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A discrete variational identity on semi-direct sums of Lie algebras

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Abstract

The discrete variational identity under general bilinear forms on semi-direct sums of Lie algebras is established. The constant γ involved in the variational identity is determined through the corresponding solution to the stationary discrete zero-curvature equation. An application of the resulting variational identity to a class of semi-direct sums of Lie algebras in the Volterra lattice case furnishes Hamiltonian structures for the associated integrable couplings of the Volterra lattice hierarchy.

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1. Introduction

An algebraic approach to integrable couplings [1, 2] was recently presented, based on the concept of semi-direct sums of Lie algebras [3, 4]. There exist plenty of examples of both continuous and discrete integrable couplings belonging to such a class of integrable equations [1–10]. The corresponding results show various mathematical structures that integrable equations possess and provide a powerful tool to analyze integrable equations, particularly, multi-component integrable equations and integrable couplings [9, 11]. Observe that a general Lie algebra can be decomposed into a semi-direct sum of a solvable Lie algebra and a semisimple Lie algebra [12]. The semi-direct sum decomposition of Lie algebras allows for more classifications of integrable equations supplementing existing theories [13, 14], e.g., classifications within the areas of symmetry reductions [15, 16] and Lax pairs [17].

Let G be a matrix loop algebra, E be the shift operator and D denote the forward difference operator $E-1$, i.e., $D = E - 1$. Traditionally, we write

$$(E^m f)(n) = f^{(m)}(n) = f(n+m), \qquad m, n \in \mathbb{Z}, \tag{1.1}$$

and define an inverse of the difference operator $E - E^{-1}$ as follows (see, e.g., [18]):

$$((E-E)^{-1} f)(n) = \frac{1}{2}\left(\sum_{m=-\infty}^{-1} f(n+1+2m) - \sum_{m=1}^{\infty} f(n-1+2m)\right), \qquad n \in \mathbb{Z}, \tag{1.2}$$

where f is an expression depending on the lattice variable n. The corresponding inverses of the forward and backward difference operators are determined by

$$(E-1)^{-1} = (E-E^{-1})^{-1}(1+E^{-1}), \qquad (1-E^{-1})^{-1} = (E-E^{-1})^{-1}(E+1).$$

Other kinds of inverses for difference operators are possible (see, e.g., [19]). The inverses are normally used in deriving hierarchies of soliton equations, in particular, non-isospectral hierarchies.

Let $u = u(n,t) = (u_1(n,t),\ldots,u_q(n,t))^T$ be a vector potential, in which $n \in \mathbb{Z}$ and $t \in \mathbb{R}$ are the lattice variable and the time variable, respectively. When an object P (e.g., a function or an operator) depends on u, its Gateaux derivative with respect to u in a direction $v = (v_1,\ldots,v_q)^T$ is defined by

$$P'[v] = P'(u)[v] = \frac{\partial}{\partial \varepsilon} P(u+\varepsilon v)\bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} P(u_1+\varepsilon v_1,\ldots,u_q+\varepsilon v_q)\bigg|_{\varepsilon=0}. \tag{1.3}$$

We denote by $\mathcal{B}$ the space of functions which are C^∞-differentiable with respect to n and t and C^∞-Gateaux differentiable with respect to u, and define the Lie bracket $[\cdot,\cdot]$ on $\mathcal{B}^q = \{(P_1,\ldots,P_q)^T | P_i \in \mathcal{B}, 1 \leqslant i \leqslant q\}$ as follows:

$$[K,S] = K'[S] - S'[K] = K'(u)[S] - S'(u)[K], \qquad K, S \in \mathcal{B}^q. \tag{1.4}$$

The forward difference operator $D = E - 1$ yields an equivalence relation $\sim$ on $\mathcal{B}$:

$$P \sim Q \quad \text{if} \quad \exists R \in \mathcal{B} \quad \text{such that} \quad P - Q = DR.$$

Let $\sum_{n\in\mathbb{Z}} P$ denote the equivalence class to which P belongs:

$$\sum_{n\in\mathbb{Z}} P = \{P + DR | R \in \mathcal{B}\}, \qquad P \in \mathcal{B}, \tag{1.5}$$

and $\mathcal{F}$, the quotient space: $\mathcal{F} = \mathcal{F}(\mathcal{B}) = \left\{\sum_{n\in\mathbb{Z}} P \,\middle|\, P \in \mathcal{B}\right\}$. An equivalence class of $\mathcal{B}$ by $\sim$ is called a functional. The variational derivative $\frac{\delta \mathcal{P}}{\delta u} \in \mathcal{B}^q$ of a functional $\mathcal{P} \in \mathcal{F}$ with respect to u is determined by

$$\sum_{n\in\mathbb{Z}} \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T \xi = \frac{\partial}{\partial \varepsilon} \mathcal{P}(u+\varepsilon\xi)\bigg|_{\varepsilon=0}, \qquad \xi \in \mathcal{B}^q.$$

It is easy to see that

$$\frac{\delta}{\delta u_i} \sum_{n\in\mathbb{Z}} P = \sum_{j\in\mathbb{Z}} E^{-j} \frac{\partial P}{\partial u_i^{(j)}}, \qquad 1 \leqslant i \leqslant q, \tag{1.6}$$

where $P = P(u) \in \mathcal{B}$.

The adjoint operator $J^\dagger : \mathcal{B}^q \to \mathcal{B}^q$ of a linear operator $J : \mathcal{B}^q \to \mathcal{B}^q$ is determined by

$$\sum_{n\in\mathbb{Z}} \xi^T J^\dagger \eta = \sum_{n\in\mathbb{Z}} \eta^T J \xi, \qquad \xi, \eta \in \mathcal{B}^q.$$

If $J^\dagger = -J$, then J is called to be skew symmetric. A linear skew-symmetric operator $J : \mathcal{B}^q \to \mathcal{B}^q$ is called to be Hamiltonian, if the corresponding Poisson bracket

$$\{\mathcal{P},\mathcal{Q}\} = \{\mathcal{P},\mathcal{Q}\}_J = \sum_{n\in\mathbb{Z}} \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T J \frac{\delta \mathcal{Q}}{\delta u}, \qquad \mathcal{P},\mathcal{Q} \in \mathcal{F} \tag{1.7}$$

satisfies the Jacobi identity:

$$\{\mathcal{P},\{\mathcal{Q},\mathcal{R}\}\} + \text{cycle}(\mathcal{P},\mathcal{Q},\mathcal{R}) = 0, \qquad \mathcal{P},\mathcal{Q},\mathcal{R} \in \mathcal{F}.$$

A system of evolution equations $u_t = K$, $K \in \mathcal{B}^q$, is called to be a Hamiltonian system, if there are a Hamiltonian operator $J: \mathcal{B}^q \to \mathcal{B}^q$ and a functional $\mathcal{H} \in \mathcal{F}$ such that

$$u_t = K = J\frac{\delta \mathcal{H}}{\delta u}. \tag{1.8}$$

The functional $\mathcal{H}$ is called a Hamiltonian functional of the system, and we say that the system possesses a Hamiltonian structure.

