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Riemann-Hilbert problems and soliton solutions of nonlocal real reverse-spacetime mKdV equations

Wen-Xiu Ma [a,b,c,d,*]

[a] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China*
[b] *Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[c] *Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA*
[d] *School of Mathematical and Statistical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa*

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A B S T R A C T

We would like to analyze a kind of nonlocal reverse-spacetime integrable PT-symmetric multicomponent modified Korteweg-de Vires (mKdV) equations by making a group of nonlocal reductions, and establish their associated Riemann-Hilbert problems which determine generalized Jost solutions of higher-order matrix spectral problems. The Sokhotski-Plemelj formula is used to transform the associated Riemann-Hilbert problems into Gelfand-Levitan-Marchenko type integral equations. The Riemann-Hilbert problems in the reflectionless case are solved explicitly, and the resulting formulation of solutions enables us to present solitons to the nonlocal reverse-spacetime integrable PT-symmetric mKdV equations.

1. Introduction

Nonlocal integrable equations have received increasing attention in recent years, and they can be classified into reverse-space, reverse-time and reverse-spacetime types. Notably, a couple of scalar nonlocal nonlinear Schrödinger (NLS) equations and modified Korteweg-de Vires (mKdV) equations have been identified as significant models for understanding nonlocal nonlinear physical phenomena [1,3]. On one hand, the inverse scattering technique has been successfully used to solve those nonlocal nonlinear equations, under either zero or nonzero boundary conditions [2,5,26]. On the other hand, Darboux transformations [18,21,17,36] and the Hirota bilinear method [14,15] are shown to be powerful in constructing their N-soliton solutions. Some multicomponent [26,35] and higher dimensional [8] generalizations of nonlocal integrable equations have also been proposed and studied. Such nonlocal nonlinear integrable equations share the PT symmetry – the invariance under the parity-time transformation $(x \to -x,\ t \to -t,\ i \to -i)$.

* Correspondence to: Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA.
E-mail address: mawx@cas.usf.edu.

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Riemann-Hilbert problems have become a pretty hot topic in soliton theory and are used to generate soliton solutions to integrable equations, closely connected with the inverse scattering transforms and Darboux transformations. Many integrable equations, both local and nonlocal, have been studied by analyzing associated Riemann-Hilbert problems systematically. Among illustrative examples are the multiple wave interaction equations [34], the general coupled nonlinear Schrödinger equations [37], the Harry Dym equation [38], the generalized Sasa-Satsuma equation [11], multicomponent mKdV equations [23], the nonlocal reverse-space scalar NLS equation [39] and nonlocal reverse-spacetime multicomponent NLS equations [26].

In this paper, we would like to propose and analyze a class of nonlocal reverse-spacetime integrable PT-symmetric multicomponent mKdV equations by nonlocal reductions for the multicomponent AKNS spectral problems, and develop their associated Riemann-Hilbert problems, which generate soliton solutions. One of the nonlocal integrable mKdV equations that will be analyzed is

$$\begin{cases} p_{1,t}(x,t) = p_{1,xxx}(x,t) + 3(p_1^2)_x(x,t)p_1(-x,-t) + 3(p_1p_2)_x(x,t)p_2(-x,-t), \\ p_{2,t}(x,t) = p_{2,xxx}(x,t) + 3(p_1p_2)_x(x,t)p_1(-x,-t) + 3(p_2^2)_x(x,t)p_2(-x,-t). \end{cases} \tag{1.1}$$

The rest of the paper is structured as follows. In Section 2, we generate nonlocal reverse-spacetime PT-symmetric mKdV equations from the multicomponent AKNS mKdV equations by making a group of nonlocal reductions. In Section 3, we establish associated Riemann-Hilbert problems to determine generalized Jost solutions by exploring a property of eigenfunctions, and transform the resulting Riemann-Hilbert problems into systems of Gelfand-Levitan-Marchenko type integral equations by the Sokhotski-Plemelj formula. In Section 4, we solve the Riemann-Hilbert problems with the identity jump matrix and accordingly present N-soliton solutions. The final section gives a few concluding remarks.

2. Nonlocal reductions and nonlocal mKdV equations

Let n be an arbitrary natural number. Assume that λ denotes a spectral parameter, and u, a $2n$-component potential

$$u = u(x,t) = (p, q^T)^T,\ p = (p_1, p_2, \cdots, p_n),\ q = (q_1, q_2, \cdots, q_n)^T. \tag{2.1}$$

The multicomponent local mKdV equations are generated from the AKNS matrix spectral problems with multiple potentials (see, e.g., [30]):

