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Nonlocal integrable mKdV equations by two nonlocal reductions and their soliton solutions

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ABSTRACT

We conduct two nonlocal group reductions of the AKNS matrix spectral problems to generate a class of nonlocal reverse-spacetime integrable mKdV equations. One reduction replaces the spectral parameter with its negative complex conjugate while the other does not change the spectral parameter. Beginning with the specific distribution of eigenvalues, we construct soliton solutions by solving the corresponding generalized Riemann-Hilbert problems with the identity jump matrix, where eigenvalues could equal adjoint eigenvalues.

1. Introduction

Matrix spectral problems lay the basis for the theory of integrable equations. Their group reductions can result in both local and nonlocal reduced integrable equations [2,13,22]. Starting from the Ablowitz-Kaup-Newell-Segur (AKNS) matrix spectral problems, we can generate three kinds of nonlocal nonlinear Schrödinger (NLS) equations and two kinds of nonlocal modified Korteweg-de Vries (mKdV) equations by conducting one group reduction [2,14], and other kinds of novel nonlocal reduced integrable equations by conducting two group reductions [19,20]. The inverse scattering transform has been successfully applied to analysis of soliton solutions to nonlocal integrable equations (see, e.g., [1,10]).

Integrable equations can also be solved by other efficient methods, which include Darboux transformation, the Hirota bilinear method and Riemann-Hilbert problems, and indeed, their soliton solutions can be systematically presented (see, e.g., [5,9,23,24,27]). In particular, the Riemann-Hilbert technique is used to solve nonlocal integrable NLS and mKdV equations. We refer the interested readers to the recent references [3,25,26] on local equations and [14–16,28] on nonlocal equations, which present applications of Riemann-Hilbert problems. In this paper, we would like to present a kind of novel reduced nonlocal reverse-spacetime integrable mKdV equations by conducting two nonlocal group reductions and compute their soliton solutions through generalized Riemann-Hilbert problems with the identity jump matrix.

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The rest of this paper is organized as follows. In Section 2, we conduct two nonlocal group reductions for the AKNS matrix spectral problems to present type $(-\lambda^*, \lambda)$ reduced nonlocal reverse-spacetime integrable mKdV equations, where $*$ stands for the complex conjugate. Two pairs of scalar examples are

$$p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\sigma p_1 p_1^*(-x,-t)p_{1,x} - 3\sigma p_1(-x,-t)(|p_1|^2)_x],$$

and

$$p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6\delta p_1 p_1(-x,-t)p_{1,x} + 3\delta p_1^*(-x,-t)(|p_1|^2)_x],$$

where $\sigma = \pm 1$, $\delta = \pm 1$, and α and β are arbitrary real constants. In Section 3, based on the explored distribution of eigenvalues, we solve the corresponding generalized Riemann-Hilbert problems with the identity jump matrix, where eigenvalues could equal adjoint eigenvalues, and compute soliton solutions to the resulting reduced nonlocal integrable mKdV equations. In the final section, we present a conclusion and some concluding remarks.

2. Reduced nonlocal integrable mKdV equations

2.1. The matrix AKNS integrable hierarchies revisited

Let us first recall the AKNS hierarchies of matrix integrable equations for subsequent analysis.

As usual, let λ denote the spectral parameter, and assume that p and q are two matrix potentials:

$$p = p(x,t) = (p_{jk})_{m\times n},\ q = q(x,t) = (q_{kj})_{n\times m}, \tag{2.1}$$

where $m, n \geq 1$ are two arbitrarily given integers. We consider the matrix AKNS spectral problems as follows:

$$\begin{cases} -i\phi_x = U\phi = U(u,\lambda)\phi = (\lambda\Lambda + P)\phi, \\ -i\phi_t = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi = (\lambda^r\Omega + Q^{[r]})\phi,\ r\geq 0. \end{cases} \tag{2.2}$$

