)_+^{\frac{k}{n}}, B_n \right] + \left[\sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2, B_n \right]_+, \end{aligned} \tag{3.4a}$$

$$B_n q_i = \lambda_{n,i} q_i, \qquad \bar{B}_n r_i = \lambda_{n,i} r_i, \qquad 1 \leqslant i \leqslant N. \tag{3.4b}$$

This is called the Harry Dym hierarchy with self-consistent sources. It has the Lax representation

$$B_n \psi = \mu \psi, \qquad \psi_{\tau_k} = \left[(B_n)_+^{\frac{k}{n}} + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2 \right] \psi, \tag{3.5}$$

where μ is the spectral parameter.

Example 3.1. Let us first take $n = 2$ and $k = 3$, and set $\tau_3 = t$ and $v = u_0$. Then the corresponding n-reduced Harry Dym system of equations (3.4) becomes

$$B_{2,t} = [B_3, B_2] + \left[\sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2, B_2 \right]_+ \tag{3.6a}$$

$$B_2 q_i = \lambda_{2,i} q_i, \qquad \bar{B}_2 r_i = \lambda_{2,i} r_i, \qquad 1 \leqslant i \leqslant N. \tag{3.6b}$$

Based on proposition 2.2, the above reduced system reads

$$2u_t = -u^2 (u u_{xxx} + 6 u_x u_{xx} + 3 u v_{xx} + 12 u_x v_x) - 2u \sum_{i=1}^{N} (q_i r_i)_x + 2u_x \sum_{i=1}^{N} q_i r_i, \tag{3.7a}$$

$$u^2 q_{i,xx} = \lambda_{2,i} q_i, \qquad u^2 r_{i,xx} = \lambda_{2,i} r_i, \quad 1 \leqslant i \leqslant N, \tag{3.7b}$$

which has the Lax representation

$$u^2 \partial^2 \psi = \mu \psi, \quad \psi_t = \left[u^3 \partial^3 + 3u^2 (u_x + v) \partial^2 + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2 \right] \psi.$$

Let us next take $n = 3$ and $k = 2$, and set $\tau_2 = y$ and $v = u_0$. Then the corresponding n-reduced Harry Dym system of equations (3.4) becomes

$$B_{3,y} = [B_2, B_3] + \left[\sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2, B_3 \right]_+ \tag{3.8a}$$

$$B_3 q_i = \lambda_{3,i} q_i, \qquad \bar{B}_3 r_i = \lambda_{3,i} r_i, \quad 1 \leqslant i \leqslant N. \tag{3.8b}$$

Based on proposition 2.2, the above reduced system reads

$$3[u^2(u_x+v)]_y = u^3(uu_{xxx}+6u_xu_{xx}+3uv_{xx}+12u_xv_x) - 6u^2(u_x+v)\sum_{i=1}^N (q_ir_i)_x$$
$$+3[u^2(u_x+v)]_x\sum_{i=1}^N q_ir_i - 3u^3\sum_{i=1}^N(q_{i,xx}r_i+q_{i,x}r_{i,x}), \tag{3.9a}$$

$$u_y = u^2(u_{xx}+2v_x) - u\sum_{i=1}^N (q_ir_i)_x + u_x\sum_{i=1}^N q_ir_i, \tag{3.9b}$$

$$u^3q_{i,xxx}+3u^2(u_x+v)q_{i,xx} = \lambda_{3,i}q_i, \quad 1\leqslant i\leqslant N, \tag{3.9c}$$

$$u^3r_{i,xxx}+3u^2u_xr_{i,xx}-3u^2(u_x+v)r_{i,xx} = -\lambda_{3,i}r_i, \quad 1\leqslant i\leqslant N, \tag{3.9d}$$

which has the Lax representation

$$[u^3\partial^3+3u^2(u_x+v)\partial^2]\psi=\mu\psi, \qquad \psi_t=\left(u^2\partial^2+\sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right)\psi.$$