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ -i\phi_t = V\phi = V(u,\lambda)\phi, \tag{2.2}$$

where the Lax pair reads

$$U = \lambda\Lambda + P,\ V = \lambda^3\Omega + Q,\ \Lambda = \mathrm{diag}(\alpha_1, \alpha_2 I_n),\ \Omega = \mathrm{diag}(\beta_1, \beta_2 I_n), \tag{2.3}$$

where I_n is the identity matrix of size n, and $\alpha_1, \alpha_2, \beta_1$ and β_2 are arbitrary constants. The involved other two square matrices of size $n+1$ are

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.4}$$

and

$$Q = \frac{\beta}{\alpha}\lambda^2 \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2}\lambda \begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix} - \frac{\beta}{\alpha^3} \begin{bmatrix} i(pq_x - p_xq)) & p_{xx} + 2pqp \\ q_{xx} + 2qpq & i(qp_x - q_xp) \end{bmatrix}, \tag{2.5}$$

where $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$. Obviously, when there are just a pair of nonzero potentials, e.g., p_n and q_n, the matrix spectral problems in (2.2) become the standard AKNS ones [4]. It is direct to see that the compatibility condition of the spectral problems in (2.2), i.e., the zero curvature equation

$$U_t - V_x + i[U,V] = 0, \tag{2.6}$$

presents the multicomponent local mKdV equations

$$\begin{cases} p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp), \\ q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \end{cases} \tag{2.7}$$

which is bi-Hamiltonian, and thus, typically integrable [30].

To derive nonlocal integrable counterparts, let us introduce a group of specific nonlocal reductions for the spectral matrix U:

$$U^T(-x,-t,\lambda) = CU(x,t,\lambda)C^{-1}, \tag{2.8}$$

in which T stands for the matrix transpose and the constant matrix C is a block matrix

$$C = \begin{bmatrix} 1 & 0 \\ 0 & \Sigma \end{bmatrix}, \tag{2.9}$$

where Σ is an arbitrary invertible and symmetric matrix. By (2.3), each nonlocal reduction means that

$$P^T(-x,-t) = CP(x,t)C^{-1}, \tag{2.10}$$

thereby leading to the potential reverse-spacetime reduction

$$q(x,t) = \Sigma^{-1}p^T(-x,-t). \tag{2.11}$$

It follows directly from this potential reduction that

$$\begin{cases} V^T(-x,-t,\lambda) = CV(x,t,\lambda)C^{-1}, \\ Q^T(-x,-t,\lambda) = CQ(x,t,\lambda)C^{-1}. \end{cases}$$

Now, we can see that all those imply that the nonlocal reductions in (2.8) do not cause any conflict in the zero curvature equation (2.6). Therefore, the multicomponent local mKdV equations (2.7) reduce to the following nonlocal real reverse-spacetime multicomponent mKdV equations:

$$p_t(x,t) = -\frac{\beta}{\alpha^3}[p_{xxx}(x,t) + 3p(x,t)\Sigma^{-1}p^T(-x,-t)p_x(x,t) + 3p_x(x,t)\Sigma^{-1}p^T(-x,-t)p(x,t)], \tag{2.12}$$

where Σ is an arbitrary invertible and symmetric matrix and p is a vector of potentials defined as in (2.1). Two difficulties in solving those nonlinear systems are the reverse-spacetime nonlocality and the higher-dimension of the systems.

When $n = 1$, we can obtain two scalar examples (see, e.g., [3,18]):

$$p_t(x,t) = p_{xxx}(x,t) + 6\sigma p(x,t)p(-x,-t)p_x(x,t),\ \sigma = \pm 1. \tag{2.13}$$

When $n = 2$, we can have systems of two equations:

$$\begin{cases} p_{1,t}(x,t) = p_{1,xxx}(x,t) + 3(p_1^2)_x(x,t)r_1(-x,-t) + 3(p_1p_2)_x(x,t)r_2(-x,-t), \\ p_{2,t}(x,t) = p_{2,xxx}(x,t) + 3(p_1p_2)_x(x,t)r_1(-x,-t) + 3(p_2^2)_x(x,t)r_2(-x,-t), \end{cases} \tag{2.14}$$

where

$$r_1(x,t) = c_1p_1(x,t) + c_2p_2(x,t),\ r_2(x,t) = c_2p_1(x,t) + c_3p_2(x,t), \tag{2.15}$$

the c_i's being constants satisfying $c_2^2 \neq c_1c_3$. Note that a linear transformation of p_1 and p_2 can remove the terms involving c_2, but such a general form could describe more nonlinear physical phenomena. All those equations share the PT symmetry [6]. That is, if $p(x,t)$ is a solution, so is $p^*(-x,-t)$, where $*$ denotes the complex conjugate.

3. Riemann-Hilbert problems

In this section, we analyze a class of Riemann-Hilbert problems associated with the spectral problems in (2.2) for the nonlocal reverse-spacetime integrable mKdV equations (2.12) (see, e.g., [34,12,7] for local equations). Solutions to the Riemann-Hilbert problems with the identity jump matrix generat soliton solutions for the nonlocal mKdV equations in the following section.