Here the $(m+n)$-th order square matrices, Λ and Ω, are defined by

$$\Lambda = \mathrm{diag}(\alpha_1 I_m, \alpha_2 I_n),\ \Omega = \mathrm{diag}(\beta_1 I_m, \beta_2 I_n), \tag{2.3}$$

where I_s denotes the identity matrix of size s, and α_1, α_2 and β_1, β_2 are two pairs of arbitrarily given distinct real constants. The other two $(m+n)$-th order square matrices, P and $Q^{[r]}$, are defined by

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.4}$$

which is called the potential matrix, and

$$Q^{[r]} = \sum_{s=0}^{r-1}\lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \tag{2.5}$$

where $a^{[s]}, b^{[s]}, c^{[s]}$ and $d^{[s]}$ are determined recursively by

$$b^{[0]} = 0,\ c^{[0]} = 0,\ a^{[0]} = \beta_1 I_m,\ d^{[0]} = \beta_2 I_n, \tag{2.6a}$$

$$b^{[s+1]} = \frac{1}{\alpha}(-ib_x^{[s]} - pd^{[s]} + a^{[s]}p),\ s\geq 0, \tag{2.6b}$$

$$c^{[s+1]} = \frac{1}{\alpha}(ic_x^{[s]} + qa^{[s]} - d^{[s]}q),\ s\geq 0, \tag{2.6c}$$

$$a_x^{[s]} = i(pc^{[s]} - b^{[s]}q),\ d_x^{[s]} = i(qb^{[s]} - c^{[s]}p),\ s\geq 1, \tag{2.6d}$$

with zero constants of integration being taken. In particular, we can obtain

$$Q^{[1]} = \frac{\beta}{\alpha}P,\ Q^{[2]} = \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2}I_{m,n}(P^2 + iP_x),$$

and

$$Q^{[3]} = \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3),$$

where $\alpha = \alpha_1 - \alpha_2$, $\beta = \beta_1 - \beta_2$ and $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. The recursive relations in (2.6) also mean that

$$W=\sum_{s\geq 0}\lambda^{-s}\begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{2.7}$$

provide a Laurent series solution to the stationary zero curvature equation

$$W_x=i[U,W]. \tag{2.8}$$

The compatibility conditions of the two matrix spectral problems in (2.2), i.e., the zero curvature equations:

$$U_t-V_x^{[r]}+i[U,V^{[r]}]=0,\ r\geq 0, \tag{2.9}$$

generate one matrix AKNS integrable hierarchy (see, e.g., [17] for more details):

$$p_t=i\alpha b^{[r+1]},\ q_t=-i\alpha c^{[r+1]},\ r\geq 0, \tag{2.10}$$

which can be proved to possess a bi-Hamiltonian structure and infinitely many symmetries and conservation laws. The second nonlinear integrable system in the hierarchy gives us the AKNS matrix mKdV equations:

$$p_t=-\frac{\beta}{\alpha^3}(p_{xxx}+3pqp_x+3p_xqp),\ q_t=-\frac{\beta}{\alpha^3}(q_{xxx}+3q_xpq+3qpq_x), \tag{2.11}$$

where p and q are the two matrix potentials defined by (2.1).

2.2. Reduced nonlocal integrable mKdV equations

We would like to conduct two nonlocal group reductions for the matrix AKNS spectral problems in (2.2) simultaneously (see also [11] for the basic idea on how to make group reductions), to present a kind of novel reduced nonlocal reverse-spacetime integrable mKdV equations.

Assume that Σ_1 and Σ_2 are a pair of constant invertible Hermitian matrices of sizes m and n, respectively, and Δ_1 and Δ_2 are another pair of constant invertible symmetric matrices of sizes m and n, respectively. Let us consider two nonlocal group reductions for the spectral matrix U:

$$U^\dagger(-x,-t,-\lambda^*)=(U(-x,-t,-\lambda^*))^\dagger=-\Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{2.12}$$

and

$$U^T(-x,-t,\lambda)=(U(-x,-t,\lambda))^T=\Delta U(x,t,\lambda)\Delta^{-1}, \tag{2.13}$$

where Σ and Δ are the two constant invertible matrices defined by

$$\Sigma=\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},\ \Delta=\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{2.14}$$