We now assume that a pair of matrix discrete spectral problems

$$\begin{cases} E\phi = U\phi = U(u,\lambda)\phi, \\ \phi_t = V\phi = V(u, Eu, E^{-1}u, \ldots; \lambda)\phi, \end{cases} \tag{1.9}$$

where $u = u(n,t)$ is the potential, ϕ_t denotes the derivative with respect to t, $U, V \in G$ are called a Lax pair and λ is a spectral parameter, determines a discrete soliton equation

$$u_t = K = K(n,t,u,Eu,E^{-1}u,\ldots), \qquad K \in \mathcal{B}^q, \tag{1.10}$$

through their isospectral (i.e., $\lambda_t = 0$) compatibility condition (i.e., discrete zero curvature equation)

$$U_t = (EV)U - VU. \tag{1.11}$$

This means that a triple (U, V, K) satisfies

$$U'[K] = (EV)U - VU,$$

where $U'[K]$ denotes the Gateaux derivative of U with respect to u in a direction K. The Lie algebraic structure for such triples was discussed [18] and applied to non-isospectral flows [20].

To generate integrable couplings of equation (1.10), take a semi-direct sum of G with another matrix loop algebra G_c as introduced in [4]:

$$\bar{G} = G \Subset G_c. \tag{1.12}$$

The notion of semi-direct sums implies that G and G_c satisfy

$$[G, G_c] \subseteq G_c,$$

where $[G, G_c] = \{[A,B] | A \in G, B \in G_c\}$. Obviously, G_c is an ideal Lie sub-algebra of $\bar{G}$. The subscript c indicates a contribution to the construction of integrable couplings. We also require that the closure property between G and G_c under the matrix multiplication

$$GG_c, G_cG \subseteq G_c,$$

where $G_1G_2 = \{AB | A \in G_1, B \in G_2\}$, to guarantee that the discrete zero curvature equation (1.11) over semi-direct sums of Lie algebras can engender coupling systems.

Then choose a pair of enlarged matrix discrete spectral problems

$$\begin{cases} E\bar{\phi} = \bar{U}\bar{\phi} = \bar{U}(\bar{u},\lambda)\bar{\phi}, \\ \bar{\phi}_t = \bar{V}\bar{\phi} = \bar{V}(\bar{u}, E\bar{u}, E^{-1}\bar{u}, \ldots; \lambda)\bar{\phi}, \end{cases} \tag{1.13}$$

where the enlarged Lax pair is given by

$$\bar{U} = U + U_c, \qquad \bar{V} = V + V_c, \qquad U_c, V_c \in G_c. \tag{1.14}$$

Obviously, under the soliton equation (1.10), the corresponding enlarged discrete zero-curvature equation

$$\bar{U}_t = (E\bar{V})\bar{U} - \bar{V}\bar{U} \tag{1.15}$$

is equivalent to

$$\begin{cases} U_t = (EV)U - VU, \\ U_{c,t} = [(EV)U_c - U_cV] + [(EV_c)U - UV_c] + [(EV_c)U_c - U_cV_c]. \end{cases} \tag{1.16}$$

The first equation above precisely gives equation (1.10), and thus the whole system provides a coupling system for equation (1.10). This shows the procedure of generating discrete integrable couplings through semi-direct sums of Lie algebras, proposed in [4].

As usual, a bilinear form $\langle\cdot,\cdot\rangle$ on a vector space is said to be non-degenerate when if $\langle A, B\rangle = 0$ for all vectors A, then $B = 0$, and if $\langle A, B\rangle = 0$ for all vectors B, then $A = 0$. Since semi-direct sums of Lie algebras are not semisimple, the Killing form is always degenerate on semi-direct sums of Lie algebras [12], and thus it is not helpful in analyzing Hamiltonian equations by the trace identity [21, 22]. Indeed, semi-direct sums of Lie algebras can carry particular algebraic structures [23], and the corresponding groups can extend the Poincaré group to unite geometrical with internal symmetries in a nontrivial way [24]. A natural question here for us is whether we can replace the Killing forms with general bilinear forms to establish Hamiltonian structures for discrete soliton equations associated with semi-direct sums of Lie algebras.

In this paper, we would like to answer this question. As in the case of the continuous variational identity [25], we are going to show that a discrete variational identity also ubiquitously exists in discrete spectral problems and plays important roles in constructing Hamiltonian structures and thereby conserved quantities for discrete soliton equations. The crucial step of our success is that while presenting a discrete variational identity under a general bilinear form $\langle\cdot,\cdot\rangle$ on a given algebra g we get rid of the invariance property

$$\langle\rho(A), \rho(B)\rangle = \langle A, B\rangle \tag{1.17}$$

under an isomorphism ρ of the algebra g, but keep the symmetric property

$$\langle A, B\rangle = \langle B, A\rangle \tag{1.18}$$

and the invariance property under the multiplication

$$\langle A, BC\rangle = \langle AB, C\rangle, \tag{1.19}$$

where AB denotes the product of A and B in g. If g is also associative, then g forms a Lie algebra under

$$[A, B] = AB - BA,$$

and the invariance property under the Lie bracket holds:

$$\langle A, [B, C]\rangle = \langle [A, B], C\rangle. \tag{1.20}$$

Conversely, the invariance property under the Lie bracket, (1.20), does not imply the invariance property under the multiplication, (1.19). We will show by examples that generally there are many non-degenerate bilinear forms with the properties (1.18) and (1.19) on a given semi-direct sum of Lie algebras.

The paper is organized as follows. First, in section 2, we would like to establish a discrete variational identity under general non-degenerate, symmetric and invariant bilinear forms, in order to construct Hamiltonian structures of soliton equations associated with semi-direct sums of Lie algebras. Moreover, in section 3, the constant γ appeared in the variational identity will be determined precisely. Then, in section 4, an application is given to a kind of semi-direct sums of Lie algebras in the Volterra lattice case, and consequently, Hamiltonian structures of the associated integrable couplings of the Volterra lattice hierarchy are presented. This also justifies that the approach of integrable couplings using semi-direct sums of Lie algebras [4] can engender integrable Hamiltonian equations. A few concluding remarks are given in the last section.