3.2. The k-constraint

For each $k\geqslant 1$, the k-constraint is defined by

$$L^k = B_k + \sum_{i=1}^N q_i\partial^{-1}r_i\partial^2. \tag{3.10}$$

The sub-manifold determined by the k-constraint is invariant under the t_n-flow, i.e. we have

$$(L^k)_{t_n} = B_{k,t_n} + \left(\sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right)_{t_n},$$

since we can compute that

$$\begin{aligned}
&(L^k)_{t_n} = [B_n, L^k],\\
&B_{k,t_n} = \left(L^k_{t_n}\right)_+ = [B_n, L^k]_+,\\
&\left(\sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right)_{t_n} = \sum_{i=1}^N\left(q_{i,t_n}\partial^{-1}r_i\partial^2 + q_i\partial^{-1}r_{i,t_n}\partial^2\right)\\
&\qquad = \sum_{i=1}^N[(B_nq_i)\partial^{-1}r_i\partial^2 - q_i\partial^{-1}(\bar B_nr_i)\partial^2]\\
&\qquad = \left[B_n, \sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right]_-\\
&\qquad = [B_n, L^k-B_k]_- = [B_n, L^k]_-,
\end{aligned}$$

where proposition 2.1 was used. Therefore, upon dropping the τ_k dependence from (2.10a), we reduce the extended Harry Dym hierarchy (2.10) to the following hierarchy:

$$\left(B_k+\sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right)_{t_n} = \left[\left(\left(B_k+\sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right)^{\frac{n}{k}}\right)_+, B_k+\sum_{i=1}^N q_i\partial^{-1}r_i\partial^2\right], \tag{3.11a}$$

$$q_{i,t_n} = \left(B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2\right)_+^{\frac{n}{k}} q_i, \qquad 1 \leqslant i \leqslant N, \tag{3.11b}$$

$$r_{i,t_n} = -\partial^{-2}\left(B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2\right)_+^{\frac{n}{k}*} \partial^2 r_i, \qquad 1 \leqslant i \leqslant N. \tag{3.11c}$$

By proposition 2.1, equation (3.11a) is equivalent to

$$B_{k,t_n} = [B_n, B_k] + \left[B_n, \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2\right]_+. \tag{3.12}$$

This hierarchy (3.11) is called the k-constrained Harry Dym hierarchy. It has the Lax representation

$$\left(B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2\right)\psi = \mu\psi, \qquad \psi_{t_n} = B_n \psi,$$

where μ is the spectral parameter.

Example 3.2. Let us first take $n = 2$ and $k = 3$, and set $t_2 = y$ and $v = u_0$. Then the corresponding k-constrained Harry Dym system of equations (3.11) becomes

$$B_{3,y} = [B_2, B_3] + \left[B_2, \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2\right]_+ \tag{3.13a}$$

$$q_{i,y} = B_2 q_i, \qquad r_{i,y} = -\bar{B}_2 r_i, \quad 1 \leqslant i \leqslant N. \tag{3.13b}$$

Based on proposition 2.2, this constrained system reads

$$\begin{aligned}3[u^2(u_x + v)]_y = {} & u^3(uu_{xxx} + 6u_x u_{xx} + 3uv_{xx} + 12u_x v_x) \\ & + 2u^2 \sum_{i=1}^{N} (q_i r_i)_x - 2uu_x \sum_{i=1}^{N} q_i r_i,\end{aligned} \tag{3.14a}$$

$$u_y = u^2(u_{xx} + 2v_x), \tag{3.14b}$$

$$q_{i,y} = u^2 q_{i,xx}, \qquad r_{i,y} = -u^2 r_{i,xx}, \quad 1 \leqslant i \leqslant N. \tag{3.14c}$$

It has the following Lax representation:

$$\left[u^3\partial^3 + 3u^2(u_x + v)\partial^2 + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2\right]\psi = \mu\psi, \quad \psi_y = u^2\partial^2\psi.$$

Let us next take $n = 3$ and $k = 2$, and set $t_3 = t$ and $v = u_0$. Then the corresponding k-constrained Harry Dym system of equations (3.11) becomes

$$B_{2,t} = [B_3, B_2] + \left[B_3, \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2\right]_+ \tag{3.15a}$$

$$q_{i,t} = B_3 q_i, \qquad r_{i,t} = -\bar{B}_3 r_i, \quad 1 \leqslant i \leqslant N. \tag{3.15b}$$