3.1. Property of eigenfunctions

Suppose that all the potentials rapidly vanish when x or $t \to \infty$ or $-\infty$ so that we do not have any convergence problem. Obviously, an equivalent pair of matrix spectral problems to (2.2) reads

$$\psi_x = i\lambda[\Lambda,\psi] + \check{P}\psi,\ \check{P} = iP, \tag{3.1}$$

$$\psi_t = i\lambda^3[\Omega,\psi] + \check{Q}\psi,\ \check{Q} = iQ, \tag{3.2}$$

which $\psi = \phi\,\mathrm{e}^{-i\lambda\Lambda x}$ satisfies if ϕ solves (2.2). Applying a generalized Liouville's formula [32], we can have $(\det\psi)_x = 0$, because of $\mathrm{tr}(\check{P}) = \mathrm{tr}(\check{Q}) = 0$. To establish associated Riemann-Hilbert problems, we adopt the following adjoint spectral problems, besides the spectral problems. The adjoint counterparts of the x-part problem of (2.2) and the spectral problem (3.1) are defined as

$$i\tilde{\phi}_x = \tilde{\phi}U, \tag{3.3}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi},\Lambda] + \tilde{\psi}P, \tag{3.4}$$

which $\tilde{\phi} = \phi^{-1}$ and $\tilde{\psi} = \psi^{-1}$ satisfies, respectively.

Let $\psi(\lambda)$ be a matrix eigenfunction of the spatial spectral problem (3.1) associated with an eigenvalue λ. First, it is easy to see that $C\psi^{-1}(x,t,\lambda)$ is a matrix adjoint eigenfunction associated with the same eigenvalue λ. Second, under the nonlocal reductions in (2.8), we can compute that

$$\begin{aligned} &i[\psi^T(-x,-t,\lambda)C]_x = i[-(\psi_x)^T(-x,-t,\lambda)C] \\ &= i\{-i\lambda[\Lambda,\psi(-x,-t,\lambda)] - \check{P}(-x,-t)\psi(-x,-t,\lambda)\}^TC \\ &= i\{-i\lambda[\psi^T(-x,-t,\lambda),\Lambda] - \psi^T(-x,-t,\lambda)\check{P}^T(-x,-t)\}C \\ &= \lambda[\psi^T(-x,-t,\lambda)C,\Lambda] + \psi^T(-x,-t,\lambda)C[C^{-1}P^T(-x,-t)C] \\ &= \lambda[\psi^T(-x,-t,\lambda)C,\Lambda] + \psi^T(-x,-t,\lambda)CP(x,t), \end{aligned}$$

and hence, the matrix function

$$\tilde{\psi}(x,t,\lambda) := \psi^T(-x,-t,\lambda)C,$$

presents another matrix adjoint eigenfunction associated with the same original eigenvalue λ, i.e., $\psi^T(-x,-t,\lambda)C$ solves the adjoint spectral problem (3.4).

Now, upon observing the asymptotic conditions for ψ, the uniqueness of solutions shows that

$$\psi^T(-x,-t,\lambda) = C\psi^{-1}(x,t,\lambda)C^{-1} \tag{3.5}$$

holds for a matrix eigenfunction ψ that satisfies $\psi \to I_{n+1}$, when x or $t \to \infty$ or $-\infty$. This enables us to conclude that the spectral problem (3.1) has the property (3.5) for its eigenfunctions, under the nonlocal reductions in (2.8).

3.2. Riemann-Hilbert problems

Let us now formulate a class of associated Riemann-Hilbert problems with the space variable x. The whole procedure is in actual fact the same as the one for the local case [23], since the nonlocal reductions in (2.8) do not present any problem in reducing the associated Riemann-Hilbert problems. However, we still present it below for ease of reference.

In order to facilitate the expression below, we assume that

$$\alpha = \alpha_1 - \alpha_2 < 0,\ \beta = \beta_1 - \beta_2 < 0, \tag{3.6}$$

as usual. To present the scattering problem, we take the two matrix eigenfunctions $\psi^\pm(x,\lambda)$ of (3.1) with the asymptotic conditions

$$\psi^\pm \to I_{n+1},\ \text{when}\ x \to \pm\infty, \tag{3.7}$$

respectively. From $(\det\psi)_x = 0$, we obtain that $\det\psi^\pm = 1$ for all $x \in \mathbb{R}$. Note that

$$\phi^\pm = \psi^\pm E,\ E = \mathrm{e}^{i\lambda\Lambda x}, \tag{3.8}$$

are both matrix eigenfunctions of the spectral problems (2.2). Thus, they must be linearly dependent, and accordingly, we can state

$$\psi^- E = \psi^+ E S(\lambda),\ \lambda \in \mathbb{R}. \tag{3.9}$$

This matrix $S(\lambda) = (s_{jl})_{(n+1)\times(n+1)}$ is traditionally called the scattering matrix. Moreover, we have $\det S(\lambda) = 1$ due to $\det\psi^\pm = 1$.