2. A discrete variational identity under non-Killing forms

For a given spectral matrix $U = U(u, \lambda) \in G$, where G is a matrix loop algebra, let us fix the proper ranks $\mathrm{rank}(\lambda)$ and $\mathrm{rank}(u)$ so that U is homogeneous in rank, i.e., we can define $\mathrm{rank}(U)$. The rank function satisfies

$$\mathrm{rank}(AB) = \mathrm{rank}(A) + \mathrm{rank}(B),$$

whenever an expression AB makes sense, e.g., EU. Therefore, to keep the rank balance in equations, we have to define

$$\mathrm{rank}(E) = \mathrm{rank}(U) = 0. \tag{2.1}$$

The requirement $\mathrm{rank}(E) = 0$ is due to the stationary discrete zero curvature equation

$$(EV)(EU) = UV, \tag{2.2}$$

and then the requirement $\mathrm{rank}(U) = 0$ is due to the discrete spectral problem $E\phi = U\phi$ in (1.9).

Let us next assume that if two solutions V_1 and V_2 to (2.2) possess the same rank, then they are linearly dependent on each other:

$$V_1 = \gamma V_2, \qquad \gamma = \text{const}. \tag{2.3}$$

This is a strict condition on spectral problems, also required in deducing the trace identity [21], the so-called quadratic-form identity [26] and the continuous variational identity [25], which can be used to construct Hamiltonian structures of various continuous soliton equations (see, e.g, [27–29]).

Associated with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on G with the symmetric property (1.18) and the invariance property under the multiplication, (1.19), we introduce a functional

$$\mathcal{W} = \sum_{n\in\mathbb{Z}} (\langle V, U_\lambda \rangle + \langle \Lambda, (EV)(EU) - UV \rangle), \tag{2.4}$$

while U_λ denotes the partial derivative with respect to λ, and $V, \Lambda \in G$ are two specific matrices. The variational derivative $\nabla_A \mathcal{R} \in G$ of a functional $\mathcal{R}$ with respect to $A \in G$ is defined by

$$\sum_{n\in\mathbb{Z}} \langle \nabla_A \mathcal{R}, B \rangle = \frac{\partial}{\partial \varepsilon} \mathcal{R}(A + \varepsilon B)\bigg|_{\varepsilon=0}, \qquad B \in G. \tag{2.5}$$

Obviously, based on the non-degenerate property of the bilinear form $\langle \cdot, \cdot \rangle$, we can have

$$\nabla_B \sum_{n\in\mathbb{Z}} \langle A, B \rangle = A, \qquad \nabla_B \sum_{n\in\mathbb{Z}} \langle A, EB \rangle = E^{-1} A.$$

It then follows from the symmetric property (1.18) and the invariance property under the multiplication, (1.19), that

$$\nabla_V \mathcal{W} = U_\lambda + U(E^{-1}\Lambda) - \Lambda U, \qquad \nabla_\Lambda \mathcal{W} = (EV)(EU) - UV. \tag{2.6}$$

Note that the first variational derivative formula cannot be obtained, if we only have the invariance property under the Lie bracket, (1.20).

2.1. A discrete variational identity

We are going to prove that there is a variational identity in the discrete world, similar to the continuous variational identity [25].

Theorem 2.1 (The discrete variational identity under general bilinear forms)**.** *Let G be a matrix loop algebra, and $U = U(u, \lambda) \in G$ be homogeneous in rank such that (2.2) has a unique solution $V \in G$ of a fixed rank up to a constant multiplier. Then for any solution $V \in G$ of (2.2), being homogeneous in rank, and any non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on G with the symmetric property (1.18) and the invariance property under the multiplication, (1.19), we have the following discrete variational identity:*

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\langle V, U_\lambda\rangle = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left\langle V, \frac{\partial U}{\partial u}\right\rangle, \tag{2.7}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to the potential u and γ is a constant.

Proof. Let us start with the functional $\mathcal{W}$ introduced in (2.4). For the variational calculation of $\mathcal{W}$ with respect to the potential u, we require the following constraint conditions:

$$\nabla_V \mathcal{W} = U_\lambda + U(E^{-1}\Lambda) - \Lambda U = 0, \tag{2.8}$$

$$\nabla_\Lambda \mathcal{W} = (EV)(EU) - UV = 0, \tag{2.9}$$

to determine V and Λ. These conditions also imply that both V and Λ are related to U and thus to the potential u. Immediately from the second constraint condition (2.9), we have

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\langle V, U_\lambda\rangle = \frac{\delta\mathcal{W}}{\delta u}.$$

Now using both of the constraint conditions (2.8) and (2.9), and noting the property that if $\nabla_A\mathcal{R}(A) = 0$, then $\frac{\delta}{\delta u}\mathcal{R}(A(u)) = 0$ for a functional $\mathcal{R}$, we know that only the dependence of u in U (but not in V and Λ) needs to be considered in computing $\frac{\delta\mathcal{W}}{\delta u}$. Therefore, based on the invariance property under the multiplication, (1.19) (note that the invariance property under the Lie bracket, (1.20), is not good enough), we obtain

$$\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\langle V, U_\lambda\rangle = \frac{\delta\mathcal{W}}{\delta u} = \left\langle V, \frac{\partial U_\lambda}{\partial u}\right\rangle + \left\langle \Theta, \frac{\partial U}{\partial u}\right\rangle, \tag{2.10}$$

where

$$\Theta = (E^{-1}\Lambda)V - V\Lambda. \tag{2.11}$$

This matrix Θ satisfies

$$E(\Theta - V_\lambda)(EU) - U(\Theta - V_\lambda) = 0,$$

namely, $\Theta - V_\lambda$ solves (2.2). This is because we have

$$\begin{aligned}(E\Theta)(EU) - U\Theta &= \Lambda(EV)(EU) - (EV)(E\Lambda)(EU) - U(E^{-1}\Lambda)V + UV\Lambda \\ &= \Lambda UV - (EV)(E\Lambda)(EU) - U(E^{-1}\Lambda)V + (EV)(EU)\Lambda \\ &= [\Lambda U - U(E^{-1}\Lambda)]V + (EV)[(EU)\Lambda - (E\Lambda)(EU)] \\ &= U_\lambda V - (EV)(EU_\lambda)\end{aligned}$$

from (2.8) and (2.9) and

$$(EV_\lambda)(EU) - UV_\lambda = U_\lambda V - (EV)(EU_\lambda)$$

from differentiating (2.9) with respect to λ. By taking use of the uniqueness condition (2.3) and rank $(\Theta - V_\lambda) = \text{rank}(V_\lambda) = \text{rank}\left(\frac{1}{\lambda}V\right)$, there exists a constant γ such that

$$\Theta - V_\lambda = (E^{-1}\Lambda)V - V\Lambda - V_\lambda = \frac{\gamma}{\lambda}V, \tag{2.12}$$

because $\frac{1}{\lambda}V$ is also a solution to (2.2).

