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The inverse scattering transform and soliton solutions of a combined modified Korteweg–de Vries equation

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ABSTRACT

The inverse scattering transform is developed for a combined modified Korteweg–de Vrie equation through the technique of Riemann–Hilbert problems. From special Riemann–Hilbert problems with an identity jump matrix, soliton solutions are generated, which corresponds to the inverse scattering problems with reflectionless coefficients. A specific example of two-soliton solutions is explicitly presented, together with its 3d plots, contour plots and x-curve plots.

1. Introduction

In modern soliton theory [2,37], the inverse scattering transform is one of the most powerful techniques to solve nonlinear integrable equations and particularly generate soliton solutions. The transform is also called the Fourier transform method in a nonlinear world [1], and closely connected with the Riemann–Hilbert problems associated with matrix spectral problems [37]. In the theory of Riemann–Hilbert problems, one starts from bounded eigenfunctions analytically extendable to the upper or lower half-plane and continuous in the closed upper or lower half-plane. Once taking the identity jump matrix, reduced Riemann–Hilbert problems yield soliton solutions, whose special limits can generate lump solutions, periodic solutions and complexiton solutions. A few integrable equations, including the multiple wave interaction equations [37], the general coupled nonlinear Schrödinger equations [43], the Harry Dym equation [46], and the generalized Sasa–Satsuma equation [10], have been studied by the Riemann–Hilbert technique.

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The standard procedure for establishing Riemann–Hilbert problems on the real axis is as follows. One starts from a pair of matrix spectral problems of the following form:

$$-i\phi_x = U\phi,\ -i\phi_t = V\phi,\ U = A(\lambda) + P(u,\lambda),\ V = B(\lambda) + Q(u,\lambda), \tag{1.1}$$

where i is the unit imaginary number, λ is a spectral parameter, u is a potential, ϕ is an $m \times m$ matrix eigenfunction, A, B are constant commuting $m \times m$ matrices, and P, Q are trace-less $m \times m$ matrices. It is known that the compatibility condition of the two matrix spectral problems is the zero curvature equation

$$U_t - V_x + i[U,V] = 0, \tag{1.2}$$

where $[\cdot,\cdot]$ is the matrix commutator. This zero curvature equation presents so-called soliton equations. To formulate Riemann–Hilbert problems for integrable equations, we adopt the following pair of equivalent matrix spectral problems

$$\psi_x = i[A(\lambda),\psi] + \check{P}(u,\lambda)\psi, \psi_t = i[B(\lambda),\psi] + \check{Q}(u,\lambda)\psi, \tag{1.3}$$

where ψ is an $m \times m$ matrix eigenfunction, $\check{P} = iP$ and $\check{Q} = iQ$. The commutativity of A and B guarantees this equivalence, and there is a relation between the two matrix eigenfunctions ϕ and ψ:

$$\phi = \psi E_g,\ E_g = \mathrm{e}^{iA(\lambda)x + iB(\lambda)t}.$$

For the matrix spectral problems (1.3), we can have two bounded analytical matrix eigenfunctions with the asymptotic conditions

$$\psi^{\pm} \to I_m,\ \text{when } x,t \to \pm\infty, \tag{1.4}$$

where I_m stands for the identity matrix of size m. Let $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the upper and lower half-planes:

$$\mathbb{C}^+ = \{z \in \mathbb{C} |\, \mathrm{Im}(z) > 0\},\ \mathbb{C}^- = \{z \in \mathbb{C} |\, \mathrm{Im}(z) < 0\}, \tag{1.5}$$

and $\mathbb{C}_0^+$ and $\mathbb{C}_0^-$, the closed upper and lower half-planes:

$$\mathbb{C}_0^+ = \{z \in \mathbb{C} |\, \mathrm{Im}(z) \geq 0\},\ \mathbb{C}_0^- = \{z \in \mathbb{C} |\, \mathrm{Im}(z) \leq 0\}. \tag{1.6}$$

Based on the above two matrix eigenfunctions $\psi^{\pm}$, we try to determine two matrix functions $P^{\pm}(x,t,\lambda)$, which are analytical in $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous in $\mathbb{C}_0^+$ and $\mathbb{C}_0^-$, respectively, and then formulate a Riemann–Hilbert problem on the real axis:

$$G^+(x,t,\lambda) = G^-(x,t,\lambda)G(x,t,\lambda),\ \lambda \in \mathbb{R}, \tag{1.7}$$

with

$$G^+(x,t,\lambda) = P^+(x,t,\lambda),\ \lambda \in \mathbb{C}_0^+,\ (G^-)^{-1}(x,t,\lambda) = P^-(x,t,\lambda),\ \lambda \in \mathbb{C}_0^-. \tag{1.8}$$

Upon taking the jump matrix G to be the identity matrix, the corresponding Riemann–Hilbert problem can be often solved to generate soliton solutions, through observing asymptotic behaviors of the matrix functions $P^{\pm}$ at infinity of λ, which also provide the canonical normalization conditions of the Riemann–Hilbert problems.

In this paper, we shall present an application example of the inverse scattering transform, based on the Riemann–Hilbert technique. The nonlinear equation that we shall discuss is the following combined modified Korteweg–de Vries (mKdV) equation

$$\begin{cases} p_{1,t} = -p_{1,xxx} - 6|p_1|^2 p_{1,x} + 3|p_2|^2 p_{1,x} + 3p_1\bar{p}_2 p_{2,x}, \\ p_{2,t} = -p_{2,xxx} + 6|p_2|^2 p_{2,x} - 3|p_1|^2 p_{2,x} - 3\bar{p}_1 p_2 p_{1,x}, \end{cases} \tag{1.9}$$

where $\bar{f}$ denotes the complex conjugate of f and $|f|^2 = f\bar{f}$. When p_1 and p_2 are real, the above combined mKdV equation is reduced to

$$\begin{cases} p_{1,t} = -p_{1,xxx} - 6p_1^2 p_{1,x} + 3p_2^2 p_{1,x} + 3p_1 p_2 p_{2,x}, \\ p_{2,t} = -p_{2,xxx} + 6p_2^2 p_{2,x} - 3p_1^2 p_{2,x} - 3p_1 p_2 p_{1,x}. \end{cases} \tag{1.10}$$

The cases of $p_1 = 0$ and $p_2 = 0$ further give the positive and negative mKdV equations respectively, which possess different properties (see, e.g., [20]). The equation (1.10) adds to the class of combined mKdV equations in the real field, the other two of which are discussed in [11,45,48].