It is known that the matrix eigenfunctions $\psi^\pm$ need to satisfy the following Volterra integral equations:

$$\psi^-(\lambda,x) = I_{n+1} + \int_{-\infty}^{x} \mathrm{e}^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^-(\lambda,y)\mathrm{e}^{i\lambda\Lambda(y-x)}\,dy, \tag{3.10}$$

$$\psi^+(\lambda,x) = I_{n+1} - \int_{x}^{\infty} \mathrm{e}^{i\lambda\Lambda(x-y)}\check{P}(y)\psi^+(\lambda,y)\mathrm{e}^{i\lambda\Lambda(y-x)}\,dy, \tag{3.11}$$

where the asymptotic conditions (3.7) have been imposed. It then follows from the theory of Volterra integral equations that the eigenfunctions $\psi^{\pm}$ can exist and allow analytical continuations off the real line $\lambda \in \mathbb{R}$ provided that the integrals on the right hand sides converge.

Precisely from the diagonal form of Λ, we can observe that the integral equation for the last n columns of ψ^{+} contains only the exponential factor $\mathrm{e}^{i\alpha\lambda(x-y)}$, which also decays exponentially because of $y > x$ in the integral, when λ takes values in the upper half-plane $\mathbb{C}^{+}$, and the integral equation for the first column of ψ^{-} contains only the exponential factor $\mathrm{e}^{-i\alpha\lambda(x-y)}$, which decays exponentially because of $y < x$ in the integral, if λ takes values in the upper half-plane $\mathbb{C}^{+}$. It thus follows that those $n+1$ columns are analytical in the upper half-plane $\mathbb{C}^{+}$ and continuous in the closed upper half-plane $\bar{\mathbb{C}}^{+}$. By a similar argument, we can show that the last n columns of ψ^{-} and the first column of ψ^{+} are analytical in the lower half-plane $\mathbb{C}^{-}$ and continuous in the closed lower half-plane $\bar{\mathbb{C}}^{-}$.

In order to determine two generalized matrix Jost solutions, denoted by T^{+} and T^{-}, which are analytic in $\mathbb{C}^{+}$ and $\mathbb{C}^{-}$ and continuous in $\bar{\mathbb{C}}^{+}$ and $\bar{\mathbb{C}}^{-}$, respectively, we express

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \cdots, \psi_{n+1}^{\pm}), \tag{3.12}$$

where $\psi_j^{\pm}$ denotes the jth column of $\phi^{\pm}$ $(1 \le j \le n+1)$, and

$$\tilde{\psi}^{\pm} = (\tilde{\psi}^{\pm,1}, \tilde{\psi}^{\pm,2}, \cdots, \tilde{\psi}^{\pm,n+1})^{T}, \tag{3.13}$$

where $\tilde{\psi}^{\pm,j}$ denotes the jth row of $\tilde{\psi}^{\pm}$ $(1 \le j \le n+1)$, and we denote

$$H_1 = \mathrm{diag}(1, \underbrace{0, \cdots, 0}_{n}),\ H_2 = \mathrm{diag}(0, \underbrace{1, \cdots, 1}_{n}). \tag{3.14}$$

Then, we can take the generalized matrix Jost solution T^{+} as

$$T^{+} = T^{+}(x,\lambda) = (\psi_1^{-}, \psi_2^{+}, \cdots, \psi_{n+1}^{+}) = \psi^{-} H_1 + \psi^{+} H_2, \tag{3.15}$$

which is analytic in $\lambda \in \mathbb{C}^{+}$ and continuous in $\lambda \in \bar{\mathbb{C}}^{+}$. To determine the other generalized matrix Jost solution T^{-}, i.e., the analytic counterpart of T^{+} in the lower half-plane $\mathbb{C}^{-}$, we adopt the adjoint matrix spectral problems. Note that when ϕ and ψ solve the two spectral problems, the inverse matrices $\tilde{\phi} = \phi^{-1}$ and $\tilde{\psi} = \psi^{-1}$ solve the corresponding two adjoint spectral problems, respectively. We can take the other generalized matrix Jost solution T^{-} as the adjoint matrix solution of (3.4), i.e.,

$$T^{-} = (\tilde{\psi}^{-,1}, \tilde{\psi}^{+,2}, \cdots, \tilde{\psi}^{+,n+1})^{T} = H_1 \tilde{\psi}^{-} + H_2 \tilde{\psi}^{+} = H_1 (\psi^{-})^{-1} + H_2 (\psi^{+})^{-1}, \tag{3.16}$$

which is analytic for $\lambda \in \mathbb{C}^{-}$ and continuous for $\lambda \in \bar{\mathbb{C}}^{-}$.