Based on proposition 2.2, this constrained system reads

$$2uu_t = -u^3(uu_{xxx} + 6u_xu_{xx} + 3uv_{xx} + 12u_xv_x) + 6u^2(u_x + v)\sum_{i=1}^{N}(q_ir_i)_x$$
$$- 3[u^2(u_x + v)]_x\sum_{i=1}^{N}q_ir_i + 3u^3\sum_{i=1}^{N}(q_{i,xx}r_i + q_{i,x}r_{i,x}), \tag{3.16a}$$

$$u^2(u_{xx} + 2v_x) - u\sum_{i=1}^{N}(q_ir_i)_x + u_x\sum_{i=1}^{N}q_ir_i = 0, \tag{3.16b}$$

$$q_{i,t} = u^3q_{i,xxx} + 3u^2(u_x + v)q_{i,xx}, \quad 1 \leqslant i \leqslant N, \tag{3.16c}$$

$$r_{i,t} = u^3r_{i,xxx} + 3u^2u_xr_{i,xx} - 3u^2(u_x + v)r_{i,xx}, \quad 1 \leqslant i \leqslant N. \tag{3.16d}$$

It has the following Lax representation:

$$\left(u^2\partial^2 + \sum_{i=1}^{N}q_i\partial^{-1}r_i\partial^2\right)\psi = \mu\psi, \qquad \psi_t = [u^3\partial^3 + 3u^2(u_x + v)\partial^2]\psi.$$

4. Conclusions and remarks

We constructed an extended Harry Dym hierarchy based on eigenfunctions and adjoint eigenfunctions, and the resulting Harry Dym hierarchy contains the Harry Dym hierarchy with self-consistent sources and the constrained Harry Dym hierarchy. The extended Harry Dym hierarchy is compatible with the Harry Dym hierarchy, and two examples in each of three cases were computed. Since the (2+1)-dimensional Harry Dym hierarchy can be reduced to (1+1)-dimensional soliton hierarchies, many (1+1)-dimensional extended soliton hierarchies of Harry Dym type could be generated.

We also expect that there exist complexiton solutions, besides solitons and postions, to the extended Harry Dym hierarchy. The inverse scattering technique [11] and the Darboux transformation method [12] should work for the extended Harry Dym hierarchy as well. Soliton-type solutions should be expressed as Wronskian determinants (see, e.g., [19, 25]), and through the Wronskian technique, large solution subspaces of the extended Harry Dym hierarchy may be constructed.

The productivity tool adopted in our analysis is the pseudo-differential operator technique, through which only three integrable Hamiltonian hierarchies—the KP hierarchy, the mKP hierarchy and the Harry Dym hierarchy—are generated by the r-matrix formulation [21]. Under the 2- or 3-reduction, the extended KP hierarchy [17] and the extended mKP hierarchy [19] are reduced to the KdV equations with self-consistent sources, the mKdV equations with self-consistent sources and the Harry Dym equations with self-consistent sources, respectively. Therefore, the extended integrable hierarchies generalize typical integrable equations and greatly enrich integrable structures. The key idea of extending integrable hierarchies by the pseudo-differential operator technique is to use the Lax formulation consisting of pseudo-differential Lax operators involving eigenfunctions and adjoint eigenfunctions, and it may also be applied to integrable dispersionless equations [26, 27] and hidden integrable hierarchies [28].

Finally, let us discuss a more general extending scheme. Assume that a pair of extended Lax operators is defined by

$$\tilde{B}_k = B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2, \qquad \tilde{B}_l = B_l + \sum_{i=1}^{M} \alpha_i \partial^{-1} \beta_i \partial^2,$$

where we require that two sets of eigenfunctions and adjoint eigenfunctions satisfy

$$q_{i,\tau_l} = B_l q_i, \qquad r_{i,\tau_l} = -\bar{B}_l r_i = -\partial^{-2} B_l^* \partial^2 r_i, \qquad 1 \leqslant i \leqslant N,$$

and

$$\alpha_{j,\tau_k} = B_k \alpha_j, \qquad \beta_{j,\tau_k} = -\bar{B}_k \beta_j = -\partial^{-2} B_k^* \partial^2 \beta_j, \qquad 1 \leqslant j \leqslant M.$$

