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A spectral problem based on so(3, $\mathbb{R}$) and its associated commuting soliton equations

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We present a spectral problem, based on the three-dimensional real special orthogonal Lie algebra so(3, $\mathbb{R}$), and construct a hierarchy of commuting bi-Hamiltonian soliton equations by zero curvature equations associated with the spectral problem. An illustrative example of soliton equations is computed, together with its associated bi-Hamiltonian structure. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4826104]

I. INTRODUCTION

It is known that soliton hierarchies such as the Korteweg-de Vries hierarchy, the Ablowitz-Kaup-Newell-Segur hierarchy, and the Kaup-Newell hierarchy are generated from spectral problems associated with matrix Lie algebras (see, e.g., Refs. 1–6). The so-called trace identity is used to find Hamiltonian structures of soliton hierarchies.[7,8] When associated matrix Lie algebras are non-semisimple, we obtain integrable couplings,[9,10] and the variational identity provides a basic tool for generating their Hamiltonian structures.[11,12] If there exist bi-Hamiltonian structures,[13] the corresponding Hamiltonian pairs often generate hereditary recursion operators (see, e.g., Refs. 6, 14, and 15).

Let us state a standard procedure for building soliton hierarchies (see, e.g., Refs. 7 and 8). The beginning is to introduce a spatial spectral problem

$$\phi_x = U\phi,\ U = U(u,\lambda) \in \tilde{\mathfrak{g}}, \tag{1.1}$$

where u is a dependent variable and λ is the spectral parameter, based on a matrix loop algebra $\tilde{\mathfrak{g}}$ associated with a given matrix Lie algebra $\mathfrak{g}$, often being simple or semisimple. We then take a solution

$$W = W(u,\lambda) = \sum_{i\geq 0} W_{0,i}\lambda^{-i},\ W_{0,i} \in \mathfrak{g},\ i \geq 0, \tag{1.2}$$

to the stationary zero curvature equation

$$W_x = [U, W]. \tag{1.3}$$

Further, introduce the Lax matrices

$$V^{[m]} = V^{[m]}(u,\lambda) = (\lambda^m W)_+ + \Delta_m \in \tilde{\mathfrak{g}},\ m \geq 0, \tag{1.4}$$

where P_+ denotes the polynomial part of P in λ, to formulate the temporal spectral problems

$$\phi_{t_m} = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi,\ m \geq 0. \tag{1.5}$$

The introduction of the modification terms $\Delta_m \in \tilde{\mathfrak{g}}$ aims at guaranteeing that the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,\ m \geq 0, \tag{1.6}$$

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will engender a hierarchy of soliton equations:

$$u_{t_m} = K_m(u),\ m \ge 0. \tag{1.7}$$

Such a soliton hierarchy usually possesses Hamiltonian structures

$$u_{t_m} = K_m(u) = J\frac{\delta \mathcal{H}_m}{\delta u},\ m \ge 0. \tag{1.8}$$

Those Hamiltonian functionals $\mathcal{H}_m$'s can often be generated by applying the trace identity,[7,8]

$$\frac{\delta}{\delta u}\int \mathrm{tr}(\frac{\partial U}{\partial \lambda}W)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\,\mathrm{tr}(\frac{\partial U}{\partial u}W),\ \gamma = \frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|, \tag{1.9}$$

or more generally, the variational identity,[11,12]

$$\frac{\delta}{\delta u}\int \langle\frac{\partial U}{\partial \lambda}, W\rangle\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\,\langle\frac{\partial U}{\partial u}, W\rangle,\ \gamma = \frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|, \tag{1.10}$$

where $\langle\cdot\,,\,\cdot\rangle$ is a non-degenerate, symmetric, and ad-invariant bilinear form on the underlying matrix loop algebra $\tilde{\mathfrak{g}}$.[11,12]

When a matrix loop algebra $\tilde{\mathfrak{g}}$ is semisimple, the trace identity will usually work for generating Hamiltonian structures. When $\tilde{\mathfrak{g}}$ is non-semisimple, we need to use the variational identity since the trace identity is not generally valid, and the bilinear form $\langle\cdot\,,\,\cdot\rangle$ in the variational identity (1.10) must not be of the Killing type. If $\langle A, A\rangle$ is positive for an arbitrary non-zero matrix $A \in \tilde{\mathfrak{g}}$, then $(\tilde{\mathfrak{g}}, \langle\cdot,\cdot\rangle)$ is a quadratic Lie algebra.