Finally, (2.10) can further be expressed as

$$\begin{aligned}
\frac{\delta}{\delta u}\sum_{n\in\mathbb{Z}}\langle V, U_\lambda\rangle &= \left\langle V, \frac{\partial U_\lambda}{\partial u}\right\rangle + \left\langle V_\lambda, \frac{\partial U}{\partial u}\right\rangle + \frac{\gamma}{\lambda}\left\langle V, \frac{\partial U}{\partial u}\right\rangle \\
&= \frac{\partial}{\partial \lambda}\left\langle V, \frac{\partial U}{\partial u}\right\rangle + \left(\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\right)\left\langle V, \frac{\partial U}{\partial u}\right\rangle \\
&= \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left\langle V, \frac{\partial U}{\partial u}\right\rangle.
\end{aligned}$$

This completes the proof. □

2.2. *A formula for the constant γ*

Let us consider the other form of the stationary discrete zero curvature equation

$$(E\Gamma)U - U\Gamma = 0. \tag{2.13}$$

If V is a solution to (2.2), then $\Gamma = VU$ satisfies

$$D\Gamma = [U, V], \tag{2.14}$$

where $D = E - 1$, as defined in the introduction. This is a counterpart of the stationary continuous zero curvature equation $V_x = [U, V]$. When U is invertible, then V is a solution to (2.2) iff $\Gamma = VU$ is a solution to (2.13).

The matrix V presents the gradient which is needed for constructing the desired Hamiltonian structure, and the matrix Γ contributes to the constant γ in the variational identity as follows.

Theorem 2.2. *Let V be a solution to (2.2) and $\Gamma = VU$. Then for any bilinear form $\langle\cdot,\cdot\rangle$ on G with the properties (1.18) and (1.20), we have*

$$D\langle\Gamma^m, \Gamma^m\rangle = (E-1)\langle\Gamma^m, \Gamma^m\rangle = 0, \qquad m \geqslant 1. \tag{2.15}$$

Proof. Noting that $E\Gamma = E(VU) = UV$, it follows from the symmetric property (1.18) and the invariance property under Lie bracket, (1.20), that

$$\begin{aligned}
D\langle\Gamma^m, \Gamma^m\rangle &= \langle (UV)^m, (UV)^m\rangle - \langle (VU)^m, (VU)^m\rangle \\
&= \langle (UV)^m - (VU)^m, (UV)^m + (VU)^m\rangle \\
&= \langle [U, V(UV)^{m-1}], (UV)^m + (VU)^m\rangle \\
&= \langle U, [V(UV)^{m-1}, (UV)^m + (VU)^m]\rangle \\
&= \langle U, [V(UV)^{m-1}(VU)^{m-1}V, U]\rangle \\
&= \langle V(UV)^{m-1}(VU)^{m-1}V, [U, U]\rangle = 0,
\end{aligned}$$

where $m \geqslant 1$. This proves the theorem. □

By (2.15), $\langle\Gamma, \Gamma\rangle$ is independent of the lattice variable n.

Theorem 2.3. *Let V be a solution to (2.2) and $\Gamma = VU$. If $\langle\Gamma,\Gamma\rangle \neq 0$, then the constant γ in the discrete variational identity (2.7) is given by*

$$\gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\langle\Gamma,\Gamma\rangle|. \tag{2.16}$$

Proof. It follows from (2.8) and (2.12) that

$$\begin{aligned}\Gamma_\lambda &= (VU)_\lambda = V_\lambda U + VU_\lambda\\ &= \left[(E^{-1}\Lambda)V - V\Lambda - \frac{\gamma}{\lambda}V\right]U + V[\Lambda U - U(E^{-1}\Lambda)]\\ &= [E^{-1}\Lambda, VU] - \frac{\gamma}{\lambda}VU\\ &= [E^{-1}\Lambda, \Gamma] - \frac{\gamma}{\lambda}\Gamma.\end{aligned}$$

Therefore, differentiating $\langle\Gamma,\Gamma\rangle$ with respect to λ yields

$$\begin{aligned}\langle\Gamma,\Gamma\rangle_\lambda &= \langle\Gamma_\lambda,\Gamma\rangle + \langle\Gamma,\Gamma_\lambda\rangle = 2\langle\Gamma_\lambda,\Gamma\rangle\\ &= 2\left\langle[E^{-1}\Lambda,\Gamma] - \frac{\gamma}{\lambda}\Gamma,\Gamma\right\rangle\\ &= 2\langle[E^{-1}\Lambda,\Gamma],\Gamma\rangle - \frac{2\gamma}{\lambda}\langle\Gamma,\Gamma\rangle\\ &= 2\langle E^{-1}\Lambda,[\Gamma,\Gamma]\rangle - \frac{2\gamma}{\lambda}\langle\Gamma,\Gamma\rangle\\ &= -\frac{2\gamma}{\lambda}\langle\Gamma,\Gamma\rangle.\end{aligned}$$

This implies that (2.16) holds. □

Note that formula (2.16) for the constant γ in (2.12) is still true, if we only have the invariance property under the Lie bracket, (1.20).

3. Symmetric and invariant bilinear forms

Let us consider the following semi-direct sum of Lie algebras of 4×4 matrices:

$$\bar{G} = G \Subset G_c = \left\{\begin{bmatrix} A_0 & 0\\ 0 & A_0\end{bmatrix}\middle| A_0 = \begin{bmatrix} a_1 & a_2\\ a_3 & a_4\end{bmatrix}\right\} \Subset \left\{\begin{bmatrix} 0 & A_1\\ 0 & 0\end{bmatrix}\middle| A_1 = \begin{bmatrix} a_5 & a_6\\ a_7 & a_8\end{bmatrix}\right\}, \tag{3.1}$$

where a_i, $1 \leqslant i \leqslant 8$, are real constants. In order to construct symmetric and invariant bilinear forms on $\bar{G}$ conveniently, we transform the semi-direct sum $\bar{G}$ into a vector form. Define the mapping

$$\sigma : \bar{G} \to R^8, \qquad A \mapsto (a_1,\ldots,a_8)^T, \qquad A = \begin{bmatrix} a_1 & a_2 & a_5 & a_6\\ a_3 & a_4 & a_7 & a_8\\ 0 & 0 & a_1 & a_2\\ 0 & 0 & a_3 & a_4\end{bmatrix} \in \bar{G}. \tag{3.2}$$