The rest of the paper is structured as follows. In Section 2, within the zero-curvature formulation, we derive a combined mKdV hierarchy, together with its recursion operator, based on a matrix spectral problem suited for the Riemann–Hilbert theory. In Sections 3 and 4, to present an inverse scattering transform for the combined mKdV equation (1.9), we analyze analytical properties of matrix eigenfunctions and build a kind of Riemann–Hilbert problems of the equivalent matrix spectral problem. In Section 5, we compute soliton solutions to the combined mKdV equation from special associated Riemann–Hilbert problems on the real axis, in which the jump matrix is taken as the identity matrix. The last section is devoted to conclusions and remarks.

2. A combined mKdV integrable hierarchy

2.1. Zero curvature formulation

We state the zero curvature formulation to generate integrable hierarchies as follows (see, e.g., [21,29,42]). Let u be a vector potential and λ, a spectral parameter. Choose a square matrix spectral matrix $U = U(u, \lambda)$ from a given matrix loop algebra, whose underlying Lie algebra could be either semisimple [21,42] or non-semisimple [29]. Assume that there is a formal Laurent series solution

$$W = W(u, \lambda) = \sum_{m=0}^{\infty} W_m \lambda^{-m} = \sum_{m=0}^{\infty} W_m(u) \lambda^{-m} \tag{2.1}$$

to the corresponding stationary zero curvature equation

$$W_x = i[U, W]. \tag{2.2}$$

Using this solution W, we introduce a series of Lax matrices

$$V^{[r]} = V^{[r]}(u, \lambda) = (\lambda^r W)_+ + \Delta_r, \ r \geq 0, \tag{2.3}$$

where the subscript $+$ denotes the operation of taking a polynomial part in λ, and Δ_r, $r \geq 0$, are appropriate modification terms. The selection of Δ_r is somewhat subtle and depends on whether an integrable hierarchy

$$u_t = K_r(u) = K_r(x, t, u, u_x, \cdots), \ r \geq 0, \tag{2.4}$$

can be generated from a series of zero curvature equations

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0,\ r \geq 0. \tag{2.5}$$

The two matrices U and $V^{[r]}$ are called a Lax pair [18] of the r-th integrable equation in the hierarchy (2.4). Obviously, the zero curvature equations in (2.5) are the compatibility conditions of the spatial and temporal matrix spectral problems

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ -i\phi_t = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi,\ r \geq 0, \tag{2.6}$$

where ϕ is the matrix eigenfunction.

To show the commutability of the hierarchy (2.4), we normally start by verifying Lax operator algebras (see, e.g., [22–24] for details):

$$[\![V^{[m]}, V^{[n]}]\!] = V^{[m]'}(u)[K_n] - V^{[n]'}(u)[K_m] = 0,\ m,n \geq 0, \tag{2.7}$$

which ensures the existence of infinitely many common commuting Lie symmetries $\{K_m\}_{m=0}^{\infty}$:

$$[K_m, K_n] = K_m'(u)[K_n] - K_m'(u)[K_n] = 0,\ m,n \geq 0. \tag{2.8}$$

In the above computations, R' stands for the Gateaux derivative of R with respect to u in a direction S:

$$R'(u)[S] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} R(u+\varepsilon S, u_x + \varepsilon S_x, \cdots).$$

When the underlying matrix loop algebra in the zero curvature formulation is simple, the associated zero curvature equations engender classical integrable hierarchies [7,13]; when semisimple, the associated zero curvature equations generate a collection of different integrable hierarchies; and when non-semisimple, we get hierarchies of integrable couplings [30], which require extra care in exploring their integrability.

2.2. A combined mKdV hierarchy

We consider the following matrix spectral problem

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ U = (U_{jl})_{3\times 3} = \begin{bmatrix} 2\lambda & p_1 & p_2 \\ \bar{p}_1 & \lambda & 0 \\ -\bar{p}_2 & 0 & \lambda \end{bmatrix},\ u = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \tag{2.9}$$

where λ is a spectral parameter. Two special cases: (a) $p_2 = 0$ and real p_1, and (b) $p_1 = 0$ and real p_2, are reduced to the spectral problems associated with the positive and negative mKdV equations, respectively.

To derive an associated combined mKdV hierarchy, we first solve the stationary zero curvature equation (2.2) corresponding to (2.9), as suggested in the general zero curvature formulation. We seek a solution W of the form

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.10}$$

where a is a real scalar, $b = (b_1, b_2)$ and $c = (\bar{b}_1, -\bar{b}_2)^T$ are two-dimensional vectors, and d is a 2×2 matrix satisfying $d^\dagger(\bar{\lambda}) = \overline{d(\lambda)}^T = \Sigma d(\bar{\lambda})\Sigma^{-1},\ \Sigma = \mathrm{diag}(1,-1)$. It is direct to show that the stationary zero curvature equation (2.2) is

$$\begin{cases} a_x = i(pc - bq), \\ b_x = i(\alpha\lambda b + pd - ap), \\ d_x = i(qb - cp), \end{cases} \tag{2.11}$$

where $q = (q_1, q_2)^T = (\bar{p}_1, -\bar{p}_2)^T$. We take W as a formal series:

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{m=0}^{\infty} W_m \lambda^{-m}, \ W_m = W_m(u) = \begin{bmatrix} a^{[m]} & b^{[m]} \\ c^{[m]} & d^{[m]} \end{bmatrix}, \ m \geq 0, \tag{2.12}$$

where $b^{[m]}, c^{[m]}$ and $d^{[m]}$ are expressed as

$$b^{[m]} = (b_1^{[m]}, b_2^{[m]}), \ c^{[m]} = \left(\overline{b_1^{[m]}}, -\overline{b_2^{[m]}}\right)^T, \ d^{[m]} = (d_{jl}^{[m]})_{2\times 2}, \ m \geq 0, \tag{2.13}$$

where the $d^{[m]}$'s satisfy $(d^{[m]})^\dagger = \Sigma d^{[m]} \Sigma^{-1}, \ m \geq 0$. Then, the system (2.11) exactly presents the following recursion relations:

$$b^{[0]} = 0, \ c^{[0]} = 0, \ a_x^{[0]} = 0, \ d_x^{[0]} = 0, \tag{2.14a}$$

$$b^{[m+1]} = -ib_x^{[m]} - pd^{[m]} + a^{[m]}p, \ m \geq 0, \tag{2.14b}$$

$$a_x^{[m]} = i(pc^{[m]} - b^{[m]}q), \ d_x^{[m]} = i(qb^{[m]} - c^{[m]}p), \ m \geq 1. \tag{2.14c}$$

Next we choose the initial values:

$$a^{[0]} = 2, \ d^{[0]} = I_2, \tag{2.15}$$

and take constants of integration in (2.14c) to be zero, that is, require

$$W_m|_{u=0} = 0, \ m \geq 1. \tag{2.16}$$

Then, with $a^{[0]}$ and $d^{[0]}$ given by (2.15), all matrices $W_m, \ m \geq 1$, are uniquely determined. For example, a direct computation, in virtue of (2.14), generates that

$$b_j^{[1]} = p_j, \ a^{[1]} = 0, \ d_{jl}^{[1]} = 0; \tag{2.17a}$$

$$b_j^{[2]} = -ip_{j,x}, a^{[2]} = -pq, \ d_{jl}^{[2]} = p_l q_j; \tag{2.17b}$$

$$b_j^{[3]} = -p_{j,xx} - 2pqp_j, \tag{2.17c}$$

$$a^{[3]} = -i(pq_x - p_x q), \ d_{jl}^{[3]} = -i(p_{l,x} q_j - p_l q_{j,x}); \tag{2.17d}$$

$$b_j^{[4]} = i(p_{j,xxx} + 3pqp_{j,x} + 3p_x qp_j), \tag{2.17e}$$

$$a^{[4]} = 3(pq)^2 + pq_{xx} - p_x q_x + p_{xx} q, \tag{2.17f}$$

$$d_{jl}^{[4]} = -3p_l pqq_j - p_{l,xx} q_j + p_{l,x} q_{j,x} - p_l q_{j,xx}; \tag{2.17g}$$

where $1 \leq j, l \leq 2$.

To generate the combined mKdV hierarchy, we introduce the following Lax matrices

$$V^{[r]} = V^{[r]}(u, \lambda) = (V_{jl}^{[r]})_{3\times 3} = (\lambda^r W)_+ = \sum_{m=0}^{r} W_m \lambda^{r-m}, \ r \geq 0, \tag{2.18}$$

where the modification terms are taken as zero. The compatibility conditions of (2.6), i.e., the zero curvature equations (2.5), engender the so-called combined mKdV hierarchy:

$$u_t = p_t^T = K_r = ib^{[r+1]T}, \ r \geq 0, \tag{2.19}$$

which can be shown, by checking the corresponding Lax operator algebra, to satisfy

$$[K_m, K_n] = 0, \ m, n \geq 0. \tag{2.20}$$

It is direct to work out the following recursion operator [38] for the combined mKdV hierarchy

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \tag{2.21}$$

with the entries being defined by

$$\begin{cases} \Phi_{11} = i[-\partial - 2p_1\partial^{-1}\bar{p}_1 + p_2\partial^{-1}\bar{p}_2 + 2p_1\partial^{-1}p_1\bar{(\cdot)}], \\ \Phi_{12} = i[p_1\partial^{-1}\bar{p}_2 - p_2\partial^{-1}p_1\bar{(\cdot)} - p_1\partial^{-1}p_2\bar{(\cdot)}], \\ \Phi_{21} = i[-p_2\partial^{-1}\bar{p}_1 + p_1\partial^{-1}p_2\bar{(\cdot)} + p_2\partial^{-1}p_1\bar{(\cdot)}], \\ \Phi_{22} = i[-\partial - p_1\partial^{-1}\bar{p}_1 + 2p_2\partial^{-1}\bar{p}_2 - 2p_2\partial^{-1}p_2\bar{(\cdot)}], \end{cases} \tag{2.22}$$

where $\bar{(\cdot)}$ denotes the conjugate operator: $\bar{(\cdot)}f = \bar{f}$.

The first nonlinear integrable equation in the hierarchy (2.19) is a combined nonlinear Schrödinger equation:

$$p_{j,t} = -i[p_{j,xx} + 2(|p_1|^2 - |p_2|^2)p_j], \ 1 \leq j \leq 2, \tag{2.23}$$

and the second equation is exactly the combined mKdV equation (1.9).

In what follows, we shall discuss the scattering and inverse scattering problems for the combined mKdV equation (1.9) using the Riemann–Hilbert technique [37] (see also [4,12]). The results will lay the groundwork for soliton solutions later on.

3. Direct scattering

The matrix spectral problems of the combined mKdV equation (1.9) are

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \ -i\phi_t = V^{[3]}\phi = V^{[3]}(u,\lambda)\phi, \tag{3.1}$$

where the Lax pair reads

$$U = \lambda\Lambda + P, \ V^{[3]} = \lambda^3\Lambda + Q, \ \Lambda = \mathrm{diag}(2,1,1), \tag{3.2}$$

with

$$P = \begin{bmatrix} 0 & p_1 & p_2 \\ \bar{p}_1 & 0 & 0 \\ -\bar{p}_2 & 0 & 0 \end{bmatrix}, \ Q = \begin{bmatrix} a^{[1]}\lambda^2 + a^{[2]}\lambda + a^{[3]} & b^{[1]}\lambda^2 + b^{[2]}\lambda + b^{[3]} \\ c^{[1]}\lambda^2 + c^{[2]}\lambda + c^{[3]} & d^{[1]}\lambda^2 + d^{[2]}\lambda + d^{[3]} \end{bmatrix}, \tag{3.3}$$

where $c^{[m]}, \ 1 \leq m \leq 3$, are defined through (2.13), and $a^{[m]}, b^{[m]}, d^{[m]}$, $1 \leq m \leq 3$, are determined in (2.17).