We will make use of the three-dimensional real special orthogonal Lie algebra $\mathrm{so}(3, \mathbb{R})$, consisting of 3×3 skew-symmetric real matrices. This Lie algebra is simple and has a basis

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.11}$$

whose commutator relations are

$$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2. \tag{1.12}$$

The derived algebra $[\mathrm{so}(3, \mathbb{R}), \mathrm{so}(3, \mathbb{R})]$ is $\mathrm{so}(3, \mathbb{R})$ itself, and $\mathrm{so}(3, \mathbb{R})$ is one of the only two three-dimensional real Lie algebras with a three-dimensional derived algebra. The other one is $\mathrm{sl}(2, \mathbb{R})$, which has been widely used in soliton theory.

The matrix loop algebra we will adopt in what follows is

$$\widetilde{\mathrm{so}}(3, \mathbb{R}) = \{M \in \mathrm{so}(3, \mathbb{R})\,|\,\text{entries of } M \text{ - Laurent series in } \lambda\}. \tag{1.13}$$

The algebra $\widetilde{\mathrm{so}}(3, \mathbb{R})$ contains matrices of the form

$$\lambda^m e_1 + \lambda^n e_2 + \lambda^l e_3$$

with arbitrary integers m, n, l. This matrix loop algebra lays a foundation for our study of soliton equations.

In this paper, we would like to introduce a spectral problem, based on the matrix loop algebra $\widetilde{\mathrm{so}}(3, \mathbb{R})$, and construct a hierarchy of commuting bi-Hamiltonian soliton equations from associated zero curvature equations. The corresponding Hamiltonian structures will be furnished by using the trace identity, thereby all equations in the resulting soliton hierarchy possessing infinitely many commuting symmetries and conservation laws. The first nonlinear system will be worked out, together with its associated bi-Hamiltonian structure. A few concluding remarks will be given in Sec. IV.

II. SPECTRAL PROBLEM AND SOLITON EQUATIONS

To construct a hierarchy of soliton equations, let us introduce a matrix spectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi,\ u = \begin{bmatrix} p \\ q \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \tag{2.1}$$

with the spectral matrix U being chosen as

$$U = \lambda^2 e_1 + \lambda p e_2 + \lambda q e_3 = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 \\ \lambda q & 0 & -\lambda p \\ \lambda^2 & \lambda p & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(3,\mathbb{R}). \tag{2.2}$$

This spectral problem is of the same type as the Kaup-Newell one,[16] but its underlying loop algebra is different. Then, we solve the stationary zero curvature Eq. (1.3), and it becomes

$$\begin{cases} a_x = \lambda p c - \lambda q b \\ b_x = -\lambda^2 c + \lambda q a \\ c_x = -\lambda p a + \lambda^2 b \end{cases}, \tag{2.3}$$

if W is assumed to be of the form

$$W = a e_1 + b e_2 + c e_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(3,\mathbb{R}). \tag{2.4}$$

Further letting

$$a = \sum_{i\ge 0} a_i \lambda^{-2i},\ b = \sum_{i\ge 0} b_i \lambda^{-2i-1},\ c = \sum_{i\ge 0} c_i \lambda^{-2i-1},\ i \ge 0, \tag{2.5}$$

and taking the initial values

$$a_0 = 1,\ b_0 = p,\ c_0 = q, \tag{2.6}$$

which are required by

$$-c_0 + q a_0 = 0,\ -p a_0 + b_0 = 0,\ a_{0,x} = p c_0 - q b_0,$$

the system (2.3) leads equivalently to

$$\begin{cases} a_{i+1,x} = -(p b_{i,x} + q c_{i,x}), \\ b_{i+1} = c_{i,x} + p a_{i+1}, \qquad i \ge 0, \\ c_{i+1} = -b_{i,x} + q a_{i+1}, \end{cases} \tag{2.7}$$

the first of which is because we have

$$p b_x + q c_x = p(-\lambda^2 c + \lambda q a) + q(-\lambda p a + \lambda^2 b) = -\lambda^2 (pc - qb) = -\lambda a_x.$$