It now follows from $\det \psi^{\pm} = 1$, the definitions of $T^{\pm}$, and the scattering relation (3.9) between ψ^{+} and ψ^{-} that

$$\det T^{+}(x,\lambda) = s_{11}(\lambda),\ \det T^{-}(x,\lambda) = \hat{s}_{11}(\lambda), \tag{3.17}$$

where $S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{jl})_{(n+1)\times(n+1)}$. This implies that

$$\lim_{x\to\infty} T^{+}(x,\lambda) = \begin{bmatrix} s_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix},\ \lambda \in \bar{\mathbb{C}}^{+};\ \lim_{x\to\infty} T^{-}(x,\lambda) = \begin{bmatrix} \hat{s}_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix},\ \lambda \in \bar{\mathbb{C}}^{-}. \tag{3.18}$$

Now taking advantage of T^{+} and T^{-}, we can introduce the following two unimodular generalized matrix Jost solutions:

$$\begin{cases} G^+(x,\lambda) = T^+(x,\lambda)\begin{bmatrix} s_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}, \ \lambda \in \bar{\mathbb{C}}^+; \\ (G^-)^{-1}(x,\lambda) = \begin{bmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} T^-(x,\lambda), \ \lambda \in \bar{\mathbb{C}}^-. \end{cases} \tag{3.19}$$

Those two generalized matrix Jost solutions yield the required matrix Riemann-Hilbert problems on the real line for the nonlocal reverse-spacetime mKdV equations (2.12):

$$G^+(x,\lambda) = G^-(x,\lambda)G_0(x,\lambda), \ \lambda \in \mathbb{R}, \tag{3.20}$$

where based on (3.9), the jump matrix G_0 reads

$$G_0(x,\lambda) = E\begin{bmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} \tilde{S}(\lambda) \begin{bmatrix} s_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} E^{-1}. \tag{3.21}$$

The matrix $\tilde{S}(\lambda)$ has the factorization:

$$\tilde{S}(\lambda) = (H_1 + H_2 S(\lambda))(H_1 + S^{-1}(\lambda)H_2), \tag{3.22}$$

which can be simplified into

$$\tilde{S}(\lambda) = (\tilde{s}_{jl})_{(n+1)\times(n+1)} = \begin{bmatrix} 1 & \hat{s}_{1,2} & \cdots & \hat{s}_{1,n+1} \\ s_{2,1} & 1 & & 0 \\ \vdots & & \ddots & \\ s_{n+1,1} & 0 & & 1 \end{bmatrix}. \tag{3.23}$$

Moreover, on one hand, directly from the Volterra integral equations (3.10) and (3.11), we get the canonical normalization conditions for the associated Riemann-Hilbert problems:

$$G^{\pm}(x,\lambda) \to I_{n+1}, \ \text{when } \lambda \in \bar{\mathbb{C}}^{\pm} \to \infty. \tag{3.24}$$

On the other hand, based on the property (3.5), we can directly show that

$$(G^+)^T(-x,-t,\lambda) = C(G^-)^{-1}(x,t,\lambda)C^{-1}, \tag{3.25}$$

and thus, the jump matrix satisfies

$$G_0^T(-x,-t,\lambda) = CG_0(x,t,\lambda)C^{-1}. \tag{3.26}$$

It is also worth mentioning that the jump matrix G_0, defined by (3.21) and (3.22), carries all basic scattering data from the scattering matrix $S(\lambda)$ that one needs to formulate the inverse scattering transforms.

3.3. Evolution of the scattering data

To develop the direct scattering transforms, we calculate the derivative of the equation (3.9) with the time variable t, and utilize the temporal spectral problem (3.2) for $\psi^{\pm}$.

It thus follows that the scattering matrix S has to obey a matrix evolution law:

$$S_t = i\lambda^3[\Omega, S], \tag{3.27}$$

where Ω is defined as in (2.3). This precisely yields the time dependence of the scattering coefficients:

$$\begin{cases} s_{12} = s_{12}(0,\lambda)\mathrm{e}^{i\beta\lambda^3 t},\ s_{13} = s_{13}(0,\lambda)\mathrm{e}^{i\beta\lambda^3 t},\ \cdots,\ s_{1,n+1} = s_{1,n+1}(0,\lambda)\mathrm{e}^{i\beta\lambda^3 t}, \\ s_{21} = s_{21}(0,\lambda)\mathrm{e}^{-i\beta\lambda^3 t},\ s_{31} = s_{31}(0,\lambda)\mathrm{e}^{-i\beta\lambda^3 t},\ \cdots,\ s_{n+1,1} = s_{n+1,1}(0,\lambda)\mathrm{e}^{-i\beta\lambda^3 t}, \end{cases} \tag{3.28}$$

and tells that all other scattering coefficients do not depend on the time variable t.