Assume that all the potentials sufficiently rapidly vanish when $x \to \pm\infty$ or $t \to \pm\infty$. From the matrix spectral problems in (3.1), we note that when $x, t \to \pm\infty$, we have the asymptotic behavior: $\phi \sim \mathrm{e}^{i\lambda\Lambda x + i\lambda^3\Lambda t}$. Therefore, if we make the variable transformation

$$\phi = \psi E_g,\ E_g = \mathrm{e}^{i\lambda\Lambda x + i\lambda^3\Lambda t},$$

then we can have the canonical normalization $\psi \to I_3$, when $x, t \to \pm\infty$. Once setting $\check{P} = iP$ and $\check{Q} = iQ$, the equivalent pair of matrix spectral problems to (3.1) reads

$$\psi_x = i\lambda[\Lambda, \psi] + \check{P}\psi, \tag{3.4}$$

$$\psi_t = i\lambda^3[\Lambda, \psi] + \check{Q}\psi. \tag{3.5}$$

Upon applying a generalized Liouville's formula [31], we can have

$$\det \psi = 1, \tag{3.6}$$

due to $\operatorname{tr}(\check{P}) = \operatorname{tr}(\check{Q}) = 0$.

Let us now formulate an associated Riemann–Hilbert problem with the variable x, under the integrable conditions:

$$\int_{-\infty}^{\infty} |x|^m \sum_{j=1}^{2} (|p_j| + |q_j|)\, dx < \infty,\ m = 0, 1. \tag{3.7}$$

In the direct scattering problem, we first introduce two matrix solutions $\psi^{\pm}(x, \lambda)$ of (3.4) with the asymptotic conditions

$$\psi^{\pm} \to I_3,\ \text{when } x \to \pm\infty, \tag{3.8}$$

respectively. The above superscripts refer to which end of the x-axis the boundary conditions are required for. Based on (3.6), we see that $\det \psi^{\pm} = 1$ for all $x \in \mathbb{R}$. Since

$$\phi^{\pm} = \psi^{\pm} E,\ E = \mathrm{e}^{i\lambda\Lambda x}, \tag{3.9}$$

are two matrix solutions of (3.1), they are linearly dependent, and as a result of the fact, one has

$$\psi^{-} E = \psi^{+} E S(\lambda),\ \lambda \in \mathbb{R}, \tag{3.10}$$

where $S(\lambda) = (s_{jl})_{3\times 3}$ is the scattering matrix. Note that $\det S(\lambda) = 1$ because of $\det \psi^{\pm} = 1$.

Through the method of variation in parameters, we can transform the x-part of (3.1) into the following Volterra integral equations for $\psi^{\pm}$ [37]:

$$\psi^{-}(\lambda, x) = I_3 + \int_{-\infty}^{x} \mathrm{e}^{i\lambda\Lambda(y-x)} \check{P}(y) \psi^{-}(\lambda, y) \mathrm{e}^{i\lambda\Lambda(x-y)}\, dy, \tag{3.11}$$

$$\psi^{+}(\lambda, x) = I_3 - \int_{x}^{\infty} \mathrm{e}^{i\lambda\Lambda(y-x)} \check{P}(y) \psi^{+}(\lambda, y) \mathrm{e}^{i\lambda\Lambda(x-y)}\, dy, \tag{3.12}$$

where the boundary condition (3.8) has been used. Therefore, under the conditions (3.7), the theory of Volterra integral equations tells that the eigenfunctions $\psi^{\pm}$ exist and allow analytical continuations off the real axis $\lambda \in \mathbb{R}$ as long as the integrals on their right hand sides converge. Based on the diagonal form of Λ, we can easily see that the integral equation for the first column of ψ^- contains only the exponential factor $\mathrm{e}^{-i\alpha\lambda(y-x)}$, which decays because of $y < x$ in the integral, when λ is in the closed upper half-plane, and the integral equation for the last two columns of ψ^+ contains only the exponential factor $\mathrm{e}^{i\alpha\lambda(y-x)}$, which also decays because of $y > x$ in the integral, when λ is in the closed upper half-plane. Thus, these three columns are analytical in the upper half-plane and continuous in the closed upper half-plane. In a similar manner, we can show that the last two columns of ψ^- and the first column of ψ^+ are analytical in the lower half-plane and continuous in the closed lower half-plane.

First, if we express

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \psi_3^{\pm}), \tag{3.13}$$

that is, $\psi_j^{\pm}$ stands for the jth column of $\phi^{\pm}$ $(1 \leq j \leq 3)$, then the matrix solution

$$P^+ = P^+(x, \lambda) = (\psi_1^-, \psi_2^+, \psi_3^+) = \psi^- H_1 + \psi^+ H_2 \tag{3.14}$$

is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \mathbb{C}_0^+$, and the matrix solution

$$(\psi_1^+, \psi_2^-, \psi_3^-) = \psi^+ H_1 + \psi^- H_2 \tag{3.15}$$

is analytic in $\lambda \in \mathbb{C}^-$ and continuous in $\lambda \in \mathbb{C}_0^-$. In the above derivation, H_1 and H_2 are the following matrices

$$H_1 = \mathrm{diag}(1, 0, 0),\ \ H_2 = \mathrm{diag}(0, 1, 1). \tag{3.16}$$

Moreover, from the Volterra integral equations (3.11) and (3.12), we find that

$$P^+(x, \lambda) \to I_3,\ \text{when}\ \lambda \in \mathbb{C}_0^+ \to \infty, \tag{3.17}$$

and

$$(\psi_1^+, \psi_2^-, \psi_3^-) \to I_3,\ \text{when}\ \lambda \in \mathbb{C}_0^- \to \infty. \tag{3.18}$$

Secondly, we construct the analytic counterpart of P^+ in the lower half-plane $\mathbb{C}^-$ from the adjoint counterparts of the matrix spectral problems. The adjoint equation of the x-part of (3.1) and the adjoint equation of (3.4) are given by

$$i\tilde{\phi}_x = \tilde{\phi} U, \tag{3.19}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi} P. \tag{3.20}$$