We impose the following conditions on constants of integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0,\ i \ge 1, \tag{2.8}$$

to determine the sequence of $\{a_i, b_i, c_i | \, i \geq 1\}$ uniquely. This way, the first few sets can be evaluated as follows:

$$\begin{aligned} a_1 &= -\tfrac{1}{2}(p^2+q^2), \\ b_1 &= -\tfrac{1}{2}p(p^2+q^2)+q_x, \\ c_1 &= -\tfrac{1}{2}(p^2+q^2)q - p_x, \\ a_2 &= \tfrac{3}{8}(p^2+q^2)^2 + p_x q - p q_x, \\ b_2 &= \tfrac{3}{8}p(p^2+q^2)^2 - p_{xx} - \tfrac{3}{2}p^2 q_x - \tfrac{3}{2}q^2 q_x, \\ c_2 &= \tfrac{3}{8}(p^2+q^2)^2 q + \tfrac{3}{2}p^2 p_x + \tfrac{3}{2}p_x q^2 - q_{xx}. \end{aligned}$$

Now, based on the recursion relations in (2.7) and the structure of the spectral matrix U in (2.2), we introduce

$$V^{[m]} = \lambda(\lambda^{2m+1}W)_+ = (\lambda^{2m+2}W)_+ - a_{m+1}e_1, \ m \geq 0, \tag{2.9}$$

and consequently, the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \ m \geq 0, \tag{2.10}$$

engender a hierarchy of soliton equations

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} \\ c_{m,x} \end{bmatrix} = \Phi^m \begin{bmatrix} b_{0,x} \\ c_{0,x} \end{bmatrix} = \Phi^m \begin{bmatrix} p_x \\ q_x \end{bmatrix}, \ m \geq 0, \tag{2.11}$$

where Φ can be determined by the recursion relations in (2.7):

$$\Phi = \begin{bmatrix} -\partial p \partial^{-1} p & \partial - \partial p \partial^{-1} q \\ -\partial - \partial q \partial^{-1} p & -\partial q \partial^{-1} q \end{bmatrix}, \ \partial = \frac{\partial}{\partial x}. \tag{2.12}$$

We will show that Φ is a hereditary recursion operator for the soliton hierarchy (2.11).

III. HAMILTONIAN STRUCTURE AND LIOUVILLE INTEGRABILITY

A. Hamiltonian structure

To furnish Hamiltonian structures, we use the trace identity (1.9) (or more generally the variational identity (1.10)). It is direct to see

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & -q & -2\lambda \\ q & 0 & -p \\ 2\lambda & p & 0 \end{bmatrix}, \ \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix}, \ \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so we have

$$\mathrm{tr}(W\frac{\partial U}{\partial \lambda}) = -4\lambda a - 2(pb+qc), \ \mathrm{tr}(W\frac{\partial U}{\partial p}) = -2\lambda b, \ \mathrm{tr}(W\frac{\partial U}{\partial q}) = -2\lambda c. \tag{3.1}$$

Now, the trace identity (1.9) in this case gives

$$\frac{\delta}{\delta u}\int (2\lambda a + pb + qc)\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} \lambda b \\ \lambda c \end{bmatrix}.$$

Balancing coefficients of all powers of λ in the equality tells

$$\frac{\delta}{\delta u}\int (2a_{m+1} + pb_m + qc_m)\, dx = (\gamma - 2m)\begin{bmatrix} b_m \\ c_m \end{bmatrix}, \ m \geq 0.$$

Checking a particular case with $m = 1$ yields $\gamma = 0$, and thus we obtain

$$\frac{\delta}{\delta u}\int(-\frac{2a_{m+1}+pb_m+qc_m}{2m})\,dx = \begin{bmatrix} b_m \\ c_m \end{bmatrix},\ m \geq 1. \tag{3.2}$$

Now, it follows that the soliton hierarchy (2.11) has the Hamiltonian structures,