3.4. Gelfand-Levitan-Marchenko type equations

To derive Gelfand-Levitan-Marchenko type integral equations for the generalized matrix Jost solutions, let us transform the Riemann-Hilbert problems in (3.20) as follows:

$$\begin{cases} G^+ - G^- = G^- v,\ v = G_0 - I_{n+1},\ \text{on } \mathbb{R}, \\ G^\pm \to I_{n+1} \text{ as } \lambda \in \bar{\mathbb{C}}^\pm \to \infty, \end{cases} \tag{3.29}$$

where G_0 is determined by (3.21) and (3.22). Let $G(\lambda) = G^\pm(\lambda)$ if $\lambda \in \mathbb{C}^\pm$. Assume that G has R simple poles: $\{\mu_j\}_{j=1}^R$, where R is an arbitrary natural number, and the poles are off the real line $\mathbb{R}$ to avoid spectral singularities. Define

$$\tilde{G}^\pm(\lambda) = G^\pm(\lambda) - \sum_{j=1}^R \frac{G_j}{\lambda - \mu_j},\ \lambda \in \bar{\mathbb{C}}^\pm;\ \tilde{G}(\lambda) = \tilde{G}^\pm(\lambda),\ \lambda \in \mathbb{C}^\pm, \tag{3.30}$$

where G_j is the residue of $G(\lambda)$ at $\lambda = \mu_j$, i.e.,

$$G_j = \mathrm{res}(G(\lambda), \mu_j) = \lim_{\lambda \to \mu_j} (\lambda - \mu_j) G(\lambda). \tag{3.31}$$

This way, we obtain

$$\begin{cases} \tilde{G}^+ - \tilde{G}^- = G^+ - G^- = G^- v,\ \text{on } \mathbb{R}, \\ \tilde{G}^\pm \to I_{n+1} \text{ as } \lambda \in \bar{\mathbb{C}}^\pm \to \infty. \end{cases} \tag{3.32}$$

Now, through applying the Sokhotski-Plemelj formula [10], we get the solution

$$\tilde{G}(\lambda) = I_{n+1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \lambda}\, d\xi. \tag{3.33}$$

Further, taking the limit as $\lambda \to \lambda_l$, we obtain

$$\mathrm{lhs} = \lim_{\lambda \to \lambda_l} \tilde{G} = F_l - \sum_{j \neq l}^R \frac{G_j}{\mu_l - \mu_j},\ \mathrm{rhs} = I_{n+1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \mu_l}\, d\xi,$$

where

$$F_l = \lim_{\lambda \to \mu_l} \frac{(\lambda - \mu_l)G(\lambda) - G_l}{\lambda - \mu_l}, \ 1 \le l \le R. \tag{3.34}$$

As a result, we finally obtain the following Gelfand-Levitan-Marchenko type equations:

$$I_{n+1} - F_l + \sum_{j \ne l}^{R} \frac{G_j}{\mu_l - \mu_j} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- G_0)(\xi)}{\xi - \mu_l} \, d\xi - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{G^-(\xi)}{\xi - \mu_l} \, d\xi = 0, \ 1 \le l \le R. \tag{3.35}$$

These equations are used to determine solutions to the associated Riemann-Hilbert problems by (3.30), and hence, produce the generalized matrix Jost solutions. The general theory of existence and uniqueness of solutions has yet to be developed. Nevertheless, a specific formulation of solutions in the case of identity jump matrices will be explicitly presented in the next section.

3.5. Recovery of the potential

To recover the potential matrix P from the generalized matrix Jost solutions, we make an asymptotic expansion for the generalized matrix Jost solution G^+:

$$G^+(x,t,\lambda) = I_{n+1} + \frac{1}{\lambda} G_1^+(x,t) + \mathrm{O}(\frac{1}{\lambda^2}), \ \lambda \to \infty. \tag{3.36}$$

Then plugging the asymptotic expansion into the matrix spectral problem (3.1), all $\mathrm{O}(1)$ terms engender

$$P = \lim_{\lambda \to \infty} \lambda [G^+(\lambda), \Lambda] = -[\Lambda, G_1^+]. \tag{3.37}$$

To get the potential for the nonlocal equations, one needs to check an involution property (2.10) for P or equivalently an involution property for G_1^+:

$$(G_1^+)^T(-x,-t) = -C G_1^+(x,t) C^{-1}. \tag{3.38}$$

Then if so, we obtain the solutions to the nonlocal reverse-spacetime integrable mKdV equations (2.12):

$$p_j = -\alpha (G_1^+)_{1,j+1}, \ 1 \le j \le n, \tag{3.39}$$

where we denote $G_1^+ = ((G_1^+)_{jl})_{(n+1)\times(n+1)}$.