Note that the inverse matrices $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and $\tilde{\psi}^{\pm} = (\psi^{\pm})^{-1}$ solve these two adjoint equations, respectively. Upon expressing $\tilde{\psi}^{\pm}$ as follows:

$$\tilde{\psi}^{\pm} = (\tilde{\psi}^{\pm,1}, \tilde{\psi}^{\pm,2}, \tilde{\psi}^{\pm,3})^T, \tag{3.21}$$

that is, $\tilde{\psi}^{\pm,j}$ stands for the jth row of $\tilde{\psi}^{\pm}$ $(1 \leq j \leq 3)$, we can verify by similar arguments that the adjoint matrix solution of (3.20),

$$P^- = (\tilde{\psi}^{-,1}, \tilde{\psi}^{+,2}, \tilde{\psi}^{+,3})^T = H_1\tilde{\psi}^- + H_2\tilde{\psi}^+ = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1}, \tag{3.22}$$

is analytic in $\lambda \in \mathbb{C}^-$ and continuous in $\lambda \in \mathbb{C}_0^-$, and the other matrix solution of (3.20),

$$(\tilde{\psi}^{+,1}, \tilde{\psi}^{-,2}, \tilde{\psi}^{-,3})^T = H_1\tilde{\psi}^+ + H_2\tilde{\psi}^- = H_1(\psi^+)^{-1} + H_2(\psi^-)^{-1}, \tag{3.23}$$

is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \mathbb{C}_0^+$. Using a similar argument, we can see that

$$P^-(x,\lambda) \to I_3, \text{ when } \lambda \in \mathbb{C}_0^- \to \infty, \tag{3.24}$$

and

$$(\tilde{\psi}^{+,1}, \tilde{\psi}^{-,2}, \tilde{\psi}^{-,3})^T \to I_3, \text{ when } \lambda \in \mathbb{C}_0^+ \to \infty. \tag{3.25}$$

Till now, we have constructed the two matrix functions, P^+ and P^-, which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous in $\mathbb{C}_0^+$ and $\mathbb{C}_0^-$, respectively. Defining

$$G^+(x,\lambda) = P^+(x,\lambda),\ \lambda \in \mathbb{C}_0^+,\ (G^-)^{-1}(x,\lambda) = P^-(x,\lambda),\ \lambda \in \mathbb{C}_0^-, \tag{3.26}$$

we can directly show that on the real axis, the two matrix functions G^+ and G^- are related by

$$G^+(x,\lambda) = G^-(x,\lambda)G(x,\lambda),\ \lambda \in \mathbb{R}, \tag{3.27}$$

where by (3.10), we have

$$\begin{aligned} G(x,\lambda) &= E(H_1 + H_2S(\lambda))(H_1 + S^{-1}(\lambda)H_2)E^{-1} \\ &= E\begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} \\ s_{21} & 1 & 0 \\ s_{31} & 0 & 1 \end{bmatrix} E^{-1} \end{aligned} \tag{3.28}$$

with $S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{jl})_{3\times 3}$. The equations (3.27) and (3.28) are exactly the associated matrix Riemann–Hilbert problems we would like to build for the combined mKdV equation (1.9). The asymptotic properties

$$P^\pm(x,\lambda) \to I_3, \text{ when } \lambda \in \mathbb{C}_0^\pm \to \infty, \tag{3.29}$$

provide the canonical normalization conditions

$$G^\pm(x,\lambda) \to I_3, \text{ when } \lambda \in \mathbb{C}_0^\pm \to \infty, \tag{3.30}$$

for the presented Riemann–Hilbert problems.

To complete the direct scattering transform, let us take the derivative of (3.10) with time t and use the vanishing conditions of the potentials at infinity of t. This way, we can verify that the scattering matrix S satisfies

$$S_t = i\lambda^3[\Lambda, S], \tag{3.31}$$

which tells the time evolution of the time-dependent scattering coefficients:

$$s_{12} = s_{12}(0,\lambda)\mathrm{e}^{i\lambda^3 t},\ s_{13} = s_{13}(0,\lambda)\mathrm{e}^{i\lambda^3 t},\ s_{21} = s_{21}(0,\lambda)\mathrm{e}^{-i\lambda^3 t},\ s_{31} = s_{31}(0,\lambda)\mathrm{e}^{-i\lambda^3 t}, \tag{3.32}$$

and all other scattering coefficients are independent of the time variable t:

$$s_{11,t} = s_{22,t} = s_{23,t} = s_{32,t} = s_{33,t} = 0. \tag{3.33}$$

4. Inverse scattering

It is known that the Riemann–Hilbert problems with zeros can be solved by transforming into the ones without zeros [37]. The uniqueness of solutions to each associated Riemann–Hilbert problem, defined by (3.27) and (3.28), does not hold unless the zeros of $\det P^{\pm}$ in the upper and lower half-planes are specified and the structures of $\ker P^{\pm}$ at these zeros are determined [37,40,41].

Based on $\det \psi^{\pm} = 1$, it follows from the definitions of $P^{\pm}$ and the scattering relation between ψ^+ and ψ^- that

$$\det P^+(x, \lambda) = s_{11}(\lambda),\ \det P^-(x, \lambda) = \hat{s}_{11}(\lambda), \tag{4.1}$$

where, due to $\det S = 1$, we have

$$\hat{s}_{11} = (S^{-1})_{11} = s_{22}s_{33} - s_{23}s_{32}.$$

We now specify the scattering data. Let N be an arbitrary natural number and assume that $\det P^+$ has N zeros $\{\lambda_k,\ 1 \le k \le N\}$ in the upper half-plane, and $\det P^-$ has N zeros $\{\hat{\lambda}_k,\ 1 \le k \le N\}$ in the lower half-plane. The numbers of zeros of $\det P^+$ and $\det P^-$ must be the same, and otherwise, the associated Riemann–Hilbert problems are not solvable. Let us further assume that

$$\ker P^+(\lambda_k) = M_k,\ \operatorname{im} P^-(\hat{\lambda}_k) = N_k\ 1 \le k \le N, \tag{4.2}$$

where two subspaces M_k and N_k of $\mathbb{C}^3$ are given and satisfy

$$M_k \oplus N_k = \mathbb{C}^3,\ 1 \le k \le N. \tag{4.3}$$

We transform the Riemann–Hilbert problems in (3.27) with zeros into the corresponding Riemann–Hilbert problems without zeros. To this end, we introduce

$$P^- = P_I^- \tilde{P}^-,\ P^+ = \tilde{P}^+ P_I^+, \tag{4.4}$$

where P_I^- and P_I^+ are determined by a reduced Riemann–Hilbert problem

$$P_I^- P_I^+ = I_3, \tag{4.5}$$

with the same zeros given as for (3.27) and the same kernel structures:

$$\ker P_I^+(\lambda_k) = \ker P^+(\lambda_k),\ \operatorname{im} P_I^-(\hat{\lambda}_k) = \operatorname{im} P^-(\hat{\lambda}_k),\ 1 \le k \le N. \tag{4.6}$$