To guarantee the compatibility of the τ_k-flow and the τ_l-flow defined by

$$L_{\tau_k} = [\tilde{B}_k, L], \qquad L_{\tau_l} = [\tilde{B}_l, L],$$

we need to let a more general pseudo-differential zero curvature equation

$$\tilde{B}_{k,\tau_l} - \tilde{B}_{l,\tau_k} + [\tilde{B}_k, \tilde{B}_l] = 0$$

hold. By proposition 2.1, this equation is equivalent to

$$B_{k,\tau_l} - B_{l,\tau_k} + [B_k, B_l] + \left[B_k, \sum_{j=1}^{M} \alpha_j \partial^{-1} \beta_j \partial^2 \right]_+ + \left[\sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2, B_l \right]_+ = 0,$$

and

$$\left[\sum_{i=1}^{N} q_i \partial^{-1} r_i \partial^2, \sum_{j=1}^{M} \alpha_j \partial^{-1} \beta_j \partial^2 \right] = 0,$$

due to $[q\partial^{-1} r \partial^2, \alpha \partial^{-1} \beta \partial^2]_+ = 0$. The resulting systems of equations extend the extended Harry Dym hierarchy (2.10), and thus, provide a larger class of nonlinear equations that possess Lax representations involving two extended Lax operators. However, the last equation above also brings infinitely many differential constraints on the eigenfunctions and adjoint eigenfunctions: q_i and r_i, $1 \leqslant i \leqslant N$, and α_j and β_j, $1 \leqslant j \leqslant M$. It is interesting how to satisfy these constraints to make the τ_k-flow and the τ_l-flow compatible with each other.

Acknowledgments

The work was supported in part by the Established Researcher Grant and the CAS faculty development grant of the University of South Florida, and Chunhui Plan of the Ministry of Education of China. The author is also grateful to the referee for helpful comments and suggestions.

References

[1] Mel'nikov V K 1983 *Lett. Math. Phys.* **7** 129
[2] Mel'nikov V K 1987 *Commun. Math. Phys.* **112** 639
[3] Cheng Y and Li Y S 1991 *Phys. Lett.* A **157** 22
[4] Konopelchenko B, Sidorenko J and Strampp W 1991 *Phys. Lett.* A **157** 17
[5] Oevel W and Strampp W 1993 *Commun. Math. Phys.* **157** 51
[6] Zeng Y B 1994 *Physica* D **73** 171
[7] Oevel W and Carillo S 1998 *J. Math. Anal. Appl.* **217** 161, 179
[8] Zeng Y B and Ma W X 1999 *J. Math. Phys.* **40** 6526

[9] Ma W X 2003 *J. Phys. Soc. Japan* **72** 3017
[10] Oevel W and Strampp W 1996 *J. Math. Phys.* **37** 6213
[11] Lin R L, Zeng Y B and Ma W X 2001 *Physica* A **291** 287
[12] Zeng Y B, Ma W X and Shao Y J 2001 *J. Math. Phys.* **42** 2113
[13] Xiao T and Zeng Y B 2005 *Phys.* A **353** 38
[14] Ma W X 2005 *Chaos Solitons Fractals* **26** 1453
[15] Ma W X and Zhou Z X 2001 *J. Math. Phys.* **42** 4345
[16] Ma W X and Zeng Y B 2002 *ANZIAM J.* **44** 129
[17] Liu X J, Zeng Y B and Lin R L 2008 *Phys. Lett.* A **372** 3819
[18] Lin R L, Liu X J and Zeng Y B 2008 *J. Nonlinear Math. Phys.* **15** 333
[19] Liu X J, Lin R L, Jin B and Zeng Y B 2009 *J. Math. Phys.* **50** 053506
[20] Ma W X and Strampp W 1994 *Phys. Lett.* A **185** 277
[21] Konopelchenko B and Oevel W 1993 *Publ. Res. Inst. Math. Sci.* **29** 581
[22] Kupershmidt B A 1985 *Commun. Math. Phys.* **99** 51
[23] Kruskal M 1975 *Dynamical Systems: Theory and Applications (Lecture Notes in Physics* vol 38*)* (Heidelberg: Springer) pp 310–54
[24] Ma W X 1992 *Chin. Sci. Bull.* **37** 1249
[25] Ma W X and You Y 2005 *Trans. Am. Math. Soc.* **357** 1753
[26] Strachan I A B 2001 *Lett. Math. Phys.* **58** 129
[27] Błaszak M and Szablikowski B M 2002 *J. Phys. A: Math. Gen.* **35** 10325
[28] Konopelchenko B and Martínez Alonso L 1997 *Phys. Lett.* A **236** 431