Once the solutions $\{G^+(\lambda), G^-(\lambda)\}$ to the associated Riemann-Hilbert problems are determined, the potential matrix P defined by (3.39), provides solutions to the nonlocal reverse-spacetime integrable PT-symmetric mKdV equations (2.12), like the inverse scattering transforms.

4. Soliton solutions

In this section, we would like to construct soliton solutions to the nonlocal reverse-spacetime integrable mKdV equations (2.12). As usual, we avoid solving Gelfand-Levitan-Marchenko type integral equations, and instead, we utilize the Riemann-Hilbert technique directly.

Let N be another arbitrary natural number. Suppose that $s_{11}(\lambda)$ has N simple zeros $\{\lambda_k \in \mathbb{C}, \ 1 \le k \le N\}$, and $\hat{s}_{11}(\lambda)$ has N simple zeros $\{\hat{\lambda}_k \in \mathbb{C}, \ 1 \le k \le N\}$. Then, each of $\ker T^+(\lambda_k)$, $1 \le k \le N$, contains only a single basis column vector, denoted by v_k, $1 \le k \le N$; and each of $\ker T^-(\hat{\lambda}_k)$, $1 \le k \le N$, a single basis row vector, denoted by $\hat{v}_k$, $1 \le k \le N$. All this tells

$$T^+(\lambda_k) v_k = 0, \ \hat{v}_k T^-(\hat{\lambda}_k) = 0, \ 1 \le k \le N. \tag{4.1}$$

Beginning with these two equations, we can determine the kernel vectors $v_k, \hat{v}_k$, $1 \le k \le N$, by taking advantage of the associated spectral problems that T^+ and T^- satisfy (see, e.g., [23] for the local case).

It is known that the Riemann-Hilbert problems, by (3.20) and (3.21), with the canonical normalization conditions in (3.24) and the zero structures given in (4.1), can be solved explicitly [34,19,20], and then, we can determine the potential matrix P that provides solutions to the nonlocal integrable mKdV equations.

4.1. Solving special Riemann-Hilbert problems

As usual, soliton solutions are constructed from the Riemann-Hilbert problems in (3.20) with the identity jump matrix $G_0 = I_{n+1}$, which can be achieved if we assume the zero reflection coefficient conditions

$$s_{i1} = \hat{s}_{1i} = 0,\ 2 \le i \le n+1, \tag{4.2}$$

in the scattering problem. However, in the case of nonlocal integrable equations, we often see that we do not have the condition

$$\{\lambda_k | 1 \le k \le N\} \cap \{\hat{\lambda}_k | 1 \le k \le N\} = \emptyset, \tag{4.3}$$

and thus, we have to establish a new formulation of solutions, which should generalize the one in the literature [34,19,20]. Direct computations show us that we can determine solutions to a kind of special Riemann-Hilbert problems without the condition (4.3) as follows.

Let us introduce G^+ and G^- by

$$G^+(\lambda) = I_{n+1} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l},\ (G^-)^{-1}(\lambda) = I_{n+1} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}, \tag{4.4}$$

where $M = (m_{kl})_{N\times N}$ is a square matrix whose entries are defined by

$$m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, & \text{when } \lambda_l \ne \hat{\lambda}_k, \\ 0, & \text{when } \lambda_l = \hat{\lambda}_k, \end{cases} \quad 1 \le k, l \le N. \tag{4.5}$$

This M-matrix is different from the traditional one in the literature, since we included the case of $\lambda_l = \hat{\lambda}_k$, which often occurs in the case of nonlocal integrable equations. We can now see that the properties in (4.1) hold. Moreover, G^+ and G^- satisfy

$$(G^-)^{-1}(\lambda) G^+(\lambda) = I_{n+1}, \tag{4.6}$$

if we additionally require an orthogonal condition

$$\hat{v}_k v_l = 0,\ \text{when } \lambda_l = \hat{\lambda}_k,\ 1 \le k, l \le N. \tag{4.7}$$

Obviously, if the condition (4.3) holds, then the above result reduces to the one presented in [34,19,20]. If $\lambda_k \in \mathbb{C}^+$ and $\hat{\lambda}_k \in \mathbb{C}^-$, $1 \le k \le N$, then the above two matrices G^+ and G^- are meromorphic in the upper and lower half-planes, respectively, and thus, they solve the Riemann-Hilbert problems with the identity jump matrix on the real line.