Then, $\tilde{P}^+$ and $\tilde{P}^-$ satisfy a Riemann–Hilbert problem without zeros

$$\tilde{P}^- \tilde{P}^+ = \tilde{G},\ \tilde{G} = (P_I^-)^{-1} G (P_I^+)^{-1} = P_I^+ G (P_I^+)^{-1}. \tag{4.7}$$

This kind of regular Riemann–Hilbert problems with canonical normalization can be systematically solved (see, e.g., [37]). The solution to the special Riemann–Hilbert problem in (4.5) with the indicated zeros and kernel structures can be determined as follows [37]:

$$\begin{aligned} P_I^- &= \Big(I_3+\frac{\lambda_1-\hat\lambda_1}{\lambda-\lambda_1}P_1\Big)\cdots\Big(I_3+\frac{\lambda_N-\hat\lambda_N}{\lambda-\lambda_N}P_N\Big),\\ P_I^+ &= \Big(I_3-\frac{\lambda_N-\hat\lambda_N}{\lambda-\hat\lambda_N}P_N\Big)\cdots\Big(I_3-\frac{\lambda_1-\hat\lambda_1}{\lambda-\hat\lambda_1}P_1\Big), \end{aligned} \tag{4.8}$$

where $P_k,\ 1\le k\le N$, are the projections (i.e., $P_k^2=P_k$) which satisfy

$$M_k=\ker P_I^+(\lambda_k)=U_k\,\mathrm{im}\,P_k,\ \ N_k=\mathrm{im}\,P_I^-(\hat\lambda_k)=U_k\ker P_k,\quad 1\le k\le N. \tag{4.9}$$

Note that a projection is uniquely determined when its kernel and image are given. In the above computations, $U_k,\ 1\le k\le N$, are determined by

$$P_I^-(\hat\lambda_k)=U_k(I_3-P_k)V_k,\ \ P_I^+(\lambda_k)=V_k^{-1}(I_3-P_k)U_k^{-1},\ 1\le k\le N. \tag{4.10}$$

Actually, those yield

$$\begin{aligned} U_k &= \Big(I_3+\frac{\lambda_1-\hat\lambda_1}{\hat\lambda_k-\lambda_1}P_1\Big)\cdots\Big(I_3+\frac{\lambda_{k-1}-\hat\lambda_{k-1}}{\hat\lambda_k-\lambda_{k-1}}P_{k-1}\Big),\\ V_k &= \Big(I_3+\frac{\lambda_{k+1}-\hat\lambda_{k+1}}{\hat\lambda_k-\lambda_{k+1}}P_{k+1}\Big)\cdots\Big(I_3+\frac{\lambda_N-\hat\lambda_N}{\hat\lambda_k-\lambda_N}P_N\Big), \end{aligned} \tag{4.11}$$

which are non-degenerate matrices, since $(I_3-cP)^{-1}=I_3-\frac{c}{c-1}P$ for $P^2=P$ when $c\ne 1$.

Since s_{11} and $\hat s_{11}$ are independent of t, we have $\lambda_{k,t}=\hat\lambda_{k,t}=0,\ 1\le k\le N$. The time evolution for M_k and N_k are determined as follows. First by using (3.5), we can show that

$$\frac{dv}{dt}-i\lambda_k^3\Lambda v\in\ker P^+(\lambda_k),\ \text{for } v\in\ker P^+(\lambda_k),\ 1\le k\le N, \tag{4.12}$$

which determines the law for the time evolution of the subspace M_k. Similarly, by using the adjoint equation of (3.5),

$$i\tilde\psi_t=\lambda^3[\tilde\psi,\Lambda]+\tilde\psi Q, \tag{4.13}$$

we can have

$$\frac{dv}{dt}+i\hat\lambda_k^3 v\Lambda\in\ker P^-(\hat\lambda_k),\ \text{for } v\in\ker P^-(\hat\lambda_k),\ 1\le k\le N, \tag{4.14}$$

which determines the law for the time evolution of the complement N_k of the subspace M_k.

Let us finally recover the potential matrix P. Note that P^+ solves the matrix spectral problem (3.4). Therefore, as long as we expand P^+ at large λ as

$$P^+(x,\lambda)=I_3+\frac{1}{\lambda}P_1^+(x)+\mathrm{O}(\frac{1}{\lambda^2}),\ \lambda\to\infty, \tag{4.15}$$

plugging this series expansion into (3.4) and comparing O(1) terms tell

$$\check P=-i[\Lambda,P_1^+]. \tag{4.16}$$

To realize the symmetric property of P, let us assume that

$$\hat\lambda_k=\bar\lambda_k,\ \ P_k^\dagger(\bar\lambda)=\overline{P_k(\lambda)}=CP_k(\bar\lambda)C^{-1},\ \ C=\mathrm{diag}(1,1,-1), \tag{4.17}$$

which guarantees

$$(P_1^+)^\dagger = -CP_1^+C^{-1}. \tag{4.18}$$

It then follows that (4.16) equivalently presents the potential matrix:

$$P = -[\Lambda, P_1^+] = \begin{bmatrix} 0 & -(P_1^+)_{12} & -(P_1^+)_{13} \\ (P_1^+)_{21} & 0 & 0 \\ (P_1^+)_{31} & 0 & 0 \end{bmatrix}, \tag{4.19}$$