Equivalently, these two nonlocal group reductions require

$$P^\dagger(-x,-t)=-\Sigma P(x,t)\Sigma^{-1}, \tag{2.15}$$

and

$$P^T(-x,-t)=\Delta P(x,t)\Delta^{-1}, \tag{2.16}$$

respectively. More precisely, they require the following reductions for the matrix potentials p and q:

$$q(x,t)=-\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1, \tag{2.17}$$

and

$$q(x,t)=\Delta_2^{-1}p^T(-x,-t)\Delta_1, \tag{2.18}$$

respectively. It therefore follows that the matrix potential p must satisfy a constraint:

$$-\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1=\Delta_2^{-1}p^T(-x,-t)\Delta_1, \tag{2.19}$$

to guarantee that both group reductions in (2.12) and (2.13) are satisfied.

Furthermore, under the group reductions in (2.12) and (2.13), we can have that

$$\begin{cases} W^\dagger(-x,-t,-\lambda^*)=(W(-x,-t,-\lambda^*))^\dagger=\Sigma W(x,t,\lambda)\Sigma^{-1}, \\ W^T(-x,-t,\lambda)=(W(-x,-t,\lambda))^T=\Delta W(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{2.20}$$

which ensures that

$$\begin{cases} V^{[2s+1]\dagger}(-x,-t,-\lambda^*) = (V^{[2s+1]}(-x,-t,-\lambda^*))^\dagger = -\Sigma V^{[2s+1]}(x,t,\lambda)\Sigma^{-1}, \\ V^{2s+1]T}(-x,-t,\lambda) = (V^{[2s+1]}(-x,-t,\lambda))^T = \Delta V^{[2s+1]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{2.21}$$

and

$$\begin{cases} Q^{[2s+1]\dagger}(-x,-t,-\lambda^*) = (Q^{[2s+1]}(-x,-t,-\lambda^*))^\dagger = -\Sigma Q^{[2s+1]}(x,t,\lambda)\Sigma^{-1}, \\ Q^{[2s+1]T}(-x,-t,\lambda) = (Q^{[2s+1]}(-x,-t,\lambda))^T = \Delta Q^{[2s+1]}(x,t,\lambda)\Delta^{-1}, \end{cases} \tag{2.22}$$

where $s \geq 0$.

Consequently, under the potential reductions (2.17) and (2.18), the integrable matrix AKNS equations in (2.10) are reduced to a hierarchy of nonlocal reverse-spacetime integrable matrix mKdV equations:

$$p_t = i\alpha b^{[2s+2]}|_{q=-\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1=\Delta_2^{-1}p^T(-x,-t)\Delta_1},\ s \geq 0, \tag{2.23}$$

where p is an $m \times n$ matrix potential satisfying (2.19), Σ_1 and Σ_2 are a pair of arbitrary invertible Hermitian matrices of sizes m and n, respectively, and Δ_1 and Δ_2 are a pair of arbitrary invertible symmetric matrices of sizes m and n, respectively. As consequences of the two group reductions, each reduced equation in the hierarchy (2.23) possesses a Lax pair of the reduced spatial and temporal matrix spectral problems in (2.2) with $r = 2s+1,\ s \geq 0$, and infinitely many symmetries and conservation laws reduced from those for the integrable matrix AKNS equations in (2.10) with $r = 2s+1, s \geq 0$.

If we fix $s = 1$, i.e., $r = 3$, then the reduced matrix integrable mKdV equations in (2.23) with $s = 1$ produce a kind of reduced nonlocal reverse-spacetime integrable matrix mKdV equations:

$$\begin{aligned} p_t &= -\frac{\beta}{\alpha^3}(p_{xxx} - 3p\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1 p_x - 3p_x\Sigma_2^{-1}p^\dagger(-x,-t)\Sigma_1 p) \\ &= -\frac{\beta}{\alpha^3}(p_{xxx} + 3p\Delta_2^{-1}p^T(-x,-t)\Delta_1 p_x + 3p_x\Delta_2^{-1}p^T(-x,-t)\Delta_1 p), \end{aligned} \tag{2.24}$$

where p is an $m \times n$ matrix potential satisfying (2.19).