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} \\ c_{m,x} \end{bmatrix} = J\begin{bmatrix} b_m \\ c_m \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u},\ m \geq 0, \tag{3.3}$$

with the Hamiltonian operator

$$J = \begin{bmatrix} \partial & 0 \\ 0 & \partial \end{bmatrix}, \tag{3.4}$$

and the Hamiltonian functionals

$$\mathcal{H}_0 = \int \frac{1}{2}(p^2+q^2)\,dx,\ \mathcal{H}_m = \int(-\frac{2a_{m+1}+pb_m+qc_m}{2m})\,dx,\ m \geq 1. \tag{3.5}$$

These functionals generate infinitely many conservation laws for each soliton system in the hierarchy (2.11). Such differential polynomial type conservation laws can also be computed systematically by computer algebra systems (see, e.g., Ref. 17) or from a Riccati equation from the underlying spectral problem (see, e.g., Refs. 5 and 6).

B. Liouville integrability

It is a direct but lengthy computation to verify by Maple that J defined by (3.4) and

$$M = \Phi J = \begin{bmatrix} -\partial p \partial^{-1} p \partial & \partial^2 - \partial p \partial^{-1} q \partial \\ -\partial^2 - \partial q \partial^{-1} p \partial & -\partial q \partial^{-1} q \partial \end{bmatrix}, \tag{3.6}$$

constitute a Hamiltonian pair (see Refs. 13 and 14 for details), i.e., any linear combination N of J and M satisfies

$$\int K^T N'(u)[NS]T\,dx + \text{cycle}(K, S, T) = 0 \tag{3.7}$$

for all vector fields K, S, and T. This implies that the operator Φ defined by (2.12) is hereditary (see Ref. 18 for definition), i.e., it satisfies

$$\Phi'(u)[\Phi K]S - \Phi\Phi'(u)[K]S = \Phi'(u)[\Phi S]K - \Phi\Phi'(u)[S]K \tag{3.8}$$

for all vector fields K and S.

The condition (3.8) for the hereditary operators is equivalent to

$$L_{\Phi K}\Phi = \Phi L_K \Phi, \tag{3.9}$$

where K is an arbitrary vector field. The Lie derivative $L_K\Phi$ here is defined by

$$(L_K\Phi)S = \Phi[K, S] - [K, \Phi S],$$

where $[\cdot,\ \cdot]$ is the Lie bracket of vector fields. Note that an autonomous operator $\Phi = \Phi(u, u_x, \cdots)$ is a recursion operator of a given evolution equation $u_t = K = K(u)$ if and only if Φ needs to satisfy

$$L_K\Phi = 0. \tag{3.10}$$

Obviously, the operator Φ defined by (2.12) satisfies

$$L_{K_0}\Phi = 0,\ K_0 = \begin{bmatrix} p_x \\ q_x \end{bmatrix}, \tag{3.11}$$

and thus

$$L_{K_m}\Phi = L_{\Phi K_{m-1}}\Phi = \Phi L_{K_{m-1}}\Phi = 0,\ m \geq 1, \tag{3.12}$$

where the K_m's are defined by (2.11). This implies that the operator Φ defined by (2.12) is a common hereditary recursion operator for the soliton hierarchy (2.11). There are also a few direct symbolic algorithms for computing recursion operators of nonlinear partial differential equations by computer algebra systems (see, e.g., Ref. 19).

It now follows that the soliton hierarchy (2.11) is bi-Hamiltonian,

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u},\ m \geq 1, \tag{3.13}$$

where J, M, and $\mathcal{H}_m$'s are defined by (3.4), (3.6), and (3.5), respectively, and thus, the hierarchy is Liouville integrable, i.e., it possesses infinitely many commuting symmetries and conservation laws. Particularly, we have the Abelian symmetry algebra,

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0,\ k, l \geq 0, \tag{3.14}$$

and the two Abelian algebras of conserved functionals,

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int \Big(\frac{\delta \mathcal{H}_k}{\delta u}\Big)^T J \frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0,\ k, l \geq 0, \tag{3.15}$$

and

$$\{\mathcal{H}_k, \mathcal{H}_l\}_M = \int \Big(\frac{\delta \mathcal{H}_k}{\delta u}\Big)^T M \frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0,\ k, l \geq 0. \tag{3.16}$$