where $P_1^+ = ((P_1^+)_{jl})_{3\times 3}$ and the symmetric property

$$P^\dagger = CPC^{-1} \tag{4.20}$$

is satisfied. Therefore, the two potentials p_1 and p_2 can be computed as follows:

$$p_1 = -(P_1^+)_{12},\ p_2 = -(P_1^+)_{13}. \tag{4.21}$$

This completes the inverse scattering problem: Given the scattering coefficients $s_{21}, s_{31}, \hat{s}_{12}, \hat{s}_{13}$, zeros $\lambda_k \in \mathbb{C}_0^+$ and $\hat{\lambda}_k = \bar{\lambda}_k \in \mathbb{C}_0^-$, and subspaces M_k and N_k satisfying $M_k \oplus N_k = \mathbb{C}^3,\ 1 \le k \le N$, we can get the potentials from (4.21), where $P^+ = G^+$ solves the Riemann–Hilbert problem (3.27) with $\ker G^+(\lambda_k) = M_k$ and $\operatorname{im} (G^-)^{-1}(\hat{\lambda}_k) = N_k,\ 1 \le k \le N$.

5. Soliton solutions

To generate soliton solutions, we assume that all these zeros, λ_k and $\hat{\lambda}_k = \bar{\lambda}_k,\ 1 \le k \le N$, are simple. Therefore, each of $\ker P^+(\lambda_k),\ 1 \le k \le N$, contains only a single basis column vector, denoted by v_k, $1 \le k \le N$; and each of $\ker P^-(\hat{\lambda}_k)$, $1 \le k \le N$, a single basis row vector, denoted by $\hat{v}_k$, $1 \le k \le N$:

$$P^+(\lambda_k)v_k = 0,\ \hat{v}_k P^-(\hat{\lambda}_k) = 0,\ 1 \le k \le N. \tag{5.1}$$

The Riemann–Hilbert problems, by (3.27) and (3.28), with the canonical normalization conditions in (3.30) and the zero structures in (5.1) can be solved as explained in the last section, and thus one can readily work out the potential u through (4.21).

To present soliton solutions, we take $G = I_3$ in each Riemann–Hilbert problem determined in (3.27). This can be achieved if we take

$$s_{21} = \hat{s}_{12} = s_{31} = \hat{s}_{13} = 0, \tag{5.2}$$

which means that no reflection exists in the scattering problem. The solution to this special Riemann–Hilbert problem can be generated by (see, e.g., [17,37]):

$$P^+(\lambda) = I_3 - \sum_{k,l=1}^N \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l},\ P^-(\lambda) = I_3 + \sum_{k,l=1}^N \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_l}, \tag{5.3}$$

where $M = (m_{kl})_{N \times N}$ is a square matrix whose entries are determined by

$$m_{kl} = \frac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k},\ 1 \le k, l \le N. \tag{5.4}$$

Since the zeros λ_k and $\hat{\lambda}_k$ are constants, i.e., space and time independent, we can easily work out the spatial and temporal evolutions for the vectors, $v_k(x,t)$ and $\hat{v}_k(x,t)$, $1 \le k \le N$, in the kernels. For example, let us evaluate the x-derivative of both sides of the first set of equations in (5.1). By using (3.4) first and then again the first set of equations in (5.1), we can arrive at

$$P^+(x,\lambda_k)\Big(\frac{dv_k}{dx} - i\lambda_k\Lambda v_k\Big) = 0,\ 1 \le k \le N. \tag{5.5}$$

This implies that for each $1 \le k \le N$, $\frac{dv_k}{dx} - i\lambda_k\Lambda v_k$ is in the kernel of $P^+(x,\lambda_k)$ as required, and so a constant multiple of v_k. For the sake of convenience, we suppose that

$$\frac{dv_k}{dx} = i\lambda_k\Lambda v_k,\ 1 \le k \le N. \tag{5.6}$$

On the other hand, we can similarly assume that the time dependence of v_k is defined by

$$\frac{dv_k}{dt} = i\lambda_k^3\Lambda v_k,\ 1 \le k \le N. \tag{5.7}$$

Therefore, we can explicitly give

$$v_k(x,t) = \mathrm{e}^{i\lambda_k\Lambda x + i\lambda_k^3\Lambda t} w_k,\ 1 \le k \le N, \tag{5.8}$$

where $w_k,\ 1 \le k \le N$, are arbitrary constant column vectors. To guarantee the symmetric property (4.20) in the spectral matrix, we need to take

$$\hat{v}_k(x,t) = \hat{w}_k \mathrm{e}^{-i\bar{\lambda}_k\Lambda x - i\bar{\lambda}_k^3\Lambda t},\ \hat{w}_k = w_k^\dagger C,\ 1 \le k \le N, \tag{5.9}$$

where C is defined as in (4.17).

Finally, from the solutions in (5.3), we have

$$P_1^+ = -\sum_{k,l=1}^N v_k(M^{-1})_{kl}\hat{v}_l, \tag{5.10}$$

which satisfies $(P_1^+)^\dagger = -CP_1^+C^{-1}$, and thus further through the presentations in (4.21), obtain an N-soliton solution to the combined mKdV equation (1.9):

$$p_1 = \sum_{k,l=1}^N v_{k,1}(M^{-1})_{kl}\hat{v}_{l,2},\ p_2 = \sum_{k,l=1}^N v_{k,1}(M^{-1})_{kl}\hat{v}_{l,3}, \tag{5.11}$$

where $v_k = (v_{k,1}, v_{k,2}, v_{k,3})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \hat{v}_{k,3})$, $1 \le k \le N$, are defined by (5.8) and (5.9), respectively.