Let us now illustrate these novel reduced nonlocal reverse-spacetime integrable matrix mKdV equations, with a few examples with different values for m, n and appropriate choices for Σ, Δ.

First, we consider the case of $m = 1$ and $n = 2$. Let us choose

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix},\ \Delta_1 = 1,\ \Delta_2^{-1} = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix},$$

where σ and δ are real constants which satisfy $\sigma^2 = \delta^2 = 1$. Then, the potential constraint (2.19) equivalently requires

$$p_2 = -\sigma\delta p_1^*,$$

where $p = (p_1, p_2)$, and at this moment, the corresponding potential matrix P becomes

$$P = \begin{bmatrix} 0 & p_1 & -\sigma\delta p_1^* \\ -\sigma p_1^*(-x,-t) & 0 & 0 \\ \delta p_1(-x,-t) & 0 & 0 \end{bmatrix}. \tag{2.25}$$

Furthermore, based on (2.11), the corresponding novel reduced nonlocal reverse-spacetime integrable mKdV equations read

$$p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\sigma p_1 p_1^*(-x,-t)p_{1,x} - 3\sigma p_1(-x,-t)(|p_1|^2)_x], \tag{2.26}$$

where $\sigma = \pm 1$ and p_1^* denotes the complex conjugate of p_1. A similar argument with

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix},\ \Delta_1 = 1,\ \Delta_2^{-1} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix},$$

where σ and δ are real constants which satisfy $\sigma^2 = \delta^2 = 1$, leads to a second pair of novel scalar nonlocal reverse-spacetime integrable mKdV equations:

$$p_{1,t} = -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6\delta p_1 p_1(-x,-t)p_{1,x} + 3\delta p_1^*(-x,-t)(|p_1|^2)_x], \tag{2.27}$$

where $\delta = \pm 1$ and p_1^* denotes the complex conjugate of p_1 as well. These two pairs of equations are totally different from the ones studied in [2,6–8], in which only one nonlocal nonlinear term appears.

Second, we consider the case of $m = 1$ and $n = 4$. Let us choose

$$\Sigma_1 = 1,\ \Sigma_2^{-1} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \end{bmatrix},\ \Delta_1 = 1,\ \Delta_2^{-1} = \begin{bmatrix} 0 & \delta_1 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 \\ 0 & 0 & \delta_2 & 0 \end{bmatrix},$$

where σ_j and δ_j are real constants which satisfy $\sigma_j^2 = \delta_j^2 = 1,\ \ j = 1, 2$. Then, the potential constraint (2.19) leads equivalently to

$$p_2 = \sigma_1\delta_1 p_1^*(-x, -t),\ \ p_4 = \sigma_2\delta_2 p_3^*(-x, -t),$$

where $p = (p_1, p_2, p_3, p_4)$, and thus, the corresponding potential matrix P reads

$$P = \begin{bmatrix} 0 & p_1 & -\sigma_1\delta_1 p_1^* & p_3 & -\sigma_2\delta_2 p_3^* \\ -\sigma_1 p_1^*(-x, -t) & 0 & 0 & 0 & 0 \\ \delta_1 p_1(-x, -t) & 0 & 0 & 0 & 0 \\ -\sigma_2 p_3^*(-x, -t) & 0 & 0 & 0 & 0 \\ \delta_2 p_3(-x, -t) & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.28}$$