This mapping σ induces a Lie algebraic structure on $\mathbb{R}^8$, isomorphic to the matrix loop algebra $\bar{G}$. The corresponding Lie bracket $[\cdot,\cdot]$ on $\mathbb{R}^8$ can be computed as follows:

$$[a,b]^T = a^T R(b), \qquad a = (a_1,\ldots,a_8)^T, \qquad b = (b_1,\ldots,b_8)^T \in \mathbb{R}^8, \tag{3.3}$$

where

$$R(b) = \begin{bmatrix} 0 & b_2 & -b_3 & 0 & 0 & b_6 & -b_7 & 0 \\ b_3 & b_4 - b_1 & 0 & -b_3 & b_7 & b_8 - b_5 & 0 & -b_7 \\ -b_2 & 0 & b_1 - b_4 & b_2 & -b_6 & 0 & b_5 - b_8 & b_6 \\ 0 & -b_2 & b_3 & 0 & 0 & -b_6 & b_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2 & -b_3 & 0 \\ 0 & 0 & 0 & 0 & b_3 & b_4 - b_1 & 0 & -b_3 \\ 0 & 0 & 0 & 0 & -b_2 & 0 & b_1 - b_4 & b_2 \\ 0 & 0 & 0 & 0 & 0 & -b_2 & b_3 & 0 \end{bmatrix}.$$

This Lie algebra $(\mathbb{R}^8, [\cdot, \cdot])$ is isomorphic to the matrix Lie algebra $\bar{G}$, and the mapping σ, defined by (3.2), is a Lie isomorphism between the two Lie algebras.

A bilinear form on $\mathbb{R}^8$ can be defined by

$$\langle a, b\rangle = a^T F b, \tag{3.4}$$

where F is a constant matrix (actually, $F = (\langle \mathbf{e}_i, \mathbf{e}_j\rangle)_{8\times 8}$, where $\mathbf{e}_1, \ldots, \mathbf{e}_8$ are the standard basis of $\mathbb{R}^8$). The symmetric property $\langle a, b\rangle = \langle b, a\rangle$ requires that

$$F^T = F. \tag{3.5}$$

Under this symmetric condition, the invariance property under the Lie bracket

$$\langle a, [b, c]\rangle = \langle [a, b], c\rangle$$

equivalently requires that

$$F(R(b))^T = -R(b)F, \qquad b \in \mathbb{R}^8. \tag{3.6}$$

This matrix equation leads to a linear system of equations on the elements of F. Solving the resulting system yields

$$F = \begin{bmatrix} \eta_1 & 0 & 0 & \eta_2 & \eta_3 & 0 & 0 & \eta_4 \\ 0 & 0 & \eta_1 - \eta_2 & 0 & 0 & 0 & \eta_3 - \eta_4 & 0 \\ 0 & \eta_1 - \eta_2 & 0 & 0 & 0 & \eta_3 - \eta_4 & 0 & 0 \\ \eta_2 & 0 & 0 & \eta_1 & \eta_4 & 0 & 0 & \eta_3 \\ \eta_3 & 0 & 0 & \eta_4 & \eta_5 & 0 & 0 & \eta_5 \\ 0 & 0 & \eta_3 - \eta_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_3 - \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_4 & 0 & 0 & \eta_3 & \eta_5 & 0 & 0 & \eta_5 \end{bmatrix}, \tag{3.7}$$

where η_i, $1 \leqslant i \leqslant 5$, are arbitrary constants. Now, the corresponding bilinear form on the semi-direct sum $\bar{G}$ of Lie algebras defined by (3.1) is given as follows:

$$\begin{aligned} \langle A, B\rangle_{\bar{G}} &= \langle \sigma^{-1}(A), \sigma^{-1}(B)\rangle_{\mathbb{R}^8} = (a_1, \ldots, a_8) F (b_1, \ldots, b_8)^T \\ &= (\eta_1 a_1 + \eta_2 a_4 + \eta_3 a_5 + \eta_4 a_8)\, b_1 + [(\eta_1 - \eta_2)\, a_3 + (\eta_3 - \eta_4)\, a_7]\, b_2 \\ &\quad + [(\eta_1 - \eta_2)\, a_2 + (\eta_3 - \eta_4)\, a_6]\, b_3 + (\eta_2 a_1 + \eta_1 a_4 + \eta_4 a_5 + \eta_3 a_8)\, b_4 \\ &\quad + (\eta_3 a_1 + \eta_4 a_4 + \eta_5 a_5 + \eta_5 a_8)\, b_5 + (\eta_3 - \eta_4)\, a_3 b_6 \\ &\quad + (\eta_3 - \eta_4)\, a_2 b_7 + (\eta_4 a_1 + \eta_3 a_4 + \eta_5 a_5 + \eta_5 a_8)\, b_8, \end{aligned} \tag{3.8}$$

where

$$A = \begin{bmatrix} a_1 & a_2 & a_5 & a_6 \\ a_3 & a_4 & a_7 & a_8 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_3 & a_4 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & b_2 & b_5 & b_6 \\ b_3 & b_4 & b_7 & b_8 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & b_3 & b_4 \end{bmatrix} \in \bar{G}.$$

The bilinear form (3.8) is symmetric and invariant under the Lie bracket of the matrix Lie algebra:

$$\langle A, B\rangle = \langle B, A\rangle, \qquad \langle A, [B, C]\rangle = \langle [A, B], C\rangle, \qquad A, B, C \in \bar{G}.$$

But this kind of bilinear forms is not of Killing type, since the matrix Lie algebra $\bar{G}$ is not semisimple. A direct computation shows that the bilinear form (3.8) is also invariant under the matrix multiplication:

$$\langle A, BC\rangle = \langle AB, C\rangle, \qquad A, B, C \in \bar{G}.$$

We started with the invariance property under the Lie bracket but not under the multiplication, since it is easier to express the invariance property under the Lie bracket as an equation like (3.6).

The bilinear forms defined by (3.8) contain plenty of non-degenerate cases. A particular non-degenerate bilinear form with $\eta_1 = \eta_2 = \eta_3 = 1$ and $\eta_4 = \eta_5 = 0$ will be used to establish Hamiltonian structures for the integrable couplings of the Volterra lattice hierarchy associated with the above semi-direct sum of Lie algebras.