Particularly, taking

$$\begin{cases} \lambda_1 = 5i,\ \lambda_2 = 3i,\ \lambda_3 = i,\ \hat{\lambda}_1 = -5i,\ \hat{\lambda}_2 = -3i,\ \hat{\lambda}_3 = -i, \\ w_1 = (3+2i, 1, i)^T,\ w_2 = (0, 2-i, 2+2i)^T,\ w_3 = (2, -2, 1+i)^T, \\ \hat{w}_1 = (3-2i, 1, i),\ \hat{w}_2 = (0, 2+i, -2+2i),\ \hat{w}_3 = (2, -2, -1+i), \end{cases} \tag{5.12}$$

we obtain one two-soliton solution to the combined mKdV equation (1.9):

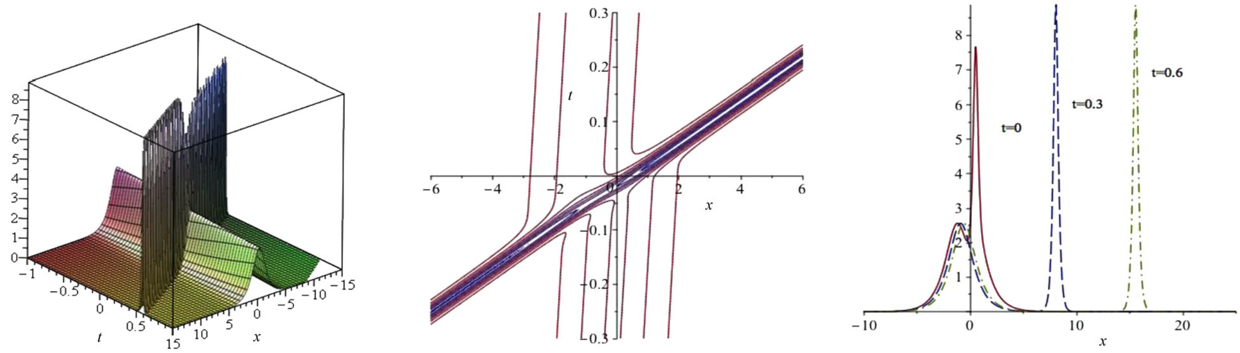

Fig. 1. Profiles of $|p_1|$: 3d plot (left), contour plot (middle) and x-curves (right).

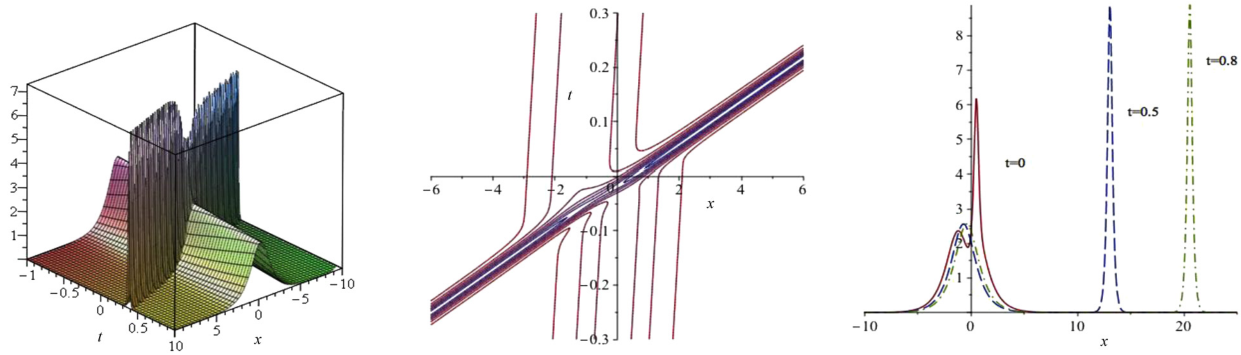

Fig. 2. Profiles of $|p_2|$: 3d plot (left), contour plot (middle) and x-curves (right).

$$p_1 = \frac{f_1}{g},\ p_2 = \frac{f_2}{g}, \tag{5.13}$$

where

$$\begin{aligned} f_1 &= (-4992 + 27456i)\,\mathrm{e}^{-23x+503t} - (5760 - 2400i)\,\mathrm{e}^{-19x+379t} \\ &\quad - (12000 - 14880i)\,\mathrm{e}^{-17x+377t} + (360 + 300i)\,\mathrm{e}^{-13x+253t}, \\ f_2 &= (-22464 - 12480i)\,\mathrm{e}^{-23x+503t} - (960 + 4800i)\,\mathrm{e}^{-19x+379t} \\ &\quad - (13500 + 8220i)\,\mathrm{e}^{-17x+377t} - (210 - 390i)\,\mathrm{e}^{-13x+253t}, \\ g &= 17784\,\mathrm{e}^{-22x+502t} + 1664\,\mathrm{e}^{-24x+504t} + 440\,\mathrm{e}^{-18x+378t} \\ &\quad + 65\,\mathrm{e}^{-12x+252t} + 90\,\mathrm{e}^{-14x+254t}. \end{aligned}$$

Three-dimensional plots, contour plots and x-curves of this set of solutions are made in Fig. 1 and Fig. 2.

6. Concluding remarks

We have considered a combined modified Korteweg–de Vries (mKdV) equation and its inverse scattering transform in terms of the Riemann–Hilbert problems. From special Riemann–Hilbert problems with the identity jump matrix, we have successfully worked out soliton solutions to the considered combined mKdV equation. As a specific example, we have presented a specific two soliton solution explicitly and made 3d plots, contour plots and x-curve plots to shed light on the characteristics of the presented soliton solution.

We remark that it would be interesting to present other kinds of exact solutions to integrable equations, including position and complexiton solutions [25,35], lump solutions [34,39,50], and algebro-geometric solutions [3,14,27,28], by applying the inverse scattering transform. It is expected that our studies would be helpful in recognizing those exact solutions from the perspective of the inverse scattering transform based on Riemann–Hilbert problems. About coupled mKdV systems, there are many recent studies such as integrable couplings [44,47], super hierarchies [6] and fractional analogous equations [5]. Therefore, another important topic for further study is to present the inverse scattering transform through Riemann–Hilbert problems for solving those generalized integrable counterparts.

The inverse scattering transform is very powerful in generating soliton solutions (see also, e.g., [16,49]). It has been recently generalized to solve initial-boundary value problems of integrable equations on the half-line and the finite interval [8,19]. Many other approaches to soliton solutions are available in the field of integrable equations, among which are the Hirota direct method [15], the generalized bilinear technique [26], the Wronskian technique [9,32] and the Darboux transformation [33,36]. It would be interesting to explore relations between those different approaches.

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