This formulation allows us to obtain a class of two-component nonlocal reverse-spacetime integrable mKdV equations:

$$\begin{cases} p_{1,t} = -\dfrac{\beta}{\alpha^3}[p_{1,xxx} - 6\sigma_1 p_1 p_1^*(-x, -t)p_{1,x} - 3\sigma_1 p_1(-x, -t)(|p_1|^2)_x \\ \qquad -3\sigma_2 p_3^*(-x, -t)(p_1 p_3)_x - 3\sigma_2 p_3(-x, -t)(p_1 p_3^*)_x], \\ p_{3,t} = -\dfrac{\beta}{\alpha^3}[p_{3,xxx} - 3\sigma_1 p_1^*(-x, -t)(p_1 p_3)_x - 3\sigma_1 p_1(-x, -t)(p_1^* p_3)_x \\ \qquad -6\sigma_2 p_3 p_3^*(-x, -t)p_{3,x} - 3\sigma_2 p_3(-x, -t)(|p_3|^2)_x], \end{cases} \tag{2.29}$$

where σ_j are real constants satisfying $\sigma_j^2 = 1,\ \ j = 1, 2$.

Third, we consider the case of $m = 2$ and $n = 2$. Let us choose

$$\Sigma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix},\ \Sigma_2^{-1} = \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix},\ \Delta_1 = \begin{bmatrix} 0 & \delta_1 \\ \delta_1 & 0 \end{bmatrix},\ \Delta_2^{-1} = \begin{bmatrix} 0 & \delta_2 \\ \delta_2 & 0 \end{bmatrix},$$

where σ_j and δ_j are real constants which satisfy $\sigma_j^2 = \delta_j^2 = 1$. Then, the potential constraint (2.19) leads precisely to

$$p_{12} = -\sigma_1\delta_1\sigma_2\delta_2 p_{11}^*,\ \ p_{22} = -\sigma_1\delta_1\sigma_2\delta_2 p_{21}^*,$$

and thus, the corresponding matrix potentials become

$$p = \begin{bmatrix} p_{11} & -\sigma_1\delta_1\sigma_2\delta_2 p_{11}^* \\ p_{21} & -\sigma_1\delta_1\sigma_2\delta_2 p_{21}^* \end{bmatrix},\ q = \begin{bmatrix} -\sigma_1\sigma_2 p_{21}^*(-x, -t) & -\sigma_1\sigma_2 p_{11}^*(-x, -t) \\ \delta_1\delta_2 p_{21}(-x, -t) & \delta_1\delta_2 p_{11}(-x, -t) \end{bmatrix}. \tag{2.30}$$

This enables us to obtain another class of two-component nonlocal reverse-spacetime integrable mKdV equations:

$$\begin{cases} p_{11,t} = -\dfrac{\beta}{\alpha^3}[p_{11,xxx} - 6\sigma p_{11} p_{21}^*(-x, -t)p_{11,x} - 3\sigma p_{21}(-x, -t)(|p_{11}|^2)_x \\ \qquad -3\sigma p_{21}^*(-x, -t)(p_{11} p_{21})_x - 3\sigma p_{11}(-x, -t)(p_{11}^* p_{21})_x], \\ p_{21,t} = -\dfrac{\beta}{\alpha^3}[p_{21,xxx} - 3\sigma p_{21}^*(-x, -t)(p_{11} p_{21})_x - 3\sigma p_{21}(-x, -t)(p_{11} p_{21}^*)_x \\ \qquad -6\sigma p_{11}^*(-x, -t)p_{21} p_{21,x} - 3\sigma p_{11}(-x, -t)(|p_{21}|^2)_x], \end{cases} \tag{2.31}$$

where $\sigma = \sigma_1\sigma_2 = \pm 1$. Obviously, the nonlinearity pattern in these two equations is different from the one in (2.29).

3. Soliton solutions

3.1. Distribution of eigenvalues

Under the group reduction in (2.12) (or (2.13)), we can observe that λ is an eigenvalue of the matrix spectral problems in (2.2) if and only if $\hat{\lambda}=-\lambda^*$ (or $\hat{\lambda}=\lambda$) is an adjoint eigenvalue, namely, the adjoint matrix spectral problems hold:

$$i\tilde{\phi}_x=\tilde{\phi}U=\tilde{\phi}U(u,\hat{\lambda}),\ i\tilde{\phi}_t=\tilde{\phi}V^{[2s+1]}=\tilde{\phi}V^{[2s+1]}(u,\hat{\lambda}), \tag{3.1}$$

where $s\geq 0$. As a consequence, we can assume to have eigenvalues $\lambda:\ \mu,\ -\mu^*$, and adjoint eigenvalues $\hat{\lambda}:\ -\mu^*,\ \mu$, where $\mu\in\mathbb{C}$.