4.2. Soliton solutions

In order to generate soliton solutions to the nonlocal reverse-spacetime integrable mKdV equations (2.12), we have to check that the involution property (2.10) or (3.38) holds. To achieve this, motivated by the previous property of eigenfunctions in Subsection 3.1, we can assume [23] that $\ker T^{+}(\lambda_k)$ and $\ker T^{-}(\lambda_k)$, $1 \leq k \leq N$, are spanned by

$$v_k(x,t) = v_k(x,t,\lambda_k) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^3 \Omega t} w_k, \ 1 \leq k \leq N, \tag{4.8}$$

and

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\lambda}_k) = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^3 \Omega t} C, \ 1 \leq k \leq N, \tag{4.9}$$

respectively. Here w_k and $\hat{w}_k 1, \leq k \leq N$, are arbitrary column and row vectors, but the orthogonal condition (4.7) requires

$$\hat{w}_k C w_l = 0, \ \text{if } \lambda_l = \hat{\lambda}_k, \ 1 \leq k, l \leq N. \tag{4.10}$$

Now, based on the above construction, we can readily see under (4.10) that if G_1^{+} satisfies the involution property (3.38), i.e., the solution to the special Riemann-Hilbert problem, determined by (4.4) and (4.5), satisfies (3.25), then the potential defined by (3.39) provides soliton solutions to the nonlocal reverse-spacetime integrable PT-symmetric mKdV equations (2.12):

$$p_j = \alpha \sum_{k,l=1}^{N} v_{k,1} (M^{-1})_{kl} \hat{v}_{l,j+1}, \ 1 \leq j \leq n, \tag{4.11}$$

where M is defined by (4.5), and $v_k = (v_{k,1}, v_{k,2}, \cdots, v_{k,n+1})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \cdots, \hat{v}_{k,n+1})$ are determined by (4.8) and (4.9), respectively.

Let us denote $w_1 = (w_{1,1}, w_{1,2}, \cdots, w_{1,n+1})^T$ and $\hat{w}_1 = (\hat{w}_{1,1}, \hat{w}_{1,2}, \hat{w}_{1,n+1})$ and set $\Sigma = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_n)$. When $N = 1$, a direct computation tells the following one-soliton solution to the nonlocal real reverse-spacetime integrable mKdV equations (2.12):

$$p_j = \frac{\alpha(\lambda_1 - \hat{\lambda}_1) w_{1,1} \hat{w}_{1,j+1} \gamma_j}{w_{1,1}\hat{w}_{1,1}\, \mathrm{e}^{-i\alpha\hat{\lambda}_1 x - i\beta\hat{\lambda}_1^3 t} + w_{1,2}\hat{w}_{1,2}\gamma_1\, \mathrm{e}^{-i\alpha\lambda_1 x - i\beta\lambda_1^3 t} + \cdots + w_{1,n+1}\hat{w}_{1,n+1}\gamma_n\, \mathrm{e}^{-i\alpha\lambda_1 x - i\beta\lambda_1^3 t}}, \tag{4.12}$$

where $1 \leq j \leq n$, and λ_1 and $\hat{\lambda}_1$ are arbitrary but the vectors w_1 and $\hat{w}_1$ need to satisfy

$$\begin{cases} w_{1,1}^2 + \gamma_1 w_{1,2}^2 + \cdots + \gamma_n w_{1,n+1}^2 = 0, \\ \hat{w}_{1,1}^2 + \gamma_1 \hat{w}_{1,2}^2 + \cdots + \gamma_n \hat{w}_{1,n+1}^2 = 0, \\ w_{1,1}\hat{w}_{1,1} + \gamma_1 w_{1,2}\hat{w}_{1,2} + \cdots + \gamma_n w_{1,n+1}\hat{w}_{1,n+1} = 0, \end{cases}$$

which represents three orthogonal conditions, corresponding to the involution property.

5. Concluding remarks

Nonlocal reverse-spacetime integrable PT-symmetric multicomponent modified Korteweg-de Vires (mKdV) equations were presented from a group of nonlocal group reductions, and their associated Riemann-Hilbert problems were formulated via matrix spectral problems and adjoint matrix spectral problems. The

Sokhotski-Plemelj formula was used to transform the associated Riemann-Hilbert problems into systems of Gelfand-Levitan-Marchenko type integral equations. Soliton solutions to the nonlocal reverse-spacetime integrable mKdV equations were generated from the Riemann-Hilbert problems with the identity jump matrix (or equivalently the reflectionless inverse scattering transforms).

It is also worth mentioning that it would be particularly interesting to find a certain kind of connections among different solution approaches, including the Hirota direct method [16], the Wronskian technique [9,28] and the Darboux transformation [33]. Moreover, various recent studies have exhibited great richness of other kinds of solutions to nonlinear dispersive wave integrable equations, such as lump and rogue wave solutions and their interaction solutions [31,40,24,29], Rossby wave solutions [42,41], algebro-geometric solutions [13,22] and solitonless solutions [25,27]. Definitely, it would be very interesting to understand how to construct those exact solutions, for example, lump solutions and rogue wave solutions, through the Riemann-Hilbert perspective or a generalized Riemann-Hilbert perspective.

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