4. Application to the Volterra lattice hierarchy

4.1. The Volterra lattice hierarchy

Let us recall the Volterra lattice hierarchy [18, 30]. A discrete spectral problem for the Volterra lattice hierarchy is given by

$$E\phi = U\phi, \qquad U = U(u, \lambda) = \begin{bmatrix} 1 & u \\ \lambda^{-1} & 0 \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{4.1}$$

This is equivalent to

$$\lambda(E^2 - E)\phi_2 = u\phi_2.$$

Upon setting

$$\Gamma = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\geqslant 0} \Gamma_i \lambda^{-i} = \sum_{i\geqslant 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}, \tag{4.2}$$

the discrete stationary zero curvature equation (2.13) gives rise to

$$\begin{cases} a^{(1)} + \lambda^{-1} b^{(1)} - a - uc = 0, \\ ua^{(1)} - b + ua = 0, \\ c^{(1)} - \lambda^{-1} a^{(1)} - \lambda^{-1} a = 0, \\ uc^{(1)} - \lambda^{-1} b = 0, \end{cases}$$

which equivalently leads to

$$\begin{cases} b = u(a^{(1)} + a), \qquad c = \lambda^{-1}(a + a^{(-1)}), \\ a^{(1)} - a + \lambda^{-1}[u^{(1)}(a^{(2)} + a^{(1)}) - u(a + a^{(-1)})] = 0. \end{cases} \tag{4.3}$$

This system can uniquely determine all sets of functions a_i, b_i and c_i, upon choosing

$$a_0 = \tfrac{1}{2}, \qquad a_i|_{u=0} = 0, \qquad i \geqslant 1. \tag{4.4}$$

In particular, the first two sets are

$$\begin{cases} a_0 = \frac{1}{2}, \quad b_0 = u, \quad c_0 = 0; \\ a_1 = -u, \quad b_1 = -u(u^{(1)} + u), \quad c_1 = 1. \end{cases}$$

The compatibility conditions of the matrix discrete spectral problems

$$\begin{cases} E\phi = U\phi, \qquad \phi_t = V^{[m]}\phi, \qquad V^{[m]} = (\lambda^{m+1}\Gamma)_+ + \Delta_m, \\ \Delta_m = \begin{bmatrix} 0 & -b_{m+1} \\ 0 & a_{m+1} + a_{m+1}^{(-1)} \end{bmatrix}, \qquad m \geqslant 0, \end{cases} \tag{4.5}$$

where $(\lambda^{m+1}\Gamma)_+$ denotes the polynomial part of $\lambda^{m+1}\Gamma$ in λ, determine (see, e.g., [18]) the Volterra lattice hierarchy of soliton equations

$$u_{t_m} = K_m = \Phi^m K_0 = u\big(a_{m+1}^{(1)} - a_{m+1}^{(-1)}\big), \qquad K_0 = u(u^{(-1)} - u^{(1)}), \qquad m \geqslant 0, \tag{4.6}$$

where the hereditary recursion operator Φ is given by

$$\Phi = u(1 + E^{-1})(-u^{(1)}E^2 + u)(E - 1)^{-1}u^{-1}. \tag{4.7}$$

Since we have

$$\langle V, U_\lambda \rangle = \mathrm{tr}(VU_\lambda) = \lambda^{-1}a^{(1)}, \qquad \langle V, U_u \rangle = \mathrm{tr}(VU_u) = -\frac{a}{u},$$

where $V = \Gamma U^{-1}$, an application of the trace identity with $\gamma = 0$ in [21] (corresponding to a particular case of (3.8): $\eta_1 = 1$ and $\eta_i = 0, 2 \leqslant i \leqslant 5$) presents the Hamiltonian structures for the Volterra lattice hierarchy:

$$u_{t_m} = J\frac{\delta \mathcal{H}_m}{\delta u}, \qquad J = u(E^{-1} - E)u, \qquad \mathcal{H}_m = \sum_{n\in\mathbb{Z}} \left[-\frac{a_{m+1}}{m+1}\right], \qquad m \geqslant 0. \tag{4.8}$$

4.2. Hierarchy of integrable couplings and their Hamiltonian structure

As in [4], introduce two Lie algebras of 4×4 matrices:

$$G = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \middle| A \in \mathbb{R}[\lambda] \otimes \mathrm{gl}(2) \right\}, \qquad G_c = \left\{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \middle| B \in \mathbb{R}[\lambda] \otimes \mathrm{gl}(2) \right\}, \tag{4.9}$$

where the loop algebra $\mathbb{R}[\lambda] \otimes \mathrm{gl}(2)$ is defined by $\mathrm{span}\{\lambda^n A | n \geqslant 0, A \in \mathrm{gl}(2)\}$, and form a semi-direct sum $\bar{G} = G \Subset G_c$ of these two Lie algebras G and G_c. In this case, G_c is an Abelian ideal of $\bar{G}$. For the Volterra spectral problem (4.1), we define the corresponding enlarged spectral matrix as follows:

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{bmatrix} U & U_a \\ 0 & U \end{bmatrix} \in G \Subset G_c, \qquad U_a = U_a(v) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}, \tag{4.10}$$

where v is a new dependent variable and the enlarged potential $\bar{u}$ reads

$$\bar{u} = (u, v)^T. \tag{4.11}$$

To solve the corresponding enlarged stationary discrete zero curvature equation

$$(E\bar{\Gamma})\bar{U} - \bar{U}\bar{\Gamma} = 0, \tag{4.12}$$

we set

$$\bar{\Gamma} = \begin{bmatrix} \Gamma & \Gamma_a \\ 0 & \Gamma \end{bmatrix}, \qquad \Gamma_a = \Gamma_a(\bar{u}, \lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix}, \tag{4.13}$$

where Γ is a solution to (2.13), defined by (4.2). Then, the enlarged stationary discrete zero curvature equation gives

$$[(E\Gamma_a)U - U\Gamma_a] + [(E\Gamma)U_a - U_a\Gamma] = 0,$$

together with (2.13). This equation equivalently leads to

$$\begin{cases} e^{(1)} + \lambda^{-1} f^{(1)} - e - ug - vc = 0, \\ va^{(1)} + ue^{(1)} - f + ue + va = 0, \\ g^{(1)} - \lambda^{-1} e^{(1)} - \lambda^{-1} e = 0, \\ vc^{(1)} + ug^{(1)} - \lambda^{-1} f = 0. \end{cases}$$