Moreover, under the group reductions in (2.12) and (2.13), we can see that

$$\phi^\dagger(-x,-t,-\lambda^*)\Sigma \text{ and } \phi^T(-x,-t,\lambda)\Delta, \tag{3.2}$$

will be two adjoint eigenfunctions associated with the same original eigenvalue λ, provided that $\phi(\lambda)$ is an eigenfunction of the matrix spectral problems in (2.2) associated with an eigenvalue λ.

3.2. Solution formulation by generalized Riemann-Hilbert problems

We would like to propose a general formulation of soliton solutions to the resulting reduced nonlocal reverse-spacetime integrable mKdV equations by solving the corresponding generalized Riemann-Hilbert problems with the identity jump matrix. Let $N_1,N_2\geq 0$ be two integers such that $N=2N_1+N_2\geq 1$.

First, let us take N eigenvalues λ_k and N adjoint eigenvalues $\hat{\lambda}_k$ as follows:

$$\lambda_k,\ 1\leq k\leq N:\ \mu_1,\ \cdots,\ \mu_{N_1},-\mu_1^*,\ \cdots,\ -\mu_{N_1}^*,\ \nu_1,\ \cdots,\ \nu_{N_2}, \tag{3.3}$$

and

$$\hat{\lambda}_k,\ 1\leq k\leq N:\ -\mu_1^*,\ \cdots,\ -\mu_{N_1}^*,\mu_1,\ \cdots,\ \mu_{N_1},\ -\nu_1,\ \cdots,\ -\nu_{N_2}, \tag{3.4}$$

where $\mu_k\in\mathbb{C},\ \ 1\leq k\leq N_1$, and $\nu_k\in\mathbb{R},\ \ 1\leq k\leq N_2$, and assume that their corresponding eigenfunctions and adjoint eigenfunctions are defined by

$$v_k,\ 1\leq k\leq N, \text{ and } \hat{v}_k,\ 1\leq k\leq N, \tag{3.5}$$

respectively. Obviously, in this nonlocal case, the following condition:

$$\{\lambda_k\,|\,1\leq k\leq N\}\cap\{\hat{\lambda}_k\,|\,1\leq k\leq N\}=\emptyset, \tag{3.6}$$

doesn't hold.

Next, we introduce two matrices:

$$G^+(\lambda)=I_{m+n}-\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\hat{\lambda}_l},\ (G^-)^{-1}(\lambda)=I_{m+n}+\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\lambda_k}, \tag{3.7}$$

where M is a square matrix $M=(m_{kl})_{N\times N}$, whose entries are defined by

$$m_{kl}=\begin{cases}\dfrac{\hat{v}_k v_l}{\lambda_l-\hat{\lambda}_k}, & \text{if } \lambda_l\neq\hat{\lambda}_k,\\ 0, & \text{if } \lambda_l=\hat{\lambda}_k,\end{cases}\quad \text{where } 1\leq k,l\leq N. \tag{3.8}$$

It has been shown in [15] that these two matrices $G^+(\lambda)$ and $G^-(\lambda)$ solve the corresponding generalized Riemann-Hilbert problem with the identity jump matrix, i.e., they satisfy

$$(G^-)^{-1}(\lambda)G^+(\lambda)=I_{m+n},\ \lambda\in\mathbb{R}, \tag{3.9}$$

provided that an orthogonal condition:

$$\hat{v}_k v_l=0 \text{ if } \lambda_l=\hat{\lambda}_k,\ \text{ where } 1\leq k,l\leq N, \tag{3.10}$$

holds.