The first nonlinear integrable system in the hierarchy (2.11) is as follows:

$$u_{t_1} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 = \begin{bmatrix} q_{xx} - \frac{3}{2}p^2 p_x - \frac{1}{2}p_x q^2 - pqq_x \\ -p_{xx} - pp_x q - \frac{1}{2}p^2 q_x - \frac{3}{2}q^2 q_x \end{bmatrix} = J\frac{\delta \mathcal{H}_1}{\delta u} = M\frac{\delta \mathcal{H}_0}{\delta u}, \tag{3.17}$$

where the Hamiltonian functional $\mathcal{H}_0$ is defined as in (3.5) and $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 = \int [-\frac{1}{8}(p^2 + q^2)^2 + \frac{1}{2}(pq_x - p_x q)]\, dx. \tag{3.18}$$

This is different from the Kaup-Newell system of nonlinear derivative Schrödinger equations in Ref. 16.

IV. CONCLUDING REMARKS

Based on the real loop algebra $\widetilde{\text{so}}(3, \mathbb{R})$, we introduced a spectral problem and generated a hierarchy of soliton equations from the associated zero curvature equations. The Liouville integrability of the resulting soliton equations has been shown upon furnishing a bi-Hamiltonian formulation by the trace identity.

The real Lie algebra of the special orthogonal group, so$(3, \mathbb{R})$, is not isomorphic to the real Lie algebra sl$(2, \mathbb{R})$ over the real field $\mathbb{R}$, and thus the newly presented soliton hierarchy (2.11) is not gauge equivalent to the Kaup-Newell soliton hierarchy[16] over the real field $\mathbb{R}$. The main difference between the presented soliton hierarchy and the Kaup-Newell soliton hierarchy[16] is that the first Hamiltonian operators are different, which are

$$J = \begin{bmatrix} \partial & 0 \\ 0 & \partial \end{bmatrix},\ J = \begin{bmatrix} 0 & \partial \\ \partial & 0 \end{bmatrix},$$

respectively. Together with the second Hamiltonian operators,

$$M = \begin{bmatrix} -\partial p \partial^{-1} p \partial & \partial^2 - \partial p \partial^{-1} q \partial \\ -\partial^2 - \partial q \partial^{-1} p \partial & -\partial q \partial^{-1} q \partial \end{bmatrix},\ M = \frac{1}{2}\begin{bmatrix} -\partial p \partial^{-1} p \partial & \partial^2 - \partial p \partial^{-1} q \partial \\ -\partial^2 - \partial q \partial^{-1} p \partial & -\partial q \partial^{-1} q \partial \end{bmatrix},$$

they generate different hereditary recursion operators,

$$\Phi = \begin{bmatrix} -\partial p \partial^{-1} p & \partial - \partial p \partial^{-1} q \\ -\partial - \partial q \partial^{-1} p & -\partial q \partial^{-1} q \end{bmatrix}, \ \Phi = \frac{1}{2} \begin{bmatrix} \partial - \partial p \partial^{-1} q & -\partial p \partial^{-1} p \\ -\partial q \partial^{-1} q & -\partial - \partial q \partial^{-1} p \end{bmatrix},$$

respectively (see, e.g., Ref. 20 for the recursion operator for the Kaup-Newell hierarchy). The loop algebra so(3, $\mathbb{R}$) has also been used to generate a soliton hierarchy with a similar spectral problem to the AKNS spectral problem on sl(2, $\mathbb{R}$).[21] This algebra can also be applied to generalizations of other typical soliton hierarchies such as the Levi and Dirac hierarchies.

There is recently a growing interest in soliton hierarchies generating from spectral problems associated with non-semisimple Lie algebras. Various examples of bi-integrable couplings and tri-integrable couplings bring us inspiring thoughts and ideas to classify integrable systems with multi-components.[22] Multi-integrable couplings generate even more diverse recursion operators in block matrix form. Clearly, mathematical structures behind integrable couplings are indeed rich and interesting, though complicated.[12,22]

It is known that Hamiltonian structures exist for the perturbation equations,[23,24] but there is no guarantee that there will exist non-degenerate bilinear forms required in the variational identity on non-semisimple matrix Lie algebras and one does not know how to generate Hamiltonian structures of integrable couplings generally.[25,26] It is an interesting question to see when Hamiltonian structures can exist for bi- or tri-integrable couplings, based on non-semisimple matrix loop algebras. To answer this, one should generalize the variational identity.

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