Since the second equation is always satisfied if the last two equations hold, this system is consistent and determines

$$\begin{cases} f = u(e^{(1)} + e) + v(a^{(1)} + a), \ g = \lambda^{-1}(e + e^{(-1)}), \\ e^{(1)} - e + \lambda^{-1}[u^{(1)}(e^{(2)} + e^{(1)}) + v^{(1)}(a^{(2)} + a^{(1)})] - vc = 0. \end{cases} \tag{4.14}$$

Trying a solution

$$e = \sum_{i \geqslant 0} e_i \lambda^{-i}, \qquad f = \sum_{i \geqslant 0} f_i \lambda^{-i}, \qquad g = \sum_{i \geqslant 0} g_i \lambda^{-i}, \tag{4.15}$$

and choosing

$$e_0 = 0, \qquad e_i|_{\bar{u}=0} = 0, \qquad i \geqslant 1, \tag{4.16}$$

we see that all sets of functions e_i, f_i and g_i are uniquely determined. In particular, the first two sets are

$$\begin{cases} e_0 = 0, \quad f_0 = v, \quad g_0 = 0; \\ e_1 = -v, \quad f_1 = -u(v^{(1)} + v) - v(u^{(1)} + u), \quad g_1 = 0. \end{cases}$$

Let us now define

$$\bar{V}^{[m]} = \begin{bmatrix} V^{[m]} & V_a^{[m]} \\ 0 & V^{[m]} \end{bmatrix} \in \bar{G}, \qquad V_a^{[m]} = (\lambda^{m+1}\Gamma_a)_+ + \Delta_{m,a}, \qquad m \geqslant 0, \tag{4.17}$$

where $V^{[m]}$ is defined as in (4.5) and $(\lambda^{m+1}\Gamma_a)_+$ denotes the polynomial part of $\lambda^{m+1}\Gamma_a$ in λ, and choose $\Delta_{m,a}$ as

$$\Delta_{m,a} = \begin{bmatrix} 0 & -f_{m+1} \\ 0 & e_{m+1} + e_{m+1}^{(-1)} \end{bmatrix}, \qquad m \geqslant 0. \tag{4.18}$$

Then, the mth enlarged discrete zero curvature equation

$$\bar{U}_{t_m} = (E\bar{V}^{[m]})\bar{U} - \bar{U}\bar{V}^{[m]}$$

leads to

$$\begin{aligned} v_{t_m} &= f_{m+1} - u\big(e_{m+1} + e_{m+1}^{(-1)}\big) - v\big(a_{m+1} + a_{m+1}^{(-1)}\big) \\ &= u\big(e_{m+1}^{(1)} - e_{m+1}^{(-1)}\big) + v\big(a_{m+1}^{(1)} - a_{m+1}^{(-1)}\big), \end{aligned} \tag{4.19}$$

together with the mth Volterra lattice equation in (4.6). Here in the last equality, we used (4.14). A hierarchy of coupling systems is thus generated for the Volterra lattice hierarchy (4.6):

$$\bar{u}_{t_m} = \begin{bmatrix} u \\ v \end{bmatrix}_{t_m} = \bar{K}_m(u) = \bar{\Phi}^m \bar{K}_0 = \begin{bmatrix} u\big(a_{m+1}^{(1)} - a_{m+1}^{(-1)}\big) \\ u\big(e_{m+1}^{(1)} - e_{m+1}^{(-1)}\big) + v\big(a_{m+1}^{(1)} - a_{m+1}^{(-1)}\big) \end{bmatrix}, \qquad m \geqslant 0, \tag{4.20}$$

in which the first system $\bar{u}_{t_0} = \bar{K}_0$ reads

$$u_{t_0} = u(u^{(-1)} - u^{(1)}), \qquad v_{t_0} = v(u^{(-1)} - u^{(1)}) + u(v^{(-1)} - v^{(1)}),$$

and the hereditary recursion operator $\bar{\Phi}$ is defined by

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 \\ \Phi_c & \Phi \end{bmatrix}, \tag{4.21}$$

where Φ is given by (4.7) and

$$\Phi_c = v(1 + E^{-1})(-u^{(1)}E^2 + u)(E - 1)^{-1}u^{-1} + u(1 + E^{-1})(-v^{(1)}E^2 + v)(E - 1)^{-1}u^{-1} \\ - u(1 + E^{-1})(-u^{(1)}E^2 + u)(E - 1)^{-1}vu^{-2}.$$

To construct Hamiltonian structures of these integrable couplings by using the discrete variational identity (2.7), we consider a non-degenerate bilinear form on $\bar{G} = G \in G_c$ defined by (3.8) under the selection of

$$\eta_1 = \eta_2 = \eta_3 = 1, \qquad \eta_4 = \eta_5 = 0. \tag{4.22}$$

Then, a direct computation tells

$$\langle \bar{V}, \bar{U}_\lambda \rangle = -\frac{vb + u^2e - uf}{\lambda u^2} = \lambda^{-1}e^{(1)}, \tag{4.23}$$

$$\langle \bar{V}, \bar{U}_u \rangle = \frac{va}{u^2} - \frac{e}{u}, \qquad \langle \bar{V}, \bar{U}_v \rangle = -\frac{a}{u}, \tag{4.24}$$

where $\bar{U}$ is defined by (4.10) and $\bar{V} = \bar{\Gamma}\bar{U}^{-1}$ with $\bar{\Gamma}$ being defined by (4.13). Now an application of the discrete variational identity (2.7) with $\gamma = 0$ engenders

$$\frac{\delta}{\delta \bar{u}}\bar{\mathcal{H}}_m = \left(\frac{va_{m+1}}{u^2} - \frac{e_{m+1}}{u}, -\frac{a_{m+1}}{u}\right)^T, \qquad \bar{\mathcal{H}}_m = \sum_{n\in\mathbb{Z}}\left[-\frac{e_{m+1}}{m+1}\right], \qquad m \geqslant 0. \tag{4.25}$$

Consequently, we obtain the Hamiltonian structure for the hierarchy of integrable couplings in (4.21):

$$\bar{u}_{t_m} = \bar{J}\frac{\delta}{\delta \bar{u}}\bar{\mathcal{H}}_m, \qquad m \geqslant 0, \tag{4.26}$$

where the Hamiltonian functionals $\bar{\mathcal{H}}_m, m \geqslant 0$, are given in (4.25) and the Hamiltonian operator J is determined by

$$\bar{J} = \begin{bmatrix} 0 & u(E^{-1} - E)u \\ u(E^{-1} - E)u & u(E^{-1} - E)v + v(E^{-1} - E)u \end{bmatrix}. \tag{4.27}$$