Now, let us make an asymptotic expansion

$$G^+(\lambda)=I_{m+n}+\frac{1}{\lambda}G_1^+ +O(\frac{1}{\lambda^2}), \tag{3.11}$$

as $\lambda \to \infty$, to obtain

$$G_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{3.12}$$

and substituting this into the matrix spatial spectral problems in (2.2) yields

$$P = -[\Lambda, G_1^+] = \lim_{\lambda\to\infty} [G^+(\lambda), \Lambda]. \tag{3.13}$$

Clearly, this generates N-soliton solutions to the matrix AKNS integrable equations (2.10):

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2, \; q = -\alpha \sum_{k,l=1}^{N} v_k^2 (M^{-1})_{kl} \hat{v}_l^1, \tag{3.14}$$

where for each $1 \le k \le N$, we have split v_k and $\hat{v}_k$ into $v_k = ((v_k^1)^T, (v_k^2)^T)^T$ and $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2)$, where v_k^1 and $\hat{v}_k^1$ are column and row vectors of dimension m, respectively, and v_k^2 and $\hat{v}_k^2$ are column and row vectors of dimension n, respectively.

When zero potentials are taken, i.e., $p = 0$ and $q = 0$ are chosen, the corresponding matrix spectral problems in (2.2) engender

$$v_k = v_k(x, t, \lambda_k) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^r \Omega t} w_k, \; 1 \le k \le N, \tag{3.15}$$

where w_k, $1 \le k \le N$, are constant column vectors. According to the preceding analysis in subsection 3.1, we can take the corresponding adjoint eigenfunctions as follows:

$$\hat{v}_k = \hat{v}_k(x, t, \hat{\lambda}_k) = v_k^\dagger(x, t, \lambda_k)\Sigma = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^r \Omega t}, \; 1 \le k \le N, \tag{3.16}$$

where

$$\hat{w}_k = w_k^\dagger \Sigma, \; 1 \le k \le N. \tag{3.17}$$

Then, the orthogonal condition (3.10) becomes

$$w_k^\dagger \Sigma w_l = 0 \text{ if } \lambda_l = \hat{\lambda}_k, \text{ where } 1 \le k, l \le N. \tag{3.18}$$

Finally, to present N-soliton solutions for the reduced nonlocal matrix integrable mKdV equations (2.23), we need to check whether G_1^+ defined by (3.12) satisfies the two involution properties:

$$(G_1^+)^\dagger(-x, -t) = \Sigma G_1^+ \Sigma^{-1}, \; (G_1^+)^T(-x, -t) = -\Delta G_1^+ \Delta^{-1}. \tag{3.19}$$

If so, the resulting potential matrix P given by (3.13) will satisfy the two nonlocal group reduction conditions in (2.15) and (2.16). Further, as a consequence of these conditions, we obtain the following N-soliton solutions:

$$p = \alpha \sum_{k,l=1}^{N} v_k^1 (M^{-1})_{kl} \hat{v}_l^2, \tag{3.20}$$

for the reduced nonlocal reverse-spacetime matrix integrable mKdV equations (2.23). These solutions are reduced from the N-soliton solutions in (3.14) for the matrix AKNS equations (2.10).

3.3. *Realizing the involution properties*

We would now like to check how to realize the involution properties in (3.19).

First, following the preceding analysis in subsection 3.1, the adjoint eigenfunctions $\hat{v}_k$, $1 \le k \le 2N_1$, can be determined as follows:

$$\hat{v}_k = \hat{v}_k(x, t, \hat{\lambda}_k) = v_k^\dagger(-x, -t, \lambda_k)\Sigma = v_{N_1+k}^T(-x, -t, -\lambda_k^*)\Delta, \; 1 \le k \le N_1, \tag{3.21}$$

and

$$\hat{v}_{N_1+k} = \hat{v}_{N_1+k}(x, t, \hat{\lambda}_{N_1+k}) = v_{N_1+k}^\dagger(-x, -t, \lambda_{N_1+k})\Sigma = v_k^T(-x, -t, \lambda_k)\Delta, \; 1 \le k \le N_1. \tag{3.22}$$