Now, noting that $\bar{\Phi}\bar{J} = \bar{J}\bar{\Phi}^\dagger$, it follows that each Hamiltonian coupling system in the above hierarchy (4.26) possesses infinitely many conserved functionals $\{\bar{\mathcal{H}}_n\}_{n=0}^\infty$ and infinitely many symmetries $\{\bar{K}_n\}_{n=0}^\infty$, which commute with each other:

$$\{\bar{\mathcal{H}}_k, \bar{\mathcal{H}}_l\} = 0, \qquad [\bar{K}_k, \bar{K}_l] = 0, \qquad k, l \geqslant 0. \tag{4.28}$$

5. Concluding remarks

The trace identity has been generalized to discrete spectral problems associated with non-semisimple Lie algebras or, equivalently, Lie algebras possessing degenerate Killing forms. The constant γ in the discrete variational identity was determined precisely by the corresponding solution to the stationary discrete zero curvature equation. The resulting discrete variational identity was applied to a class of semi-direct sums of Lie algebras in the Volterra lattice case and furnished Hamiltonian structures for the associated integrable couplings of the Volterra lattice hierarchy.

The proof of the variational identity involves, in an essential way, the invariance property of bilinear forms. We remark that there is a difference between the continuous and discrete cases. In the continuous case, we only require the invariance property of bilinear forms under the Lie bracket of the underlying algebras, but in the discrete case, we require the invariance property of bilinear forms under the multiplication. In theory, the invariance property under the multiplication is stronger than the invariance property under the Lie bracket, because

$$\langle A, [B, C]\rangle = \langle [A, B], C\rangle$$

is just a consequence of

$$\langle A, BC\rangle = \langle AB, C\rangle.$$

One such example is the Lie algebra

$$\bar{G} = \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \middle| a_i \in \mathbb{R}, 1 \leqslant i \leqslant 3 \right\},$$

for which the corresponding matrix F is

$$F = \begin{bmatrix} \eta_1 & 0 & \eta_2 \\ 0 & 0 & 0 \\ \eta_2 & 0 & \eta_3 \end{bmatrix},$$

where η_i, $1 \leqslant i \leqslant 3$, are arbitrary constants. Though all associated bilinear forms on this three-dimensional Lie algebra

$$\langle A, B\rangle = (\eta_1 a_1 + \eta_2 a_3) b_1 + (\eta_2 a_2 + \eta_3 a_3) b_3, \qquad A = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix}$$

are invariant under the Lie bracket of matrices, they are varied under the matrix multiplication. Nevertheless, we see that both invariance properties are equivalent to each other in the case of the semi-direct sum of matrix Lie algebras in section 3. We also remark that more choices of non-degenerate bilinear forms in (3.8) could lead to more Hamiltonian integrable couplings for the Volterra lattice hierarchy, and similar applications could be made for other lattice hierarchies [31–36].

To conclude, the discrete variational identity ubiquitously exists in discrete spectral problems associated with both semisimple Lie algebras and non-semisimple Lie algebras. It brings us a powerful tool for exploring Hamiltonian structures of discrete soliton equations.

References

[1] Ma W X and Fuchssteiner B 1996 *Chaos Solitons Fractals* **7** 1227
[2] Ma W X 2000 *Methods Appl. Anal.* **7** 21
[3] Ma W X, Xu X X and Zhang Y F 2006 *Phys. Lett.* A **351** 125
[4] Ma W X, Xu X X and Zhang Y F 2006 *J. Math. Phys.* **47** 053501
[5] Ma W X and Zhou R G 2002 *Physica* A **296** 60
[6] Ma W X 2003 *Phys. Lett.* A **316** 72
[7] Guo F K and Zhang Y F 2004 *Chaos Solitons Fractals* **19** 1207
[8] Xia T C, Yu F J and Zhang Y 2004 *Physica* A **343** 238
[9] Ma W X 2005 *J. Math. Phys.* **46** 033507
[10] Ding H Y, Sun Y P and Xu X X 2006 *Chaos Solitons Fractals* **30** 227

[11] Ma W X and Zhou R G 2002 *Chin. Ann. Math. Ser.* B **23** 373
[12] Jacobson N 1962 *Lie Algebras* (New York: Interscience)
[13] Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* C **71** 313
[14] Calogero F 2001 *Classical Many-Body Problems Amenable to Exact Treatments (Lecture Notes in Physics, New Series m: Monographs* vol 66*)* (Berlin: Springer)
[15] Clarkson P A and Winternitz P 1999 *The Painlevé Property (CRM Ser. Math. Phys.)* (New York: Springer) pp 591–660
[16] Basarab-Horwath P, Lahno V and Zhdanov R 2001 *Acta Appl. Math.* **69** 43
[17] Hurtubise J C and Markman E 2001 *Commun. Math. Phys.* **223** 533
[18] Ma W X and Fuchssteiner B 1999 *J. Math. Phys.* **40** 2400
[19] Ma W X and Xu X X 2004 *J. Phys. A: Math. Gen.* **37** 1323
[20] Ma W X 1993 *Phys. Lett.* A **179** 179
[21] Tu G Z 1990 *J. Phys. A: Math. Gen.* **23** 3903
[22] Ma W X 1992 *Chin. Ann. Math. Ser.* A **13** 115
[23] Luks E 1970 *J. Algebra* **15** 280
[24] O'Raifeartaigh L 1965 *Phys. Rev.* B **139** 1052
[25] Ma W X and Chen M 2006 *J. Phys. A: Math. Gen.* **39** 10787
[26] Guo F K and Zhang Y F 2005 *J. Phys. A: Math. Gen.* **38** 8537
[27] Antonowicz M, Fordy A P and Liu Q P 1991 *Nonlinearity* **4** 669
[28] Hu X B 1994 *J. Phys. A: Math. Gen.* **27** 2497
[29] Zhang Y F 2007 *Mod. Phys. Lett.* B **21** 37
[30] Zhang H W, Tu G Z, Oevel W and Fuchssteiner B 1991 *J. Math. Phys.* **32** 1908
[31] Ragnisco O and Santini P M 1990 *Inverse Problems* **6** 441
[32] Kodama Y and Ye J 1996 *Commun. Math. Phys.* **178** 765
[33] Tsuchida T, Ujino H and Wadati M 1999 *J. Phys. A: Math. Gen.* **32** 2239
[34] Zhang D J and Chen D Y 2002 *J. Phys. A: Math. Gen.* **35** 7225
[35] Maruno K and Biondini G 2004 *J. Phys. A: Math. Gen.* **37** 1819
[36] Zhang J B and Zhou R G 2005 *J. Xuzhou Norm. Univ. Nat. Sci. Ed.* **23** 15