These selections in (3.21) and (3.22) generate the conditions on w_k, $1 \le k \le N$:

$$\begin{cases} w_k^T(\Sigma^* \Delta^{*-1} - \Delta \Sigma^{-1}) = 0, \; 1 \le k \le N_1, \\ w_k = \Sigma^{-1} \Delta^* w_{k-N_1}^*, \; N_1 + 1 \le k \le 2N_1, \end{cases} \tag{3.23}$$

where A^* denotes the complex conjugate of a matrix A. Note that all these conditions aim to satisfy the reduction conditions in (2.15) and (2.16).

Next, note that when the solutions to the generalized Riemann-Hilbert problems with the identity jump matrix, defined by (3.7) and (3.8), possess the involution properties

$$(G^+)^\dagger(-\lambda^*) = \Sigma(G^-)^{-1}(\lambda)\Sigma^{-1},\ (G^+)^T(\lambda) = \Delta(G^-)^{-1}(\lambda)\Delta^{-1}, \tag{3.24}$$

the corresponding relevant matrix G_1^+ will satisfy the involution properties in (3.19), which are consequences of the group reductions in (2.12) and (2.13). Consequently, when the conditions in (3.23) and the orthogonal condition in (3.18) are satisfied for w_k, $1 \le k \le N$, the formula (3.20), together with (3.7), (3.8), (3.15) and (3.16), presents N-soliton solutions to the reduced nonlocal reverse-spacetime matrix integrable mKdV equations (2.23).

Lastly, for the case of $m = n/2 = N = 1$, let us compute an example of periodic solutions to the reduced nonlocal integrable mKdV equations. We choose $\lambda_1 = \nu,\ \hat{\lambda}_1 = -\nu,\ \ \nu \in \mathbb{R}$, and set

$$w_1 = (w_{1,1}, w_{1,2}, w_{1,3})^T, \tag{3.25}$$

where $w_{1,1}, w_{1,2}, w_{1,3} \in \mathbb{R}$ are arbitrary but satisfy $w_{1,3}^2 = w_{1,2}^2$. These choices yield a class of periodic solutions to the nonlocal reverse-spacetime integrable mKdV equation (2.26):

$$p_1 = \frac{2\sigma\nu(\alpha_1-\alpha_2)w_{1,1}w_{1,2}}{w_{1,1}^2 e^{i(\alpha_1-\alpha_2)\nu x + i(\beta_1-\beta_2)\nu^3 t} + 2\sigma w_{1,2}^2 e^{-i(\alpha_1-\alpha_2)\nu x - i(\beta_1-\beta_2)\nu^3 t}}, \tag{3.26}$$

where ν is an arbitrary real constant, and $w_{1,1}$ and $w_{1,2}$ are arbitrary real constants but required to satisfy the condition $w_{1,1}^2 = 2w_{1,2}^2$, generated from the involution properties in (3.19).

4. Concluding remarks

Type $(-\lambda^*, \lambda)$ reduced nonlocal reverse-spacetime integrable mKdV equations were generated and their soliton solutions were computed through the corresponding generalized Riemann-Hilbert problems with the identity jump matrix, where eigenvalues could equal adjoint eigenvalues. The analysis is based on the two nonlocal group reductions of the AKNS matrix spectral problems. The resulting nonlocal integrable mKdV equations are a type of novel nonlocal reverse-spacetime integrable equations, which possess one nonlocal factor in the nonlinear terms.

We remark that it would be interesting to explore soliton solutions by other approaches, including the Hirota direct method, the Wronskian technique and the Darboux transformation. Another interesting problem is to search for other kinds of reduced nonlocal integrable equations associated with other classes of matrix spectral problems (see, e.g., [18]). It is also of significant importance to study dynamical properties of diverse exact solutions in the nonlocal case, including lump solutions [21], solitonless solutions [12] and algebro-geometric solutions [4], from a perspective of Riemann-Hilbert problems. By contrast, comparatively little has been known about nonlocal integrable equations.

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